Mathematics AA HL

Non-calculator topic test

Binomial theorem, functions and polynomials

Time allowed	**90 minutes**
Maximum mark	**60**
Questions	**10; answer all questions**
Calculator	**Not permitted**
Formula reference	Included on pages 12–13
Solutions	Included at the back, from page 14

Instructions

1. Show all necessary working in the spaces provided. Correct answers without appropriate working may not receive full marks.
2. Give exact answers unless instructed otherwise. Apply domain restrictions and reject inadmissible solutions.
3. Label important features on sketches. Use a pencil and ruler where appropriate; a sketch does not need to be plotted point by point.
4. Unless otherwise stated, functions have their largest real domains. Use radians for arguments of complex numbers.
5. The maximum mark for each part is shown in square brackets. You may refer to the formula appendix throughout the test.

This is an original Radford Mathematics assessment, not an official IB examination.

This publication contains 18 pages. Questions and the formula appendix occupy pages 1–13; solutions and the mark scheme begin on page 14.

Question 1 — 6 marks

Let $f(x) = x^2 + 2x - 3$.

(a) Write $f(x)$ in vertex form and state the coordinates of its vertex. [2]

(b) The line $y = 2x + k$ is tangent to $y = f(x)$. Use the discriminant to find k. [2]

(c) Find the equation of the line through the vertex of $y = f(x)$ that is perpendicular to this tangent. [2]

Question 2 — 2 marks

Solve $|3x + 1| \leq 2$. [2]

Question 3 5 marks

The coefficient of x^2 in the expansion of $(1+2x)^n$ is 40, where n is an integer and $n \geq 2$.

(a) Find n. [3]

(b) Hence find the coefficient of x^3 in this expansion. [2]

Question 4 — 7 marks

(a) Expand $(1-2x)^{-2}$ up to and including the term in x^2. [2]

(b) Expand $\sqrt{1+4x}$ up to and including the term in x^2. [2]

(c) Hence find the expansion of $\dfrac{\sqrt{1+4x}}{(1-2x)^2}$ up to and including the term in x^2. [2]

(d) State the common open interval of validity for the expansions in parts (a) and (b), obtained from the binomial condition $|u|<1$ for $(1+u)^n$. [1]

Question 5 — 6 marks

Let

$$f(x) = \frac{2x+3}{x-2} \qquad \text{and} \qquad g(x) = \sqrt{x+1}.$$

(a) Find $f^{-1}(x)$ and state its domain. Hence explain why f is self-inverse. [3]

(b) Find $(f \circ g)(x)$ and state its domain. [3]

Question 6 — 6 marks

Let $h(x) = 3^{x-1} - 2$, for $x \in \mathbb{R}$.

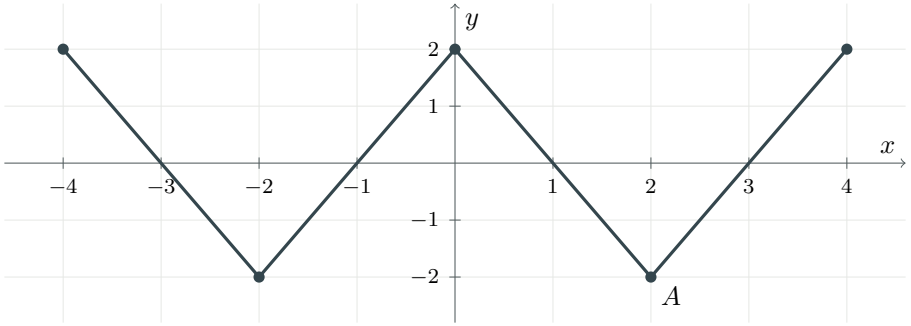
(a) State the range of h and the equation of its horizontal asymptote. [2]

(b) Solve $h(x) = 7$. [1]

(c) Solve $\log_2 x + \log_2(x-2) = 3$. [3]

Question 7 — 8 marks

The graph shows part of a continuous periodic function f, defined for all real x. The same pattern repeats in both directions. Between the labelled points, the graph consists of straight-line segments.



(a) State the least positive period of f and whether f is odd, even or neither. [2]

(b) The point $A = (2, -2)$ lies on $y = f(x)$. Find the coordinates of its corresponding point on

$$y = -2f\left(\frac{1-x}{2}\right) + 3.$$

[2]

Question 7 continues on the next page.

Question 7 continued

(c) On the axes below, sketch $y = |f(x)|$ for $-2 \leq x \leq 2$. Label all intercepts. [2]



(d) State the coordinates of the y-intercept of each graph: $y = [f(x)]^2$ and $y = \dfrac{1}{f(x)}$. [1]

(e) Explain why $y = f(|x|)$ and $y = f(x)$ have identical graphs. [1]

Question 8 6 marks

Let $r(x) = \dfrac{x^2 + 3x + 4}{x + 1}$.

(a) Use polynomial division to find the oblique and vertical asymptotes of $y = r(x)$. [2]

(b) Solve $r(x) \geq x + 3$. [3]

(c) Find the coordinates of the y-intercept of $y = r(x)$. [1]

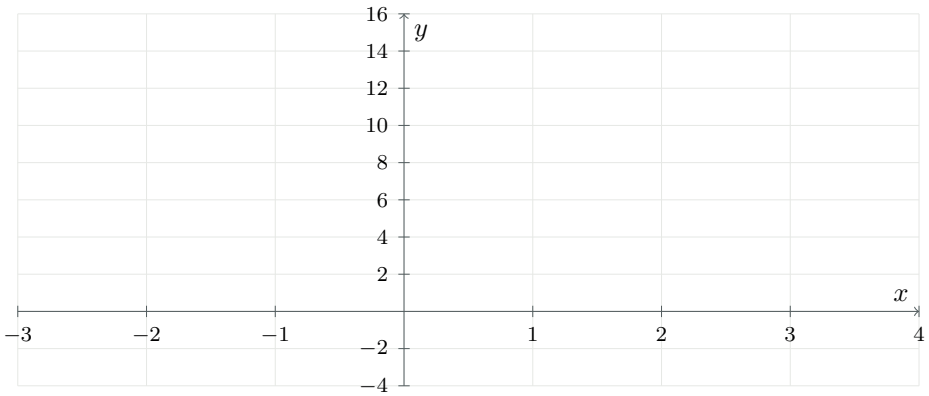
Question 9 — 7 marks

Let $P(x) = x^3 - 3x^2 - 4x + 12$.

(a) Show that $x - 2$ is a factor of $P(x)$, and factorize $P(x)$ completely over $\mathbb{R}$. [3]

(b) Find the remainder when $P(x)$ is divided by $x + 1$. [1]

(c) Sketch $y = P(x)$, showing all intercepts and its behaviour at both ends. Exact coordinates of turning points are not required. [2]



(d) Hence solve $P(x) \leq 0$. [1]

Question 10 — 7 marks

The polynomial $P(z) = z^3 + az^2 + bz + 16$ has real coefficients and a root $2 + 2i$.

(a) State its other non-real root. [1]

(b) Find the third root. Hence find a and b. [3]

(c) Express $2 + 2i$ in modulus-argument form. Hence find $(2 + 2i)^4$. [3]

End of questions

Formula appendix

Selected IB formulae | 1 of 2

The mathematical formulae below are retypeset from the IB Mathematics AA HL formula booklet (2023, version 1.0), printed pages 2–3. Headings and layout have been adapted.

Logarithms and exponentials 1.5, 1.7

$$a^x = b \iff x = \log_a b, \qquad a > 0, \quad b > 0, \quad a \neq 1.$$

$$\log_a(xy) = \log_a x + \log_a y,$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y,$$

$$\log_a(x^m) = m \log_a x,$$

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

$$a^x = e^{x \ln a}, \qquad a^{\log_a x} = x, \qquad \log_a(a^x) = x.$$

Integer binomial expansions 1.9

For positive integers n,

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + \cdots + {}^nC_r a^{n-r}b^r + \cdots + b^n.$$

$${}^nC_r = \frac{n!}{r!(n-r)!}.$$

Rational-index binomial expansions 1.10

For rational n,

$$(a+b)^n = a^n\left(1 + n\left(\frac{b}{a}\right) + \frac{n(n-1)}{2!}\left(\frac{b}{a}\right)^2 + \cdots\right).$$

Source: International Baccalaureate Organization, *Mathematics: analysis and approaches HL formula booklet*, 2023, version 1.0.

Source booklet (public copy). These are selected formulae, not a complete booklet.

Formula appendix

Selected IB formulae | 2 of 2

Complex-number representations and powers 1.12–1.14

$$z = a + bi.$$

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta} = r\,\mathrm{cis}\,\theta.$$

$$[r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta) = r^n e^{in\theta} = r^n\,\mathrm{cis}(n\theta).$$

Lines 2.1

$$y = mx + c, \qquad ax + by + d = 0, \qquad y - y_1 = m(x - x_1).$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Quadratics 2.6, 2.7

For $f(x) = ax^2 + bx + c$, the symmetry axis is

$$x = -\frac{b}{2a}.$$

$$ax^2 + bx + c = 0 \implies x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \qquad a \neq 0.$$

$$\Delta = b^2 - 4ac.$$

Polynomial-root sums and products 2.12

For $\sum_{r=0}^{n} a_r x^r = 0$,

$$\text{sum of roots} = -\frac{a_{n-1}}{a_n}, \qquad \text{product of roots} = (-1)^n \frac{a_0}{a_n}.$$

Selected mathematical formulae from the IB AA HL formula booklet (2023, version 1.0), printed page 3. Headings and layout adapted.

Solutions and mark scheme

Binomial theorem, functions and polynomials

Attempt the questions before consulting these solutions. Use the marks beside each step to check your working. **Mark key.** M1: one mark for a valid method. A1: one mark for an accurate result following an appropriate method. B1: one mark for an independent fact, feature or answer. Equivalent forms and valid alternative methods can receive full credit.

Question 1 — 6 marks

(a) $f(x) = x^2 + 2x + 1 - 4 = \boxed{(x+1)^2 - 4}$. A1

Vertex $\boxed{(-1, -4)}$. B1

(b) At an intersection,

$$x^2 + 2x - 3 = 2x + k \implies x^2 - (3 + k) = 0.$$

Tangency requires $\Delta = 0$, so $0^2 - 4(1)(-3 - k) = 0$. M1

Thus $4(3 + k) = 0$ and $\boxed{k = -3}$. A1

The method mark requires a repeated-root/discriminant argument.

(c) The tangent has gradient 2; a perpendicular line has gradient $-1/2$. M1

Through $(-1, -4)$, $\boxed{y + 4 = -\frac{1}{2}(x + 1)}$, equivalently $\boxed{x + 2y + 9 = 0}$. A1

A consistent calculation using the vertex from part (a) can earn follow-through credit.

Question 2 — 2 marks

$|3x + 1| \leq 2 \iff -2 \leq 3x + 1 \leq 2$. M1

Hence $-3 \leq 3x \leq 1$, giving $\boxed{-1 \leq x \leq \frac{1}{3}}$. A1

An equivalent correct case split is acceptable. The endpoints must be included.

Question 3 — 5 marks

(a) The coefficient condition is ${}^nC_2\, 2^2 = 40$. M1

Thus $4\frac{n(n-1)}{2} = 40$, so $n^2 - n - 20 = 0$. A1

$(n - 5)(n + 4) = 0$; since $n \geq 2$, $\boxed{n = 5}$. A1

(b) The coefficient of x^3 is ${}^5C_3\, 2^3$. M1

It is $10 \times 8 = \boxed{80}$. A1

Follow-through credit is available for a correctly calculated coefficient using an integer $n \geq 3$ obtained in part (a).

Question 4 — 7 marks

(a) With index -2 and $u = -2x$,

$$(1-2x)^{-2} = 1 + (-2)(-2x) + \frac{(-2)(-3)}{2}(-2x)^2 + \cdots .$$

Correct substitution into the binomial formula. M1
Hence $\boxed{1 + 4x + 12x^2 + \cdots}$. A1

(b) With index $1/2$ and $u = 4x$,

$$(1+4x)^{1/2} = 1 + \frac{1}{2}(4x) + \frac{(1/2)(-1/2)}{2}(4x)^2 + \cdots .$$

Correct substitution into the binomial formula. M1
Hence $\boxed{1 + 2x - 2x^2 + \cdots}$. A1

(c) Multiply the two truncated expansions, retaining all contributions through degree two:

$$(1 + 4x + 12x^2)(1 + 2x - 2x^2) = 1 + (4+2)x + (12 + 8 - 2)x^2 + \cdots .$$

Correct collection of terms, including the product of the linear terms. M1
Therefore $\boxed{1 + 6x + 18x^2 + \cdots}$. A1
Consistent use of the expansions from parts (a) and (b) can earn follow-through credit.

(d) Conditions are $|-2x| < 1$ and $|4x| < 1$. Their common open interval is $\boxed{-\frac{1}{4} < x < \frac{1}{4}}$. B1
No endpoint-convergence discussion is required: the question explicitly asks for the interval obtained from $|u| < 1$.

Question 5 — 6 marks

(a) $y = \frac{2x+3}{x-2}$ gives $yx - 2y = 2x + 3$, hence $x(y-2) = 2y + 3$. M1

Interchanging input and output, $\boxed{f^{-1}(x) = \dfrac{2x+3}{x-2}}$. A1

Its domain is $\boxed{\mathbb{R} \setminus \{2\}}$; it equals f on the same domain, so f is self-inverse. B1
The final B1 requires both the correct domain and the self-inverse explanation.

(b) $\boxed{(f \circ g)(x) = \dfrac{2\sqrt{x+1} + 3}{\sqrt{x+1} - 2}}$. A1

Require $x + 1 \geq 0$ and $\sqrt{x+1} \neq 2$, i.e. $x \geq -1$ and $x \neq 3$. M1
Domain $\boxed{[-1, 3) \cup (3, \infty)}$. A1

Question 6 — 6 marks

(a) Since $3^{x-1} > 0$, the range is $\boxed{(-2, \infty)}$. B1

Horizontal asymptote $\boxed{y = -2}$. B1

(b) $3^{x-1} - 2 = 7 \implies 3^{x-1} = 9 = 3^2$, so $\boxed{x = 3}$. B1

(c) The logarithms require $x > 2$. Combining them gives $\log_2[x(x-2)] = 3$, hence $x(x-2) = 8$. M1

$x^2 - 2x - 8 = (x-4)(x+2) = 0$, giving $x = 4$ or $x = -2$. A1

Reject -2 because it is outside the domain; $\boxed{x = 4}$. A1

The final A1 requires a correct domain check and no inadmissible root in the answer.

Question 7(a)–(b) — 4 of 8 marks

(a) Least positive period $\boxed{4}$. B1

f is $\boxed{\text{even}}$. B1

(b) A point (u, v) maps to $(1 - 2u, 3 - 2v)$. For $A = (2, -2)$, solve $(1 - x)/2 = 2$ to give $x = -3$. A1

The new ordinate is $-2(-2) + 3 = 7$, so $\boxed{A' = (-3, 7)}$. A1

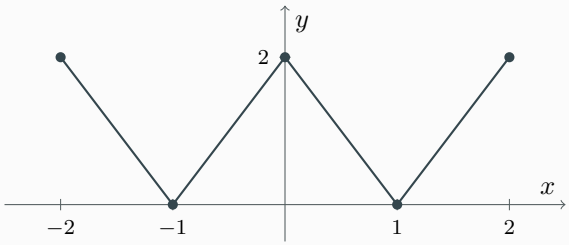
Question 7(c)–(e) — 4 of 8 marks

(c) Reflect the portions of f below the x-axis upwards. Join the points

$$(-2, 2),\ (-1, 0),\ (0, 2),\ (1, 0),\ (2, 2).$$

Correct shape across the complete stated domain. B1

Intercepts correctly labelled: $\boxed{(-1, 0), (1, 0), (0, 2)}$. B1



(d) The y-intercepts are $\boxed{(0, 4)}$ and $\boxed{(0, 1/2)}$, respectively. B1

Both must be correct for this mark.

(e) f is even, so $f(-x) = f(x)$. Replacing a negative input by its absolute value leaves the output unchanged; for non-negative inputs, $|x| = x$. B1

Question 8 — 6 marks

(a) $x^2+3x+4=(x+1)(x+2)+2$, so $\boxed{r(x)=x+2+\dfrac{2}{x+1}}$. M1

Asymptotes $\boxed{y=x+2}$ and $\boxed{x=-1}$. A1

Both asymptotes are required for A1; there is no cancellation at $x=-1$.

(b) $r(x)\geq x+3$ gives

$$\frac{2}{x+1}\geq 1 \iff \frac{1-x}{x+1}\geq 0, \qquad x\neq -1.$$

Correct single rational inequality. M1

A valid sign analysis, or separate cases according to the sign of $x+1$. M1

x	$(-\infty,-1)$	$(-1,1)$	$(1,\infty)$
$(1-x)/(x+1)$	$-$	$+$	$-$

Include the zero at $x=1$ and exclude the pole at $x=-1$: $\boxed{-1<x\leq 1}$. A1

(c) $r(0)=4$, giving $\boxed{(0,4)}$. B1

Question 9 — 7 marks

(a) $P(2)=8-12-8+12=0$, so $x-2$ is a factor. B1

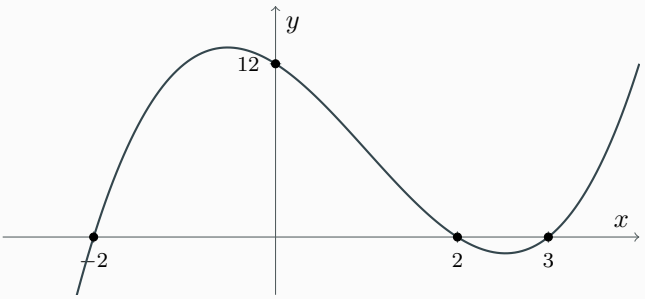
Division gives $P(x)=(x-2)(x^2-x-6)$. M1

Thus $\boxed{P(x)=(x-2)(x-3)(x+2)}$. A1

(b) The remainder is $P(-1)=-1-3+4+12=\boxed{12}$. B1

(c) All intercepts: $\boxed{(-2,0),(2,0),(3,0),(0,12)}$. B1

A smooth cubic crossing at each root, with left end down and right end up. B1



(d) $P(x)$ is non-positive to the left of -2 and between 2 and 3: $\boxed{(-\infty,-2]\cup[2,3]}$. B1

A consistently factorized and sketched cubic with three distinct real roots can earn follow-through credit.

Question 10 — 7 marks

(a) The other non-real root is $\boxed{2-2i}$. B1

Real polynomial coefficients imply conjugate pairs.

(b) The product of the conjugate roots is

$$(2+2i)(2-2i) = 4+4 = 8.$$

The product of all three roots is -16, so the third root is $-16/8 = \boxed{-2}$. A1

The sum of the roots is $2+2-2 = 2 = -a$, hence $\boxed{a=-2}$. A1

Also,

$$\begin{aligned} P(z) &= (z+2)[(z-2)^2+4] \\ &= (z+2)(z^2-4z+8) \\ &= z^3-2z^2+16, \end{aligned}$$

so $\boxed{b=0}$. A1

Equivalent methods include substituting the complex root and comparing real and imaginary parts.

(c) The modulus is $\sqrt{2^2+2^2} = 2\sqrt{2}$ and an argument is $\pi/4$. Thus

$$\boxed{2+2i = 2\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)}.$$

Correct modulus-argument form. A1

By De Moivre,

$$(2+2i)^4 = (2\sqrt{2})^4(\cos\pi + i\sin\pi).$$

Raise the modulus to the fourth power and multiply the argument by four. M1

Since $(2\sqrt{2})^4 = 64$, the result is $\boxed{-64}$. A1

Equivalent arguments differing by 2π are acceptable. Cartesian squaring can verify the result, but the requested modulus-argument representation must still be given.