Continuous random variables

IB Mathematics AA HL ▪ AHL 4.14

50 questions ▪ Densities, centre, spread and fair games

Overview

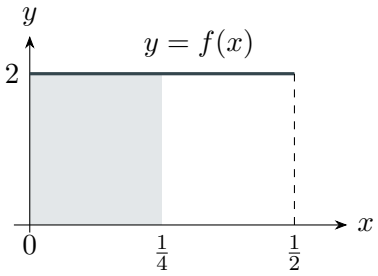
Learning goal	Apply continuous probability models and explain their results.
Syllabus link	Continuous component of AHL 4.14; probability and integration prerequisites.
Structure	15 short response; 15 extended response; 20 original IB-style questions.
Technology	Non-calculator throughout. Give exact values; trigonometric angles are in radians.

Instructions

Work on separate paper. Show your integrals, limits and algebra. Include units in contextual answers. All densities are **zero outside the stated intervals**, including any gaps. In probability questions, $<$ and $\leq$ give the same result for a continuous random variable. Solutions begin on p. 10.

1 Short response

1. The graph shows the density of a continuous random variable X.



(a) Explain what the height $f(x)$ represents.

(b) State $\mathrm{P}(X = \frac{1}{4})$.

(c) Find $\mathrm{P}(0 \leq X \leq \frac{1}{4})$.

(d) Explain why a density height of 2 is allowed.

2. State whether each function is a valid probability density. Give a reason. Each function is zero outside the stated interval.

(a) $f(x) = \frac{3}{4}(1 - x^2)$, $-1 \leq x \leq 1$.

(b) $g(x) = x$, $-1 \leq x \leq 1$.

(c) $h(x) = 2x$, $0 \leq x \leq 2$.

3. The density of X is $f(x) = kx$ for $0 \leq x \leq 2$. Find k.

4. X has density $f(x) = \frac{x}{2}$ for $0 \leq x \leq 2$. Find $\mathrm{P}(\frac{1}{2} < X < \frac{3}{2})$.

5. X has density $f(x) = 2(1-x)$ for $0 \le x \le 1$. Find $\mathrm{P}(X > \frac{1}{2})$.

6. X has density $f(x) = \begin{cases} \frac{1}{4}, & 0 \le x < 2, \\ \frac{1}{2}, & 2 \le x \le 3, \\ 0, & \text{otherwise.} \end{cases}$ Find $\mathrm{P}(1 < X < \frac{5}{2})$.

7. A waiting time X, in minutes, has constant density $\frac{1}{6}$ on $[-2, 4]$. Explain why this is an unsuitable model for a waiting time. If X instead represents a signed timing error in minutes, find its mean and median.

8. X has density $f(x) = 3x^2$ for $0 \le x \le 1$. Find $\mathrm{E}(X)$.

9. X has density $f(x) = 2x$ for $0 \le x \le 1$. Find $\mathrm{E}(X^2)$, displaying the factor x^2 in your integral.

10. A duration X, in seconds, satisfies $\mathrm{E}(X) = 3$ and $\mathrm{E}(X^2) = 11$. Find its variance and standard deviation, including units.

11. Find the median of X when $f(x) = 2x$ for $0 \le x \le 1$.

12. Find the mode of X when $f(x) = 6x(1-x)$ for $0 \le x \le 1$. Justify that the density is greatest at your answer.

13. A continuous random variable X has mean 5 and variance 4. Let $Y = 3X - 2$. Find the mean, variance and standard deviation of Y.

14. A continuous random variable X has mean 2 and standard deviation 3. Let $Y = 7 - 2X$. Find the mean and standard deviation of Y.

15. In a game, X is a measured length, in metres, with density $2x$ on $[0, 1]$. The payout is $R = 6X + 1$ euros and the fee is 5 euros. Let $G = R - 5$ be the player's net gain. Determine whether the game is fair.

2 Extended response

16. X has density $f(x) = k(4 - x)$ for $0 \le x \le 4$.

(a) Find k.

(b) Find $\mathrm{P}(1 < X < 3)$.

(c) Find $\mathrm{E}(X)$.

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17. X has density $f(x) = kx^2$ for $0 \le x \le 2$.

(a) Find k.

(b) Find $\mathrm{P}(X > 1)$.

(c) Find $\mathrm{E}(X)$ and explain why it need not divide the probability in half.

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18. A delivery delay X, in minutes, has density $f(x) = k(x - 1)$ for $1 \le x \le 3$.

(a) Find k.

(b) Find $\mathrm{P}(X < 2)$.

(c) Find the median delay.

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19. A signed measurement error X, in centimetres, has density $f(x) = \frac{3}{4}(1 - x^2)$ for $-1 \le x \le 1$.

(a) Find $\mathrm{P}(|X| < \frac{1}{2})$.

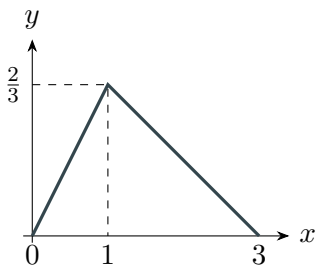
(b) Use symmetry to state the mean and mode.

(c) Find $\mathrm{E}(X^2)$, the variance and the standard deviation.

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20. X has the triangular density shown:

$$f(x) = \begin{cases} \frac{2x}{3}, & 0 \le x \le 1, \\ \frac{3-x}{3}, & 1 < x \le 3, \\ 0, & \text{otherwise.} \end{cases}$$



(a) Find $\mathrm{P}(X > 1)$.

(b) Locate the median in one of the two intervals, then find it.

(c) Find the mean.

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21. X has density $\frac{1}{6}$ on $[0, 2)$ and $\frac{1}{3}$ on $[2, 4]$.

(a) Find $\mathrm{P}(1 < X < 3)$.

(b) Find the median.

(c) Find $\mathrm{E}(X)$.

22. X has density $f(x) = k(2x+1)$ for $0 \le x \le 1$.

(a) Find k.

(b) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

(c) Find the variance and standard deviation.

23. X has density $f(x) = kx$ for $0 \le x \le a$, where $a > 0$. Its mean is 4.

(a) Find a and k.

(b) Find $\mathrm{P}(X > 2)$.

(c) Find the variance.

24. X has density $f(x) = 3x^2$ for $0 \le x \le 1$.

(a) Find c such that $\mathrm{P}(X > c) = \frac{7}{8}$.

(b) Find the median m.

(c) Explain why $c < m$, using probabilities.

25. X has density $f(x) = k\sin x$ for $0 \le x \le \pi$.

(a) Find k.

(b) Find $\mathrm{P}(\frac{\pi}{6} < X < \frac{5\pi}{6})$.

(c) State the median and mode, giving reasons.

26. X has density $f(x) = ke^{-x}$ for $0 \le x \le \ln 3$.

(a) Find k.

(b) Find $\mathrm{P}(X > \ln 2)$.

(c) Find the median.

(d) Find the mean using integration by parts.

27. A battery's operating time X, in hours, has density $f(x) = \frac{3}{8}x^2$ for $0 \le x \le 2$. Let $T = 60X$ be the same time in minutes.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

(b) Find the mean, variance and standard deviation of T, with units.

28. In a game, X is a measured distance in metres with density $2(1-x)$ on $[0,1]$. The payout is $R = 12X + 2$ euros. A player pays a constant fee c euros and has net gain $G = R - c$.

(a) Find the fair fee.

(b) Find $\text{Var}(G)$.

(c) If the fee is 5 euros, find the probability of a positive net gain.

29. A continuous measurement X has mean 2 and standard deviation $\frac{1}{2}$. A transformed score is $Y = aX + b$, where $a < 0$. Its mean is 10 and standard deviation is 2.

(a) Find a and b.

(b) Rewrite $\text{P}(Y > 12)$ as a probability involving X.

(c) Explain why the given information does not determine a numerical value for this probability.

30. For the density $f(x) = 2x$ on $[0,1]$, a student writes

$$\text{E}(X) = \int_0^1 f(x)\,\text{d}x = 1, \qquad \text{Var}(X) = \left(\int_0^1 xf(x)\,\text{d}x\right)^2, \qquad \sigma = [\text{Var}(X)]^2.$$

Correct all three statements and calculate the mean, variance and standard deviation.

3 IB exam-style / Paper 1

Original practice questions; not official IB examination questions.

31. X has density $f(x) = kx(2-x)$ for $0 \le x \le 2$.

(a) Show that $k = \frac{3}{4}$. [2 marks]

(b) Find the mode. [2 marks]

(c) Find $\mathrm{P}(\frac{1}{2} < X < \frac{3}{2})$. [3 marks]

(d) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [5 marks]

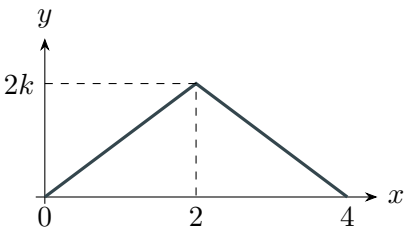
32. X has density kx on $[0,2]$ and $k(4-x)$ on $(2,4]$.

(a) Find k. [2 marks]

(b) Find $\mathrm{P}(1 < X < 3)$. [3 marks]

(c) State the mean and median, with a reason. [2 marks]

(d) Find the variance. [4 marks]



33. A distance X, in metres, has density $x/6$ on $[0,2]$ and $(6-x)/12$ on $(2,6]$. A game pays $R = 3X$ euros; a player pays c euros and gains $G = R - c$.

(a) Find the median distance. [4 marks]

(b) Show that $\mathrm{E}(X) = \frac{8}{3}$. [4 marks]

(c) Find the fair fee c. [2 marks]

(d) For this fee, find $\mathrm{P}(G > 0)$ and comment on its relationship to fairness. [3 marks]

34. X has density $f(x) = a + bx$ for $0 \le x \le 1$, where a, b are constants. It is given that $\mathrm{E}(X) = \frac{5}{9}$.

(a) Find a and b. [5 marks]

(b) Verify that the resulting function is non-negative on $[0,1]$. [1 mark]

(c) Find the variance. [4 marks]

35. X has density $f(x) = k(x-1)(3-x)$ for $1 \le x \le 3$. Let $Y = 5 - 2X$.

(a) Find k. [3 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [5 marks]

(c) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [2 marks]

(d) Find $\mathrm{P}(Y > 1)$. [2 marks]

36. A continuous random variable X has constant density $1/(b-a)$ for $a \le x \le b$, where $a < b$. Its mean is 5 and variance is 3.

(a) Use integration to show that $\mathrm{E}(X) = \frac{a+b}{2}$ and $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$. [6 marks]

(b) Find a and b. [3 marks]

(c) Find $\mathrm{P}(3 < X < 7)$. [2 marks]

37. X has density $f(x) = k/x$ for $1 \le x \le e$.

(a) Find k. [2 marks]

(b) Find the median. [3 marks]

(c) Find the mean and variance, leaving answers in terms of e. [5 marks]

38. X has density $f(x) = k\cos x$ for $0 \le x \le \frac{\pi}{2}$.

(a) Find k. [2 marks]

(b) Find $\mathrm{P}(X > \frac{\pi}{6})$. [2 marks]

(c) Find the median and state the mode. [3 marks]

(d) Find the mean. [4 marks]

39. X has density $f(x) = 3x^2/a^3$ for $0 \le x \le a$, where $a > 0$. Its median is 2.

(a) Find a. [3 marks]

(b) Find the mean. [3 marks]

(c) Find the variance. [4 marks]

40. X has density $f(x) = cx$ for $0 \le x \le 1$ and $f(x) = c$ for $1 < x \le 2$.

(a) Find c. [2 marks]

(b) Find the median. [3 marks]

(c) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$. [5 marks]

(d) Hence find the variance. [2 marks]

41. In a game, X is the fraction of a metre recorded by a measuring device. Its density is $3(1-x)^2$ for $0 \le x \le 1$. The payout is $R = 20X + 1$ euros. The net gain after paying c euros is $G = R - c$.

(a) Show that $\mathrm{E}(X) = \frac{1}{4}$ and $\mathrm{Var}(X) = \frac{3}{80}$. [6 marks]

(b) Find the fair fee and the standard deviation of the net gain. [3 marks]

(c) At the fair fee, find the probability of a positive net gain. [3 marks]

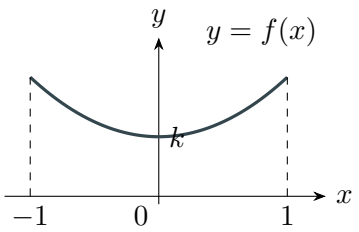
42. X has density $f(x) = k(1 + x^2)$ for $-1 \le x \le 1$.

(a) Find k. [2 marks]

(b) State the mean, median and all modes. [3 marks]

(c) Find $\text{Var}(X)$. [3 marks]

(d) For $H = 3X - 4$, find $\text{E}(H)$ and $\text{Var}(H)$. [2 marks]



43. X has density $f(x) = 2x$ for $0 \le x \le 1$. Let $Y = 1 - X$.

(a) Find the mean and variance of Y. [5 marks]

(b) Find $\text{P}(Y < \frac{1}{4})$. [3 marks]

(c) Find the median of Y, explaining how the inequality changes. [3 marks]

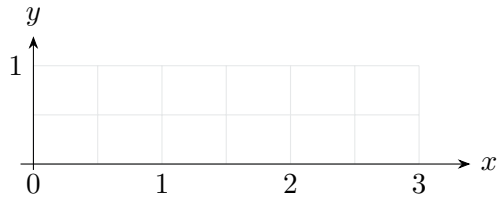
44. X has density c on $[0, 1]$, density $2c$ on $[2, 3]$, and density zero elsewhere.

(a) Find c and sketch $y = f(x)$. [3 marks]

(b) Find $\text{P}(\frac{1}{2} < X < \frac{5}{2})$. [2 marks]

(c) Find the median. [3 marks]

(d) Find the mean and explain why it need not be a value with positive density. [4 marks]



45. X has density $f(x) = \frac{1+\alpha x}{2}$ for $-1 \le x \le 1$.

(a) Find the range of α for which f is a valid density. [3 marks]

(b) Show that $\text{E}(X) = \alpha/3$. [2 marks]

(c) Given that $\text{E}(X) = \frac{1}{6}$, find $\text{Var}(X)$ and $\text{P}(X > 0)$. [5 marks]

46. X has density c on $[0, a)$ and density $c/2$ on $[a, 3a]$, where $a > 0$. Its mean is 5.

(a) Find a and c. [5 marks]

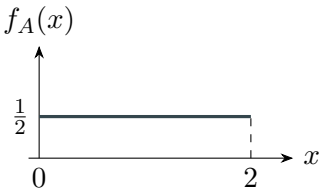
(b) Find the median. [2 marks]

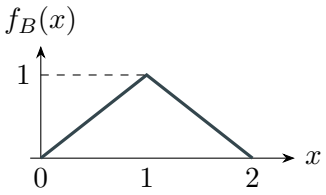
(c) Find $\text{P}(X > 2a)$. [2 marks]

(d) Find the variance. [4 marks]

47. Two machines produce continuous measurement errors A and B (in millimetres), with densities

$$f_A(x) = \tfrac{1}{2} \quad (0 \le x \le 2), \qquad f_B(x) = \begin{cases} x, & 0 \le x \le 1, \\ 2 - x, & 1 < x \le 2. \end{cases}$$





(a) State the mean of each distribution. [2 marks]

(b) Find the variance of each distribution. [5 marks]

(c) Find $\mathrm{P}(\frac{1}{2} < A < \frac{3}{2})$ and $\mathrm{P}(\frac{1}{2} < B < \frac{3}{2})$. [3 marks]

(d) Which machine is more consistent? Justify your answer. [1 mark]

48. In a game, a continuous distance X in metres has density $x/2$ on $[0, 2]$. The payout is $R = aX + b$ euros, with $a > 0$. Its mean is 8 euros and standard deviation is $\sqrt{2}$ euros.

(a) Find the mean and variance of X. [4 marks]

(b) Find a and b. [4 marks]

(c) A player pays 9 euros. Define the net gain G and find its mean. [2 marks]

(d) Find the probability of a positive net gain. [3 marks]

49. X has density $f(x) = k(x + 1)$ for $-1 \le x \le b$, where $b > -1$. Its median is 1.

(a) Find k and b. [5 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [6 marks]

(c) Find $\mathrm{P}(X > 0)$. [2 marks]

50. A game records a continuous distance X in metres with density

$$f(x) = \begin{cases} ax, & 0 \le x < 1, \\ b(2 - x), & 1 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

It is given that $\mathrm{P}(X < 1) = \frac{1}{4}$. The payout is $R = 6X$ euros; the entry fee is c euros and the net gain is $G = R - c$.

(a) Find a and b. [4 marks]

(b) Find the median and mode. [4 marks]

(c) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [6 marks]

(d) Find the fair entry fee and the standard deviation of G. [3 marks]

Answer key and solution outlines

Section 1: Short response

1.

(a) $f(x)$ is probability density, the height of the graph $y = f(x)$; probability is an area under it.

(b) 0.

(c) $2(\frac{1}{4}) = \boxed{\frac{1}{2}}$.

(d) Density need not be at most 1. The total area is $2(\frac{1}{2}) = 1$, and the density is non-negative.

2.

(a) Valid: $f \geq 0$ and $\frac{3}{4}[x - x^3/3]_{-1}^{1} = 1$.

(b) Invalid: $g < 0$ for $-1 \leq x < 0$ (also, its integral is 0).

(c) Invalid: $\int_0^2 2x\,\mathrm{d}x = 4 \neq 1$.

3. $1 = \int_0^2 kx\,\mathrm{d}x = 2k$, so $\boxed{k = \frac{1}{2}}$.

4. $\int_{1/2}^{3/2} \frac{x}{2}\,\mathrm{d}x = [x^2/4]_{1/2}^{3/2} = \boxed{\frac{1}{2}}$.

5. $\int_{1/2}^{1} 2(1-x)\,\mathrm{d}x = [2x - x^2]_{1/2}^{1} = \boxed{\frac{1}{4}}$.

6. Split at 2: $\int_1^2 \frac{1}{4}\,\mathrm{d}x + \int_2^{5/2} \frac{1}{2}\,\mathrm{d}x = \frac{1}{4} + \frac{1}{4} = \boxed{\frac{1}{2}}$.

7. A waiting time cannot be negative, but the model assigns positive probability to negative values. As a signed error, its mean and median are both the midpoint $\boxed{1 \text{ minute}}$.

8. $\mathrm{E}(X) = \int_0^1 x(3x^2)\,\mathrm{d}x = [3x^4/4]_0^1 = \boxed{\frac{3}{4}}$.

9. $\mathrm{E}(X^2) = \int_0^1 \underbrace{x^2}_{\text{squared value}} \underbrace{(2x)}_{f(x)}\,\mathrm{d}x = [x^4/2]_0^1 = \boxed{\frac{1}{2}}$.

10. $\mathrm{Var}(X) = 11 - 3^2 = \boxed{2\text{ s}^2}$; $\sigma = \sqrt{\mathrm{Var}(X)} = \boxed{\sqrt{2}\text{ s}}$.

11. $\int_0^m 2x\,\mathrm{d}x = \frac{1}{2} \Rightarrow m^2 = \frac{1}{2}$. Since $0 \leq m \leq 1$, $\boxed{m = 1/\sqrt{2}}$.

12. $f'(x) = 6 - 12x = 0 \Rightarrow x = \frac{1}{2}$. Since $f'' = -12 < 0$ and the endpoint heights are 0, the mode is $\boxed{\frac{1}{2}}$.

13. $\mathrm{E}(Y) = 3(5) - 2 = \boxed{13}$; $\mathrm{Var}(Y) = 3^2(4) = \boxed{36}$; $\sigma_Y = \boxed{6}$.

14. $\mathrm{E}(Y) = 7 - 2(2) = \boxed{3}$; $\sigma_Y = |-2|(3) = \boxed{6}$.

15. $\mathrm{E}(X) = \int_0^1 x(2x)\,\mathrm{d}x = \frac{2}{3}$. Thus $\mathrm{E}(G) = 6(\frac{2}{3}) + 1 - 5 = \boxed{0}$ euros: the game is fair.

Section 2: Extended response

16.

(a) $1 = k[4x - x^2/2]_0^4 = 8k \Rightarrow \boxed{k = \frac{1}{8}}$.

(b) $\frac{1}{8}[4x - x^2/2]_1^3 = \boxed{\frac{1}{2}}$.

(c) $\mathrm{E}(X) = \frac{1}{8}\int_0^4 x(4-x)\,\mathrm{d}x = \frac{1}{8}[2x^2 - x^3/3]_0^4 = \boxed{\frac{4}{3}}$.

17.

(a) $k[x^3/3]_0^2 = 1 \Rightarrow \boxed{k = \frac{3}{8}}$.

(b) $[x^3/8]_1^2 = \boxed{\frac{7}{8}}$.

(c) $\mathrm{E}(X) = \frac{3}{8}\int_0^2 x^3\,\mathrm{d}x = \boxed{\frac{3}{2}}$. This is a weighted average, not the median: $\mathrm{P}(X \leq \frac{3}{2}) = \frac{27}{64} \neq \frac{1}{2}$.

18.

(a) $k[(x-1)^2/2]_1^3 = 1 \Rightarrow \boxed{k = \frac{1}{2}}$.

(b) $[(x-1)^2/4]_1^2 = \boxed{\frac{1}{4}}$.

(c) $(m-1)^2/4 = \frac{1}{2}$; $m \in [1,3]$ gives $\boxed{1+\sqrt{2} \text{ minutes}}$.

19.

(a) $\frac{3}{4}\int_{-1/2}^{1/2}(1-x^2)\,\mathrm{d}x = \boxed{\frac{11}{16}}$.

(b) The mean is $\boxed{0 \text{ cm}}$; the mode is $\boxed{0 \text{ cm}}$, where $1-x^2$ is greatest.

(c) $\mathrm{E}(X^2) = \frac{3}{4}\int_{-1}^{1} x^2(1-x^2)\,\mathrm{d}x = \frac{3}{2}(\frac{1}{3} - \frac{1}{5}) = \frac{1}{5}$ cm^2. Hence $\boxed{\mathrm{Var}(X) = \frac{1}{5} \text{ cm}^2}$ and $\boxed{\sigma = 1/\sqrt{5} \text{ cm}}$.

20.

(a) Right triangle area $\frac{1}{2}(2)(\frac{2}{3}) = \boxed{\frac{2}{3}}$.

(b) The area to the left of 1 is $\frac{1}{3} < \frac{1}{2}$, so $m \in (1,3)$. Right-tail area $(3-m)^2/6 = \frac{1}{2}$ gives $\boxed{m = 3 - \sqrt{3}}$.

(c) $\mathrm{E}(X) = \int_0^1 x\frac{2x}{3}\,\mathrm{d}x + \int_1^3 x\frac{3-x}{3}\,\mathrm{d}x = \frac{2}{9} + \frac{10}{9} = \boxed{\frac{4}{3}}$.

21.

(a) $\frac{1}{6}(1) + \frac{1}{3}(1) = \boxed{\frac{1}{2}}$.

(b) $\mathrm{P}(X < 2) = \frac{1}{3}$; $\frac{1}{3} + (m-2)/3 = \frac{1}{2}$ gives $\boxed{m = \frac{5}{2}}$.

(c) $\int_0^2 x/6\,\mathrm{d}x + \int_2^4 x/3\,\mathrm{d}x = \frac{1}{3} + 2 = \boxed{\frac{7}{3}}$.

22.

(a) $k[x^2 + x]_0^1 = 1 \Rightarrow \boxed{k = \frac{1}{2}}$.

(b) $\mathrm{E}(X) = \int_0^1 x(x+\frac{1}{2})\,\mathrm{d}x = \boxed{\frac{7}{12}}$; $\mathrm{E}(X^2) = \int_0^1 x^2(x+\frac{1}{2})\,\mathrm{d}x = \boxed{\frac{5}{12}}$.

(c) $\mathrm{Var}(X) = \frac{5}{12} - (\frac{7}{12})^2 = \boxed{\frac{11}{144}}$; $\boxed{\sigma = \frac{\sqrt{11}}{12}}$.

23.

(a) Normalisation gives $ka^2/2 = 1$, so $k = 2/a^2$. Then $\mathrm{E}(X) = ka^3/3 = 2a/3 = 4$, giving $\boxed{a = 6,\ k = \frac{1}{18}}$.

(b) $1 - \int_0^2 x/18\,\mathrm{d}x = 1 - \frac{1}{9} = \boxed{\frac{8}{9}}$.

(c) $\mathrm{E}(X^2) = \int_0^6 x^2(x/18)\,\mathrm{d}x = 18$. Thus $\boxed{\mathrm{Var}(X) = 18 - 4^2 = 2}$.

24.

(a) $1 - c^3 = \frac{7}{8} \Rightarrow \boxed{c = \frac{1}{2}}$.

(b) $m^3 = \frac{1}{2} \Rightarrow \boxed{m = \sqrt[3]{\frac{1}{2}}}$.

(c) Only $\frac{1}{8}$ of the probability lies below c, whereas $\frac{1}{2}$ lies below m; the density is positive between them, so $c < m$.

25.

(a) $k[-\cos x]_0^{\pi} = 1 \Rightarrow \boxed{k = \frac{1}{2}}$.

(b) $\frac{1}{2}[-\cos x]_{\pi/6}^{5\pi/6} = \boxed{\frac{\sqrt{3}}{2}}$.

(c) Median $\boxed{\frac{\pi}{2}}$ by symmetry; mode $\boxed{\frac{\pi}{2}}$ because $\sin x$ is greatest there.

26.

(a) $k(1 - \frac{1}{3}) = 1 \Rightarrow \boxed{k = \frac{3}{2}}$.

(b) $\frac{3}{2}(\frac{1}{2} - \frac{1}{3}) = \boxed{\frac{1}{4}}$.

(c) $\frac{3}{2}(1 - e^{-m}) = \frac{1}{2} \Rightarrow e^{-m} = \frac{2}{3}$, so $\boxed{m = \ln(\frac{3}{2})}$.

(d) $\mathrm{E}(X) = \frac{3}{2}[-(x+1)e^{-x}]_0^{\ln 3} = \boxed{1 - \frac{1}{2}\ln 3}$.

27.

(a) $\mathrm{E}(X) = \frac{3}{8}\int_0^2 x^3\,\mathrm{d}x = \frac{3}{2}$ hours; $\mathrm{E}(X^2) = \frac{3}{8}\int_0^2 x^4\,\mathrm{d}x = \frac{12}{5}$. Thus $\mathrm{Var}(X) = \frac{12}{5} - \frac{9}{4} = \boxed{\frac{3}{20}\ \mathrm{h}^2}$.

(b) $\boxed{\mathrm{E}(T) = 90\ \mathrm{min}}$; $\boxed{\mathrm{Var}(T) = 60^2(\frac{3}{20}) = 540\ \mathrm{min}^2}$; $\boxed{\sigma_T = 6\sqrt{15}\ \mathrm{min}}$.

28.

(a) $\mathrm{E}(X) = 2\int_0^1 x(1-x)\,\mathrm{d}x = \frac{1}{3}$. Fairness requires $\mathrm{E}(G) = 12(\frac{1}{3}) + 2 - c = 0$, so $\boxed{c = 6\ \mathrm{euros}}$.

(b) $\mathrm{E}(X^2) = 2(\frac{1}{3} - \frac{1}{4}) = \frac{1}{6}$; $\mathrm{Var}(X) = \frac{1}{18}$. Thus $\boxed{\mathrm{Var}(G) = 12^2/18 = 8\ \mathrm{euros}^2}$.

(c) $G > 0 \Leftrightarrow X > \frac{1}{4}$. Probability $\int_{1/4}^1 2(1-x)\,\mathrm{d}x = (\frac{3}{4})^2 = \boxed{\frac{9}{16}}$.

29.

(a) $|a|/2 = 2$ and $a < 0$ give $\boxed{a = -4}$. Then $-8 + b = 10 \Rightarrow \boxed{b = 18}$.

(b) $-4X + 18 > 12 \Leftrightarrow X < \frac{3}{2}$, so $\boxed{\mathrm{P}(Y > 12) = \mathrm{P}(X < \frac{3}{2})}$.

(c) A mean and standard deviation alone do not specify the distribution or the area below $\frac{3}{2}$.

30. The integral of f gives total probability. Instead,

$$\mathrm{E}(X) = \int_0^1 x(2x)\,\mathrm{d}x = \tfrac{2}{3}, \qquad \mathrm{E}(X^2) = \int_0^1 x^2(2x)\,\mathrm{d}x = \tfrac{1}{2}.$$

Subtract the square of the mean: $\boxed{\mathrm{Var}(X) = \tfrac{1}{2} - (\tfrac{2}{3})^2 = \tfrac{1}{18}}$. The standard deviation is the square root: $\boxed{\sigma = \sqrt{1/18} = \tfrac{\sqrt{2}}{6}}$.

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31.

(a) $1 = k[x^2 - x^3/3]_0^2 = 4k/3 \Rightarrow k = \tfrac{3}{4}$.

(b) $f' = \tfrac{3}{2}(1-x) = 0$ at $x = 1$, a maximum since $f'' = -\tfrac{3}{2}$; mode $\boxed{1}$.

(c) $\tfrac{3}{4}[x^2 - x^3/3]_{1/2}^{3/2} = \boxed{\tfrac{11}{16}}$.

(d) $\mathrm{E}(X) = \tfrac{3}{4}\int_0^2 x^2(2-x)\,\mathrm{d}x = \boxed{1}$; $\mathrm{E}(X^2) = \tfrac{3}{4}\int_0^2 x^3(2-x)\,\mathrm{d}x = \tfrac{6}{5}$, so $\boxed{\mathrm{Var}(X) = \tfrac{1}{5}}$.

32.

(a) Triangle area $\tfrac{1}{2}(4)(2k) = 1 \Rightarrow \boxed{k = \tfrac{1}{4}}$.

(b) $1 - 2\int_0^1 x/4\,\mathrm{d}x = 1 - \tfrac{1}{4} = \boxed{\tfrac{3}{4}}$.

(c) Both are $\boxed{2}$ by symmetry about $x = 2$.

(d) $\mathrm{E}(X^2) = \tfrac{1}{4}\int_0^2 x^3\,\mathrm{d}x + \tfrac{1}{4}\int_2^4 x^2(4-x)\,\mathrm{d}x = 1 + \tfrac{11}{3} = \tfrac{14}{3}$. Thus $\boxed{\mathrm{Var}(X) = \tfrac{14}{3} - 4 = \tfrac{2}{3}}$.

33.

(a) The area below 2 is $\tfrac{1}{3}$. Hence $m > 2$ and $\int_m^6 (6-x)/12\,\mathrm{d}x = (6-m)^2/24 = \tfrac{1}{2}$. Thus $\boxed{m = 6 - 2\sqrt{3}\ \mathrm{m}}$.

(b) $\mathrm{E}(X) = \int_0^2 x^2/6\,\mathrm{d}x + \int_2^6 x(6-x)/12\,\mathrm{d}x = \tfrac{4}{9} + \tfrac{20}{9} = \tfrac{8}{3}$.

(c) $\mathrm{E}(G) = 3(\tfrac{8}{3}) - c = 0 \Rightarrow \boxed{c = 8 \text{ euros}}$.

(d) $G > 0 \Leftrightarrow X > \tfrac{8}{3}$, so $\mathrm{P}(G > 0) = (6 - \tfrac{8}{3})^2/24 = \boxed{\tfrac{25}{54}}$. Fairness means zero expected net gain, not a probability of profit of $\tfrac{1}{2}$.

34.

(a) Normalisation gives $a + b/2 = 1$; the mean gives $a/2 + b/3 = 5/9$. Substitute $a = 1 - b/2$: $\tfrac{1}{2} + b/12 = \tfrac{5}{9}$, so $\boxed{a = b = \tfrac{2}{3}}$.

(b) $f(x) = \tfrac{2}{3}(1+x) \geq \tfrac{2}{3} > 0$.

(c) $\mathrm{E}(X^2) = \int_0^1 x^2 \tfrac{2}{3}(1+x)\,\mathrm{d}x = \tfrac{7}{18}$. Thus $\boxed{\mathrm{Var}(X) = \tfrac{7}{18} - (\tfrac{5}{9})^2 = \tfrac{13}{162}}$.

35.

(a) Put $u = x - 2$, so $(x-1)(3-x) = 1 - u^2$. Then $1 = k\int_{-1}^{1}(1-u^2)\,du = 4k/3$, giving $\boxed{k = \frac{3}{4}}$.

(b) Symmetry gives $\boxed{\mathrm{E}(X) = 2}$. Also $\mathrm{E}(X^2) = \frac{3}{4}\int_1^3 x^2(x-1)(3-x)\,dx = \frac{21}{5}$, so $\boxed{\mathrm{Var}(X) = \frac{1}{5}}$.

(c) $\boxed{\mathrm{E}(Y) = 1}$, $\boxed{\mathrm{Var}(Y) = \frac{4}{5}}$.

(d) $Y > 1 \Leftrightarrow X < 2$; by symmetry the probability is $\boxed{\frac{1}{2}}$.

36.

(a) $\mathrm{E}(X) = [x^2/(2(b-a))]_a^b = (a+b)/2$. Also $\mathrm{E}(X^2) = (b^3-a^3)/(3(b-a)) = (a^2+ab+b^2)/3$. Subtracting $(a+b)^2/4$ gives $(b-a)^2/12$.

(b) $a + b = 10$ and $(b-a)^2 = 36$. Since $b > a$, $b - a = 6$, so $\boxed{a = 2,\ b = 8}$.

(c) $(7-3)/(8-2) = \boxed{\frac{2}{3}}$.

37.

(a) $k[\ln x]_1^e = 1 \Rightarrow \boxed{k = 1}$.

(b) $\ln m = \frac{1}{2} \Rightarrow \boxed{m = \sqrt{e}}$.

(c) $\mathrm{E}(X) = \int_1^e x(1/x)\,dx = \boxed{e-1}$; $\mathrm{E}(X^2) = \int_1^e x^2(1/x)\,dx = (e^2-1)/2$. Thus $\boxed{\mathrm{Var}(X) = (-e^2+4e-3)/2}$.

38.

(a) $k[\sin x]_0^{\pi/2} = 1 \Rightarrow \boxed{k = 1}$.

(b) $1 - \sin(\pi/6) = \boxed{\frac{1}{2}}$.

(c) $\sin m = \frac{1}{2}$ on the support gives median $\boxed{\pi/6}$; decreasing $\cos x$ has its maximum at 0, so mode $\boxed{0}$.

(d) Integration by parts: $\mathrm{E}(X) = [x\sin x + \cos x]_0^{\pi/2} = \boxed{\pi/2 - 1}$.

39.

(a) $\int_0^2 3x^2/a^3\,dx = 8/a^3 = \frac{1}{2}$, so $\boxed{a = \sqrt[3]{16} = 2\sqrt[3]{2}}$.

(b) $\mathrm{E}(X) = 3a/4 = \boxed{\frac{3}{2}\sqrt[3]{2}}$.

(c) $\mathrm{E}(X^2) = 3a^2/5$, so $\mathrm{Var}(X) = 3a^2/5 - 9a^2/16 = 3a^2/80 = \boxed{\frac{3}{20}\sqrt[3]{4}}$.

40.

(a) $c/2 + c = 1 \Rightarrow \boxed{c = \frac{2}{3}}$.

(b) The area below 1 is $\frac{1}{3}$. Thus $\frac{1}{3} + \frac{2}{3}(m-1) = \frac{1}{2}$, giving $\boxed{m = \frac{5}{4}}$.

(c) $\mathrm{E}(X) = \frac{2}{3}\int_0^1 x^2\,dx + \frac{2}{3}\int_1^2 x\,dx = \boxed{\frac{11}{9}}$. Similarly, $\mathrm{E}(X^2) = \frac{2}{3}\int_0^1 x^3\,dx + \frac{2}{3}\int_1^2 x^2\,dx = \boxed{\frac{31}{18}}$.

(d) $\boxed{\mathrm{Var}(X) = \frac{31}{18} - (\frac{11}{9})^2 = \frac{37}{162}}$.

41.

(a) $\mathrm{E}(X) = 3\int_0^1 (x - 2x^2 + x^3)\,\mathrm{d}x = \frac{1}{4}$. $\mathrm{E}(X^2) = 3\int_0^1 (x^2 - 2x^3 + x^4)\,\mathrm{d}x = \frac{1}{10}$, so $\mathrm{Var}(X) = \frac{1}{10} - \frac{1}{16} = \frac{3}{80}$.

(b) $\mathrm{E}(G) = 20(\frac{1}{4}) + 1 - c = 0 \Rightarrow \boxed{c = 6 \text{ euros}}$. $\sigma_G = 20\sqrt{3/80} = \boxed{\sqrt{15} \text{ euros}}$.

(c) $G > 0 \Leftrightarrow X > \frac{1}{4}$. Probability $\int_{1/4}^1 3(1-x)^2\,\mathrm{d}x = (\frac{3}{4})^3 = \boxed{\frac{27}{64}}$.

42.

(a) $1 = k[x + x^3/3]_{-1}^1 = 8k/3 \Rightarrow \boxed{k = \frac{3}{8}}$.

(b) Mean $\boxed{0}$ and median $\boxed{0}$ by symmetry. The density is greatest at both endpoints, so modes $\boxed{-1 \text{ and } 1}$.

(c) $\mathrm{E}(X^2) = \frac{3}{8}\int_{-1}^1 (x^2 + x^4)\,\mathrm{d}x = \frac{3}{4}(\frac{1}{3} + \frac{1}{5}) = \frac{2}{5}$. Thus $\boxed{\mathrm{Var}(X) = \frac{2}{5}}$.

(d) $\boxed{\mathrm{E}(H) = -4}$; $\boxed{\mathrm{Var}(H) = 9(\frac{2}{5}) = \frac{18}{5}}$.

43.

(a) $\mathrm{E}(X) = \frac{2}{3}$, $\mathrm{E}(X^2) = \frac{1}{2}$, so $\mathrm{Var}(X) = \frac{1}{18}$. Therefore $\boxed{\mathrm{E}(Y) = \frac{1}{3},\ \mathrm{Var}(Y) = \frac{1}{18}}$.

(b) $Y < \frac{1}{4} \Leftrightarrow X > \frac{3}{4}$; probability $1 - (\frac{3}{4})^2 = \boxed{\frac{7}{16}}$.

(c) $\mathrm{P}(Y \le m) = \mathrm{P}(X \ge 1 - m) = 1 - (1-m)^2$. Set this equal to $\frac{1}{2}$ with $m \in [0, 1]$: $\boxed{m = 1 - 1/\sqrt{2}}$. Multiplication by -1 reverses the inequality.

44.

(a) $c + 2c = 1 \Rightarrow \boxed{c = \frac{1}{3}}$. The sketch has horizontal segments of height $\frac{1}{3}$ on $[0, 1]$ and $\frac{2}{3}$ on $[2, 3]$, and lies on the x-axis elsewhere.

(b) $\frac{1}{3}(\frac{1}{2}) + \frac{2}{3}(\frac{1}{2}) = \boxed{\frac{1}{2}}$.

(c) The area below 2 is $\frac{1}{3}$; $\frac{1}{3} + \frac{2}{3}(m - 2) = \frac{1}{2} \Rightarrow \boxed{m = \frac{9}{4}}$.

(d) $\mathrm{E}(X) = \int_0^1 x/3\,\mathrm{d}x + \int_2^3 2x/3\,\mathrm{d}x = \frac{1}{6} + \frac{5}{3} = \boxed{\frac{11}{6}}$. An average can lie in the gap between possible ranges, where the density is zero.

45.

(a) The integral is 1 for every α. Non-negativity at the two endpoints requires $1 - \alpha \ge 0$ and $1 + \alpha \ge 0$. Since the function is linear, these conditions suffice: $\boxed{-1 \le \alpha \le 1}$.

(b) $\mathrm{E}(X) = \frac{1}{2}\int_{-1}^1 (x + \alpha x^2)\,\mathrm{d}x = \alpha/3$.

(c) $\alpha = \frac{1}{2}$ and $\mathrm{E}(X^2) = \frac{1}{2}\int_{-1}^1 (x^2 + \alpha x^3)\,\mathrm{d}x = \frac{1}{3}$. Thus $\boxed{\mathrm{Var}(X) = \frac{1}{3} - \frac{1}{36} = \frac{11}{36}}$. Also $\mathrm{P}(X > 0) = \int_0^1 (\frac{1}{2} + \frac{x}{4})\,\mathrm{d}x = \boxed{\frac{5}{8}}$.

46.

(a) $ca + (c/2)(2a) = 1 \Rightarrow c = 1/(2a)$. Then $\mathrm{E}(X) = c\int_0^a x\,\mathrm{d}x + (c/2)\int_a^{3a} x\,\mathrm{d}x = 5ca^2/2 = 5a/4 = 5$. Thus $\boxed{a = 4,\ c = \frac{1}{8}}$.

(b) The area below a is $ca = \frac{1}{2}$, so $\boxed{m = 4}$.

(c) $(c/2)(3a - 2a) = ca/2 = \boxed{\frac{1}{4}}$.

(d) $\mathrm{E}(X^2) = c(a^3/3) + (c/2)(26a^3/3) = 7a^2/3 = 112/3$. Thus $\boxed{\mathrm{Var}(X) = \frac{112}{3} - 25 = \frac{37}{3}}$.

47.

(a) Both means are $\boxed{1\text{ mm}}$ by symmetry.

(b) $\mathrm{E}(A^2) = \frac{1}{2}\int_0^2 x^2\,\mathrm{d}x = \frac{4}{3}$, so $\boxed{\mathrm{Var}(A) = \frac{1}{3}\text{ mm}^2}$. $\mathrm{E}(B^2) = \int_0^1 x^3\,\mathrm{d}x + \int_1^2 x^2(2-x)\,\mathrm{d}x = \frac{7}{6}$, so $\boxed{\mathrm{Var}(B) = \frac{1}{6}\text{ mm}^2}$.

(c) For A: $(1)(\frac{1}{2}) = \boxed{\frac{1}{2}}$. For B: $1 - 2\int_0^{1/2} x\,\mathrm{d}x = \boxed{\frac{3}{4}}$.

(d) Machine B: its errors have smaller variance and more probability close to their common mean.

48.

(a) $\mathrm{E}(X) = \int_0^2 x^2/2\,\mathrm{d}x = \boxed{\frac{4}{3}}$; $\mathrm{E}(X^2) = \int_0^2 x^3/2\,\mathrm{d}x = 2$, so $\boxed{\mathrm{Var}(X) = \frac{2}{9}}$.

(b) $\mathrm{Var}(R) = 2 = a^2(\frac{2}{9})$ and $a > 0$ give $\boxed{a = 3}$. Then $8 = 3(\frac{4}{3}) + b$ gives $\boxed{b = 4}$.

(c) $\boxed{G = R - 9 = 3X - 5}$ euros; $\boxed{\mathrm{E}(G) = -1\text{ euro}}$.

(d) $G > 0 \Leftrightarrow X > \frac{5}{3}$. Probability $1 - \int_0^{5/3} x/2\,\mathrm{d}x = 1 - 25/36 = \boxed{\frac{11}{36}}$.

49.

(a) The median gives $\int_{-1}^1 k(x+1)\,\mathrm{d}x = 2k = \frac{1}{2}$, so $\boxed{k = \frac{1}{4}}$. Normalisation gives $k(b+1)^2/2 = 1$, so $(b+1)^2 = 8$ and $\boxed{b = 2\sqrt{2} - 1}$.

(b) Put $L = b + 1 = 2\sqrt{2}$ and $u = x + 1$. Then $\mathrm{E}(X) = \frac{1}{4}\int_0^L (u-1)u\,\mathrm{d}u = L^3/12 - L^2/8 = \boxed{\frac{4\sqrt{2}}{3} - 1}$. Also $\mathrm{E}(X^2) = \frac{1}{4}\int_0^L (u-1)^2 u\,\mathrm{d}u = 5 - \frac{8\sqrt{2}}{3}$. Subtracting $[\mathrm{E}(X)]^2$ gives $\boxed{\mathrm{Var}(X) = \frac{4}{9}}$.

(c) $1 - \int_{-1}^0 (x+1)/4\,\mathrm{d}x = \boxed{\frac{7}{8}}$.

50.

(a) $a/2 = \frac{1}{4} \Rightarrow \boxed{a = \frac{1}{2}}$. The remaining area is $b/2 = \frac{3}{4}$, so $\boxed{b = \frac{3}{2}}$.

(b) $m > 1$; the right-tail area is $\frac{3}{4}(2-m)^2 = \frac{1}{2}$. Thus $\boxed{m = 2 - \sqrt{\frac{2}{3}}\text{ m}}$. The greatest density is $f(1) = \frac{3}{2}$, so mode $\boxed{1\text{ m}}$.

(c) $\mathrm{E}(X) = \int_0^1 x(x/2)\,\mathrm{d}x + \int_1^2 x\frac{3}{2}(2-x)\,\mathrm{d}x = \frac{1}{6} + 1 = \boxed{\frac{7}{6}\text{ m}}$. $\mathrm{E}(X^2) = \int_0^1 x^2(x/2)\,\mathrm{d}x + \int_1^2 x^2\frac{3}{2}(2-x)\,\mathrm{d}x = \frac{1}{8} + \frac{11}{8} = \frac{3}{2}$. Hence $\boxed{\mathrm{Var}(X) = \frac{3}{2} - (\frac{7}{6})^2 = \frac{5}{36}\text{ m}^2}$.

(d) Fairness requires $c = \mathrm{E}(R) = 6(\frac{7}{6}) = \boxed{7\text{ euros}}$. $\sigma_G = 6\sqrt{5/36} = \boxed{\sqrt{5}\text{ euros}}$.