Separation and integrating factors

IB Mathematics: Analysis and Approaches HL | Calculus

Goal Choose a method, solve an initial-value problem and test the result.
Syllabus AHL 5.18: separable and linear equations; AHL 5.17: volumes.
Structure Two sets of ten questions. Optional challenges are Q9–10 and Q19–20.
Technology No calculator required. All answers can be given exactly.

Instructions. Work on separate paper. Show the separation or integrating factor. If you divide by a function of y, check which solutions division could remove.

60 minutes: Q1–3 (18 min), Q4–5 (17 min), Q6 (12 min), Q7 (8 min), review (5 min).
90 minutes: Add Q8 (12 min) and Q9 (18 min). Q10 can replace Q9. A second route starts at Q11 on page 4.

1 Choosing a method — Core fluency

1. Which structure do you see?

For each equation, decide whether it is separable, linear in y, or both. Write one valid first step; do not solve. Take $x > 0$.

(a) $y' = x(1 + y^2)$ (b) $xy' + 2y = x^3$ (c) $y' = x(2 - y)$.

2. An initial condition selects a branch

Solve $(1 + x^2)y' = 2xy^2$, given $y(0) = -1$. Find any constant solution and explain why your particular solution is defined for all real x.

3. Recover a product derivative

Solve $y' + 2y = 6$, $y(0) = 1$, using an integrating factor. Check your answer in the original equation and give the limiting value as $x \to \infty$.

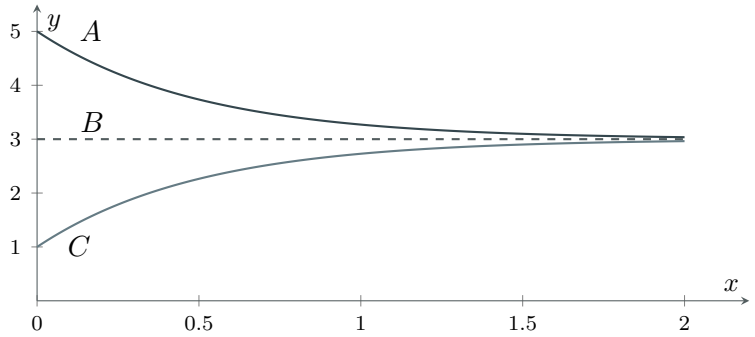
2 Models and solution curves — Mixed practice

4. The coefficient of the derivative matters

A student proposes e^{2x} as an integrating factor for $xy' + 2y = x^3$, $x > 0$. Explain the error. Solve the equation given $y(1) = 2$ and check the slope at $x = 1$ in two ways.

5. Read a family of solutions

The graph shows three members of the family solving $y' = 2(3 - y)$ for $x \geq 0$.



(a) Use the sign of y' to explain the behaviour of each curve.

(b) Use separation to find the equation of curve A, which starts at $(0, 5)$. Why does no finite x on this curve give $y = 3$?

(c) Find the exact x at which the vertical gap between A and B has halved from its initial value.

6. A changing volume

A tank initially contains 20 L of pure water. Brine containing 3 g of salt per litre enters at 2 L/min. The well-mixed liquid leaves at 1 L/min. Let $S(t)$ be the mass of salt in grams after t minutes. Assume the tank has room for the liquid.

(a) Explain why the salt leaves at $S/(20 + t)$ grams per minute.

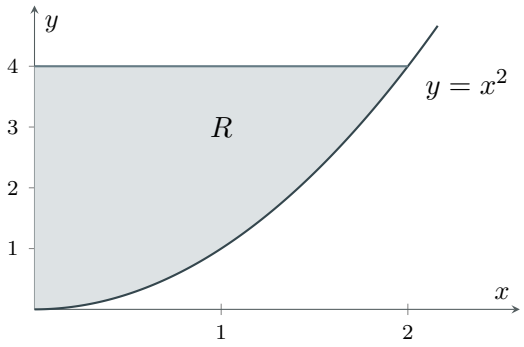
(b) Form and solve a linear differential equation for $S(t)$.

(c) Find the concentration after 20 minutes. Explain why it is still below 3 g/L.

2 Returning to volumes AHL 5.17

7. One region, two axes

The shaded region is bounded by $y = x^2$, $y = 4$ and $x = 0$ in the first quadrant.



Find the volume when R is rotated **(a)** about the y-axis and **(b)** about the x-axis. Identify the radius or radii of each cross-section.

8. Designing a nozzle

A nozzle bore has radius $r(x)$ cm at distance x cm along its axis. On $0 \leq x \leq 2$ the radius satisfies $(1 + x)r' = r/2$ and $r(0) = 2$. Find the radius profile and the interior volume. Explain why using only the mean of the two end radii as a constant radius would underestimate the volume.

3 Optional challenges Choose one

9. An initial value that removes the turning point

For $x > 0$, a family satisfies $xy' + y = 3x^2$, with $y(1) = a$.

(a) Find y in terms of a. Determine exactly which values of a give a stationary point for $x > 0$.

(b) For those values, give its coordinates and classify it. What changes if the domain is restricted to $x > 1$?

10. When does doubling become impossible?

A culture obeys $N' = kN(600 - N)$, where $k > 0$, $N(0) = 200$ and $N(2) = 300$. Time is measured in hours. Separate variables and use partial fractions to find $N(t)$. When will the starting population have doubled? Could the population at $t = 2$ ever double in finite time?

4 More practice with both methods AHL 5.18

Another 60 minutes Q11–12 (12 min), Q13–14 (14 min), Q16 (14 min), Q17 (15 min), review (5 min).
For 90 minutes Add Q15 (8 min), Q18 (7 min) and either Q19 or Q20 (15 min).

11. Using an implicit solution

A curve satisfies $(1+y^2)y' = 6x$ and passes through the origin.

(a) Separate variables and find an equation relating x and y. Leave it in implicit form.

(b) Where on the curve is $y = 3$ and $x > 0$? Find the gradient at that point.

12. An integrating factor with a negative power

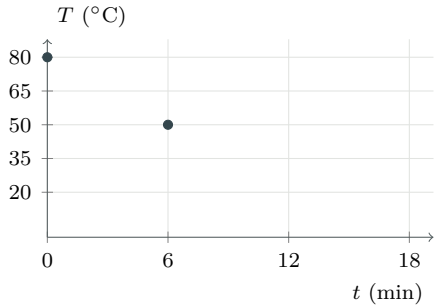
For $x \geq 0$, a curve satisfies $(1+x)y' - 2y = (1+x)^3$ with $y(0) = 2$. Find an integrating factor and hence y in terms of x. Use the original equation to find the tangent at $x = 0$.

13. Cooling above room temperature

A temperature probe reads 80°C when placed in a room at 20°C. Six minutes later it reads 50°C. Its temperature satisfies $T' = -k(T - 20)$, where $k > 0$ and t is in minutes.

(a) Find k and the temperature function $T(t)$.

(b) When does the probe read 35°C? Sketch the curve through the marked points and draw its horizontal asymptote.



14. How far does the solution extend?

Solve $y' = (1+y)^2$ with $y(0) = 0$. Give the largest interval containing 0 on which your solution is defined. Which constant solution must be checked separately when you divide by $(1+y)^2$?

15. A trigonometric integrating factor

On $0 \leq x < \pi/2$, a curve satisfies $(\cos x)y' + (\sin x)y = 1$ with $y(0) = 1$. Find y using an integrating factor. Determine the greatest value of y and where it occurs.

5 Modelling and volumes of revolution AHL 5.17–5.18

16. A tank that drains

A tank contains 24 L of water with 48 g of salt dissolved in it. Pure water enters at 1 L/min, and the well-mixed liquid leaves at 3 L/min. Let $S(t)$ be the mass of salt after t minutes.

(a) Find the volume at time t. State the time interval for which liquid remains in the tank.

(b) Form and solve a differential equation for $S(t)$.

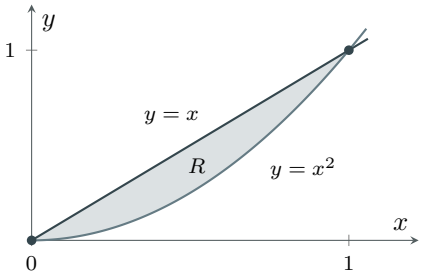
(c) When does the concentration first reach 1 g/L?

17. Two curves, two washers

The shaded region R lies between $y = x$ and $y = x^2$ for $0 \le x \le 1$.

(a) Rotate R about the x-axis. Identify the outer and inner radii and find the volume V_x.

(b) Repeat for rotation about the y-axis to find V_y. Give the exact ratio V_y/V_x.



18. When does a signal peak?

For $t \ge 0$, the output of a sensor satisfies $s' + s = 6e^{-t}$, with $s(0) = 0$ and t measured in seconds. Find $s(t)$. Determine the time and value of its maximum, justifying your answer.

6 Optional challenges Choose one

19. The area between two solution curves

Two solutions u and v of $y' + \frac{2}{1+x}y = x$ satisfy $u(0) = 5$ and $v(0) = 2$, for $x \ge 0$.

(a) Set $D = u - v$. Find $D(x)$ without solving for u and v individually. Can their graphs meet?

(b) The graphs and the lines $x = 0$ and $x = b$, where $b > 0$, enclose an area. Find b if this area is 2 square units. Can the area reach 3 square units for a finite b?

20. Equal volumes from the same region

A curve passes through the origin and satisfies $(1 + x)y' + y = 3x^2 + 2x$ for $x \ge 0$. For $a > 0$, the curve, the x-axis and $x = a$ enclose a region.

(a) Find y and the volume formed by rotating the region about the x-axis, in terms of a.

(b) For rotation about the y-axis, sketch a horizontal strip and identify its inner and outer radii. Find the positive a for which the two volumes are equal.

Answer key and solution outlines — Exact solutions

1 Choosing a method

1. (a) Separable only: $dy/(1+y^2) = x\,dx$. The y^2 term is nonlinear.
(b) Linear only: $y' + (2/x)y = x^2$. The right side of $y' = x^2 - 2y/x$ is not a product of a function of x and a function of y.
(c) Both: $dy/(2-y) = x\,dx$ for $y \neq 2$, or $y' + xy = 2x$. In the first form retain $y = 2$ separately.

2. Division by y^2 gives $dy/y^2 = 2x\,dx/(1+x^2)$, so $-1/y = \ln(1+x^2) + C$. At $(0,-1)$, $C = 1$. Thus

$$y = -\frac{1}{1+\ln(1+x^2)}.$$

The denominator is at least 1. The constant solution $y = 0$ was lost in the division; no other constant makes the equation hold for every x.

3. Multiply by e^{2x}: $(e^{2x}y)' = 6e^{2x}$. Integrate to get $y = 3 + Ce^{-2x}$. The initial condition gives $C = -2$. Then $y' = 4e^{-2x}$, so $y' + 2y = 6$, and $y \to 3$.

2 Models and solution curves

4. Divide by x first: the coefficient of y is $2/x$, giving $\mu = e^{\int (2/x)\,dx} = x^2$. Hence $(x^2y)' = x^4$, so $y = x^3/5 + C/x^2$. At $x = 1$, $C = 9/5$. The derivative at 1 is $3/5 - 18/5 = -3$. The original equation gives $y'(1) = 1 - 2(2) = -3$.

5. (a) Above 3 the slope is negative (A); at 3 it is zero (B); below 3 it is positive (C). The slope magnitude decreases as the gap closes.
(b) On A, $dy/(y-3) = -2\,dx$, so $\ln(y-3) = -2x + C$. Initially $y - 3 = 2$, giving $y = 3 + 2e^{-2x}$. Its last term is positive for every finite x.
(c) Halving the gap gives $2e^{-2x} = 1$, so $x = (\ln 2)/2$.

6. **(a) How much salt leaves?** The tank gains $2 - 1 = 1$ litre each minute, so it holds $20 + t$ litres. A one-litre sample of the well-mixed liquid contains $S/(20+t)$ grams. The outlet carries this same concentration:

$$\text{salt outflow rate} = \frac{S}{20+t}\,\frac{\text{g}}{\text{L}} \times 1\,\frac{\text{L}}{\text{min}} = \frac{S}{20+t}\,\frac{\text{g}}{\text{min}}.$$

(b) Track the salt mass. The inlet brings $3 \times 2 = 6$ grams per minute. Subtract the outflow rate to get $S' = 6 - S/(20+t)$, with $S(0) = 0$. In standard form the equation is $S' + S/(20+t) = 6$, so the integrating factor is $e^{\int 1/(20+t)\,dt} = 20 + t$. Multiplication gives

$$(20+t)S' + S = \frac{d}{dt}\big[(20+t)S\big] = 6(20+t).$$

Integrating gives $(20+t)S = 120t + 3t^2 + C$. Initially the water is pure, so $C = 0$ and $S(t) = \dfrac{120t + 3t^2}{20+t}$ grams.
(c) Convert mass to concentration. At $t = 20$, there are $S(20) = 3600/40 = 90$ grams in 40 litres. The concentration is **$90/40 = 9/4 = 2.25$ g/L**. The incoming brine mixes with the initial pure water. Indeed, for any finite $t \geq 0$,

$$\frac{S(t)}{20+t} = \frac{120t + 3t^2}{(20+t)^2} = 3 - \frac{1200}{(20+t)^2} < 3 \quad \text{g/L}.$$

Answer key and solution outlines — Volumes and challenges

2 Returning to volumes

7. (a) For a horizontal strip at height y, x runs from 0 to $\sqrt{y}$. Rotation about the y-axis produces a disc:

$$V_y = \pi \int_0^4 (\sqrt{y})^2 \, dy = 8\pi.$$

(b) A vertical strip has lower end x^2 and upper end 4. Its rotation about the x-axis leaves a hole. Subtract the two *disc areas*:

$$V_x = \pi \int_0^2 (4^2 - (x^2)^2) \, dx = \pi[16x - x^5/5]_0^2 = 128\pi/5.$$

Both answers are in cubic units.

8. Separation gives $dr/r = dx/[2(1+x)]$, so $\ln r = \frac{1}{2}\ln(1+x) + C$. Using $r(0) = 2$ gives $r = 2\sqrt{1+x}$. Hence

$$V = \pi \int_0^2 4(1+x) \, dx = 16\pi \text{ cm}^3.$$

The mean end radius is $1+\sqrt{3}$ cm. The constant-radius estimate is $2\pi(1+\sqrt{3})^2 = (8+4\sqrt{3})\pi < 16\pi$. Here the cross-sectional area is linear in x: averaging the squared radii gives the exact mean area, while squaring their mean gives a smaller one.

3 Optional challenges

9. (a) Recognize $(xy)' = 3x^2$, or first divide by x and use integrating factor x. Thus $xy = x^3 + C$ and $y = x^2 + (a-1)/x$. A stationary point requires

$$2x - (a-1)/x^2 = 0 \quad \Longleftrightarrow \quad 2x^3 = a - 1.$$

There is exactly one positive solution if $a > 1$, none otherwise.
(b) Write $h = \sqrt[3]{(a-1)/2}$. Since $a - 1 = 2h^3$, the point is $(h, 3h^2)$. At h, $y'' = 2 + 2(a-1)/h^3 = 6 > 0$, so it is a minimum. For $x > 1$ we need $h > 1$, i.e. $a > 3$; equality places it at the excluded endpoint.

10. Along the given solution $0 < N < 600$. The partial fractions identity

$$\frac{1}{N(600-N)} = \frac{1}{600}\left(\frac{1}{N} + \frac{1}{600-N}\right)$$

gives $\ln[N/(600-N)] = 600kt + C$. The initial odds are $200/400 = 1/2$, so $N/(600-N) = e^{600kt}/2$. At two hours the odds are 1: $e^{1200k} = 2$. Hence

$$N(t) = \frac{600}{1 + 2 \cdot 2^{-t/2}}.$$

At $N = 400$, the odds are 2, so $2^{t/2} = 4$ and $t = 4$ hours. Doubling 300 would require 600: the denominator is strictly greater than 1 at finite t, so 600 is only a limiting population.

Answer key and solution outlines Questions 11–15

4 More practice with both methods

11. (a) Separate to get $(1+y^2)\,dy = 6x\,dx$. Integration gives $y + y^3/3 = 3x^2 + C$. At the origin $C = 0$, so the implicit solution is

$$y + \frac{y^3}{3} = 3x^2.$$

(b) At $y = 3$ we have $3 + 27/3 = 3x^2$. The positive solution is $x = 2$, giving the point $(2, 3)$. The original equation gives the gradient $6(2)/(1+3^2) = \mathbf{6/5}$.

12. Divide by $1+x$ to get $y' - \dfrac{2}{1+x}y = (1+x)^2$. An integrating factor is $\mu = e^{-2\ln(1+x)} = (1+x)^{-2}$. On multiplying, the left side becomes a product derivative:

$$\frac{d}{dx}\left(\frac{y}{(1+x)^2}\right) = 1 \quad\Longrightarrow\quad \frac{y}{(1+x)^2} = x + C.$$

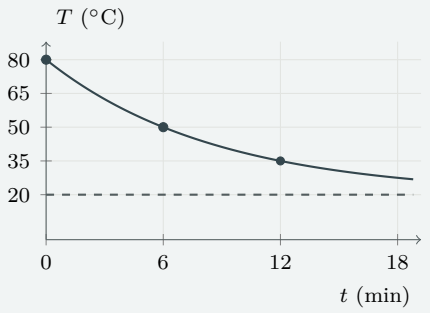
Since $y(0) = 2$, $C = 2$ and $y = (1+x)^2(x+2)$. At $x = 0$ the original equation gives $y'(0) - 2(2) = 1$. The tangent has gradient 5 through $(0, 2)$, so its equation is $y = 5x + 2$.

13. **(a)** The probe starts above room temperature, so separation gives

$$\frac{dT}{T-20} = -k\,dt, \qquad T - 20 = Ae^{-kt}.$$

Initially $A = 80 - 20 = 60$. At $t = 6$, $30 = 60e^{-6k}$, so $k = (\ln 2)/6$ per minute and

$$T(t) = 20 + 60\left(\tfrac{1}{2}\right)^{t/6}.$$



(b) The excess above room temperature halves every six minutes: $60 \to 30 \to 15$. Hence $T = 35°\text{C}$ at $\boldsymbol{t = 12}$ **min**. The curve flattens towards the horizontal asymptote $T = 20$.

14. For $y \neq -1$, separation and integration give

$$\int \frac{dy}{(1+y)^2} = \int dx \quad\Longrightarrow\quad -\frac{1}{1+y} = x + C.$$

At $(0, 0)$, $C = -1$. Rearranging gives $y = x/(1-x)$. The denominator vanishes at $x = 1$, so the largest interval containing 0 is $(-\infty, 1)$. The constant solution $y = -1$ satisfies the differential equation and was excluded during division.

15. Division by $\cos x$ gives $y' + (\tan x)y = \sec x$. Here $\cos x > 0$, so $\mu = e^{-\ln(\cos x)} = \sec x$. Then

$$(y\sec x)' = \sec^2 x \quad\Longrightarrow\quad y\sec x = \tan x + C.$$

At $x = 0$, $C = 1$, giving $y = \sin x + \cos x$. Its derivative $\cos x - \sin x$ changes from positive to negative at $x = \pi/4$. The greatest value is $y(\pi/4) = \mathbf{\sqrt{2}}$.

Answer key and solution outlines Questions 16–20

5 Modelling and volumes of revolution

16. (a) The net flow is $1-3=-2$ L/min, so $V(t)=24-2t$ litres. There is liquid in the tank for $0\le t<12$ minutes.
(b) The incoming water carries no salt. The outlet carries $3S/(24-2t)$ grams per minute. Thus $S'=-3S/(24-2t)$ and $S(0)=48$. Separate and use the initial mass:

$$\ln\frac{S}{48}=\frac{3}{2}\ln\frac{24-2t}{24}\quad\Longrightarrow\quad S(t)=48(1-t/12)^{3/2}\text{ g}.$$

(c) Divide by $V=24(1-t/12)$ to get the concentration $2\sqrt{1-t/12}$ g/L. It decreases to 1 when $1-t/12=1/4$, giving $\boldsymbol{t=9}$ **min**. At that time there are 6 g in 6 L.

17. (a) A vertical strip runs from $y=x^2$ to $y=x$. Its rotation about the x-axis has outer radius x and inner radius x^2. The volume is

$$V_x=\pi\int_0^1(x^2-x^4)\,dx=\pi\left(\tfrac{1}{3}-\tfrac{1}{5}\right)=\frac{2\pi}{15}.$$

(b) A horizontal strip runs from $x=y$ to $x=\sqrt{y}$. About the y-axis the outer radius is $\sqrt{y}$ and the inner radius is y. Thus

$$V_y=\pi\int_0^1(y-y^2)\,dy=\frac{\pi}{6},\qquad\frac{V_y}{V_x}=\frac{\pi/6}{2\pi/15}=\boldsymbol{\frac{5}{4}}.$$

18. Multiply by e^t to obtain $(e^ts)'=6$. Integrating gives $e^ts=6t+C$, and $s(0)=0$ gives $C=0$. Thus $s(t)=6te^{-t}$. Its derivative is $6e^{-t}(1-t)$, positive for $0\le t<1$ and negative for $t>1$. The maximum output is $\boldsymbol{6/e}$ at $\boldsymbol{t=1}$ **s**.

6 Optional challenges

19. (a) Subtract the equations for u and v. The common right-hand sides cancel, so $D'+2D/(1+x)=0$ and $D(0)=3$. Multiplication by $(1+x)^2$ gives $[(1+x)^2D]'=0$. Hence $D(x)=3/(1+x)^2>0$ and u stays above v for all $x\ge0$.
(b) Integrate the vertical gap: $A(b)=\displaystyle\int_0^b\frac{3}{(1+x)^2}\,dx=3-\frac{3}{1+b}$. For $A=2$, $1+b=3$, so $\boldsymbol{b=2}$. For finite $b>0$, the term $3/(1+b)$ stays positive, so the area stays below 3 square units.

20. (a) Recognize $[(1+x)y]'=3x^2+2x$. Integrating gives $(1+x)y=x^3+x^2+C$, and the origin gives $C=0$. Thus $y=x^2$. About the x-axis the discs have radius x^2, so $V_x=\pi\displaystyle\int_0^a x^4\,dx=\frac{\pi a^5}{5}$.
(b) At height y_0, the strip runs from $x=\sqrt{y_0}$ to $x=a$. Rotation about the y-axis gives a washer with inner radius $\sqrt{y_0}$ and outer radius a. Hence

$$V_y=\pi\int_0^{a^2}(a^2-y)\,dy=\frac{\pi a^4}{2}.$$

Since $a>0$, equating volumes and dividing by πa^4 gives $a/5=1/2$, so $\boldsymbol{a=5/2}$.

