Discrete random variables

IB Mathematics AA HL ▪ Section 4.14

50 original questions: mean, variance, standard deviation and fair games

Learning goal	Calculate and interpret mean and spread; find unknown parameters; use linear transformations and fair-game models.
Syllabus link	**Discrete component of AHL 4.14 only.** Uses probability-distribution and expected-value skills from SL 4.7. All questions are for AA HL.
Structure	Q1–15 short response; Q16–30 extended response; Q31–50 original AA HL Paper 1-style questions. Worked solutions begin on page 14.
Technology	Non-calculator throughout. The IB AA formula booklet may be used. Give exact answers unless stated otherwise.

Instructions. Work on separate paper and show your reasoning. For $\mathrm{E}(X^2)$, square each possible value of X and multiply by its *original probability*. Check that any probabilities you find lie in $[0, 1]$ and sum to 1. Include units where appropriate. In a game, distinguish the payout from the player's net gain after paying the entry fee.

1 Short response

Build fluency with probabilities, expected squares, variance and linear transformations.

1. Check a probability distribution

A discrete random variable X has the distribution

x	-1	1	3
$\mathrm{P}(X = x)$	$\frac{1}{5}$	k	$\frac{3}{10}$

(a) State the two conditions that the probabilities in a valid discrete probability distribution must satisfy.

(b) Find k and $\mathrm{P}(X > 0)$.

2. Use the expected square

A discrete random variable X has $\mathrm{E}(X) = 3$ and $\mathrm{E}(X^2) = 14$.

(a) Explain what $\mathrm{E}(X^2)$ means.

(b) Find $\mathrm{Var}(X)$ and the standard deviation of X.

3. A two-value distribution

The score X in one round is 0 with probability $\frac{3}{4}$ and 2 with probability $\frac{1}{4}$. Find $\mathrm{E}(X)$, $\mathrm{E}(X^2)$, $\mathrm{Var}(X)$ and the standard deviation.

4. A mean of zero

x	-2	0	1
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. Explain why a mean of zero does not imply that every outcome is zero.

5. State the units

The random variable H is the number of whole hours a hire bicycle is rented for. It has variance $\frac{9}{4}$. State the standard deviation, and give the units of both the variance and the standard deviation.

6. Shift every outcome

A discrete random variable X has mean 7 and variance 4. Let $Y = X - 3$. Find the mean, variance and standard deviation of Y.

7. Multiply every outcome

A discrete random variable X has mean -2 and variance 3. Let $Y = 4X$. Find the mean, variance and standard deviation of Y.

8. Use a negative multiplier

A discrete random variable X has mean 5 and standard deviation 3. Let $Y = 11 - 2X$. Find the mean, variance and standard deviation of Y.

9. Recover the expected square

A discrete random variable X has $\mathrm{E}(X) = -1$ and $\mathrm{Var}(X) = 8$. Find $\mathrm{E}(X^2)$.

10. Correct a transformation error

Given $\mathrm{Var}(X) = 2$ and $Y = 3X + 4$, a student writes $\mathrm{Var}(Y) = 3(2) + 4 = 10$. Explain both errors and give the correct variance and standard deviation of Y.

11. An impossible pair of values

Could a discrete random variable satisfy both $\mathrm{E}(X) = 4$ and $\mathrm{E}(X^2) = 13$? Justify your answer.

12. No variation

A random variable X equals 6 with probability 1. Find $\mathrm{E}(X)$, $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$, and interpret the variance.

13. Find a fair entry fee

In one play of a game, the payout R, in euros, has distribution

r	0	4	12
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

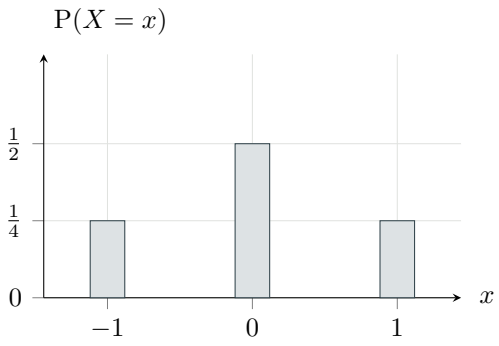
The player pays a fixed entry fee of c euros. Define the player's net gain G in terms of R and c, and find the fee that makes the game fair.

14. Increase the entry fee

The player's net gain G, in euros, in a fair game has variance 36. The payout rules stay the same but the entry fee rises by €2. Let H be the player's new net gain. Write H in terms of G, and find $\mathrm{E}(H)$ and $\mathrm{Var}(H)$.

15. Read a probability bar chart

The chart shows the distribution of a score X.



Find $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and $\mathrm{P}(|X| = 1)$.

2 Extended response

Connect distributions, interpretations and multi-step calculations.

16. Organise a variance calculation

A discrete random variable X has the following distribution. Copy and complete the last two rows.

x	-2	1	4
$p(x) = \mathrm{P}(X = x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$
$x\,p(x)$			
$x^2 p(x)$			

(a) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

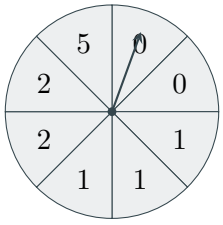
(b) Hence find the variance and standard deviation.

(c) Find $\mathrm{P}(X > \mathrm{E}(X))$.

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17. From a spinner to a distribution

The eight sectors of the spinner are equal. The spinner is fair. Let X be the number shown on one spin.



(a) Construct the probability distribution of X.

(b) Find $\mathrm{E}(X)$, $\mathrm{Var}(X)$ and the standard deviation.

(c) The payout is $R = 3X + 2$ euros. Find $\mathrm{E}(R)$ and $\mathrm{Var}(R)$.

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18. Transform the entire distribution

For $x = 0, 1, 2, 3$, the probability function of X is $\mathrm{P}(X = x) = k(4 - x)$, where k is a constant. Define $Z = 5 - 2X$.

(a) Find k.

(b) Find the mean and variance of X.

(c) Write the distribution of Z in increasing order of its values.

(d) Find $\mathrm{E}(Z)$, $\mathrm{Var}(Z)$ and $\mathrm{P}(Z \geq 3)$.

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19. Use a mean to find a probability

x	0	1	3
$\mathrm{P}(X = x)$	p	p	$1 - 2p$

(a) State the possible range of p.

(b) Given $\mathrm{E}(X) = 2$, find p.

(c) Find $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$.

20. Find an unknown outcome

x	0	2	a
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

It is given that $a > 2$ and $\mathrm{E}(X) = 3$.

(a) Find a.

(b) Find the variance and standard deviation of X.

(c) Explain why subtracting 3 from every outcome makes the mean zero but does not make the variance zero.

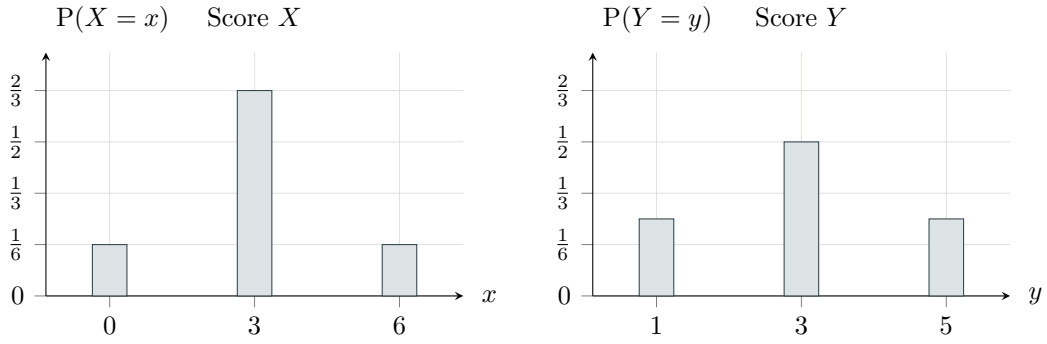
21. Recover three probabilities

x	-1	0	2
$\mathrm{P}(X = x)$	p	q	r

Given $\mathrm{E}(X) = \frac{1}{2}$ and $\mathrm{Var}(X) = \frac{5}{4}$, find p, q, r. Check that your results define a valid probability distribution.

22. Compare two distributions

The charts show the distributions of scores X and Y.



For clarity, $\mathrm{P}(Y = 1) = \mathrm{P}(Y = 5) = \frac{1}{4}$ and $\mathrm{P}(Y = 3) = \frac{1}{2}$.

(a) Show that both means equal 3.

(b) Calculate both variances. Which score has the smaller standard deviation?

(c) Find $\mathrm{P}(|X - 3| \leq 1)$ and $\mathrm{P}(|Y - 3| \leq 1)$. Does smaller variance guarantee a greater probability within every chosen distance of the mean?

23. Build a distribution from two trials

A biased coin has probability $\frac{3}{4}$ of landing heads. It is tossed twice independently. Let X be the number of heads obtained.

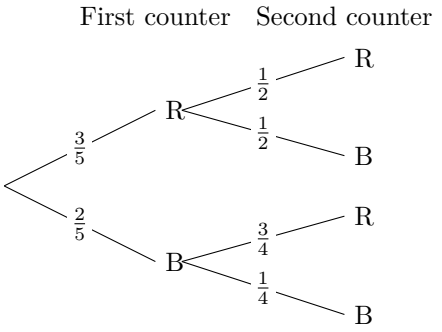
(a) Construct the distribution of X.

(b) Use this distribution to calculate its mean and variance.

(c) A score is defined by $S = 4X - 3$. Find its mean and standard deviation.

24. Sampling without replacement

A bag contains 3 red counters and 2 blue counters. Two counters are drawn at random without replacement. Let X be the number of red counters drawn. The tree gives the colour probabilities.



(a) Construct the distribution of X.

(b) Calculate its mean, variance and standard deviation.

25. Fair does not mean a fifty-fifty chance

The payout R, in euros, from one raffle ticket has distribution

r	0	5	25
$\mathrm{P}(R=r)$	$\frac{3}{5}$	$\frac{3}{10}$	$\frac{1}{10}$

The ticket costs c euros and the buyer's net gain is $G = R - c$.

(a) Find $\mathrm{E}(R)$ and $\mathrm{Var}(R)$.

(b) Find the fair ticket price.

(c) At this price, find $\mathrm{P}(G > 0)$ and $\mathrm{P}(G < 0)$. Explain why the game is fair even though these probabilities differ.

26. An insurance model

For one insurance policy, the company's claim payment C, in euros, has distribution

c	0	100	500
$\mathrm{P}(C=c)$	$\frac{4}{5}$	$\frac{3}{20}$	$\frac{1}{20}$

The company charges a premium of €50 and incurs no other costs in this model. Its profit is $P = 50 - C$.

(a) Find the mean and variance of C.

(b) Find the mean and standard deviation of P.

(c) What premium gives an expected profit of €30? Does changing only the premium change the standard deviation of profit?

27. Find unknown transformation constants

A discrete random variable X has mean 8 and standard deviation 2. The variable $Y = aX + b$ has mean 20 and standard deviation 6.

(a) Find both possible pairs (a, b).

(b) For the pair with $a < 0$, rewrite the event $X \leq 9$ as an event involving Y.

28. Restrict to a given event

x	0	2	4
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

(a) Find the mean and variance of X.

(b) Given that $X > 0$, write the conditional probabilities of the remaining values.

(c) Calculate the mean and variance of this conditional distribution.

29. Compare two fair games

Game A pays €8 with probability $\frac{1}{4}$ and €0 otherwise. Game B pays €4 with probability $\frac{1}{2}$ and €0 otherwise. Let R_A and R_B be the respective payouts, in euros.

(a) Find the fair entry fee for each game.

(b) At these fees, define the net gains G_A, G_B and calculate their variances.

(c) A player prefers the smaller standard deviation of net gain. Which game meets this preference?

(d) Find the probability of a positive net gain in each game.

30. A family of symmetric distributions

x	-1	0	1
$\mathrm{P}(X = x)$	p	$1 - 2p$	p

(a) State the allowable range of p.

(b) Show that $\mathrm{E}(X) = 0$ and $\mathrm{Var}(X) = 2p$.

(c) Find p when the standard deviation is $\frac{1}{2}$.

(d) Find the largest possible variance and the value of p at which it occurs.

3 Original AA HL Paper 1-style questions

Use clear working and exact values. Each part has a suggested mark allocation; total for this section: 204 marks.

31. Probabilities in a fixed ratio

[Maximum mark: 9]

x	0	1	2	4
$\mathrm{P}(X = x)$	k	$2k$	$3k$	$4k$

(a) Find k. [2 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$. [3 marks]

(c) Hence find $\mathrm{Var}(X)$ and the standard deviation. [2 marks]

(d) Let $Y = 5X - 7$. Find its mean and variance. [2 marks]

32. Reject an inadmissible root

[Maximum mark: 10]

x	-2	1	3
$\mathrm{P}(X = x)$	p	$1 - 2p$	p

(a) State the possible range of p. [1 mark]

(b) Show that $\mathrm{Var}(X) = 13p - p^2$. [3 marks]

(c) Given $\mathrm{Var}(X) = \frac{51}{16}$, find p, explaining why one algebraic root must be rejected. [3 marks]

(d) Let $Y = 8 - 4X$. Find its mean and standard deviation. [3 marks]

33. Use symmetry and variance together

[Maximum mark: 9]

x	0	1	2	3
$\mathrm{P}(X = x)$	a	b	b	a

(a) Write an equation relating a and b. [1 mark]

(b) Show that $\mathrm{E}(X) = \frac{3}{2}$ and $\mathrm{Var}(X) = \frac{1}{4} + 4a$. [4 marks]

(c) Given $\mathrm{Var}(X) = \frac{17}{20}$, find a and b. [2 marks]

(d) Find $\mathrm{P}(X \geq 2)$ and the standard deviation of X. [2 marks]

34. Score, payout and net gain [Maximum mark: 11]

A game awards a score X with distribution

x	0	1	2	5
$\mathrm{P}(X = x)$	$\frac{2}{5}$	$\frac{3}{10}$	$\frac{1}{5}$	$\frac{1}{10}$

The payout is $R = 3X + 1$ euros. One play costs €5, so the player's net gain is $G = R - 5$.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

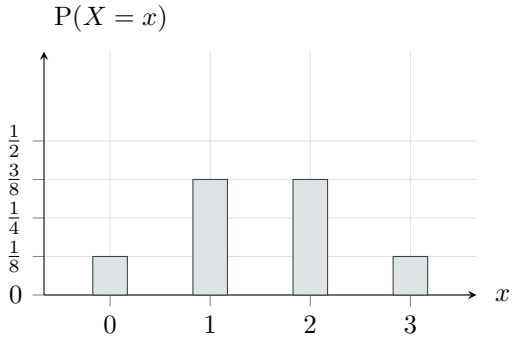
(b) Find $\mathrm{E}(G)$ and $\mathrm{Var}(G)$. [3 marks]

(c) Find the probability that the player makes a positive net gain. [2 marks]

(d) Keeping the payout rules unchanged, find the fair entry fee. [2 marks]

35. Transform a probability bar chart [Maximum mark: 9]

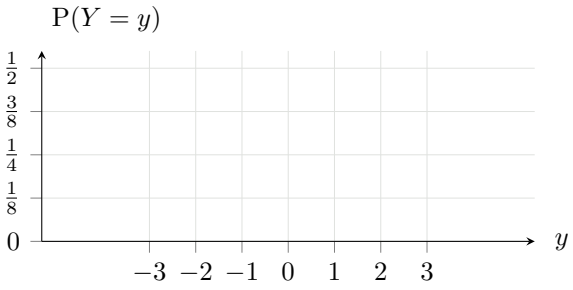
The chart shows the distribution of X. Let $Y = 2X - 3$.



(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(b) On the supplied axes, draw the probability bar chart of Y. [3 marks]

(c) Find the mean and variance of Y. [2 marks]



36. Two cards drawn without replacement [Maximum mark: 10]

Four cards are numbered 1, 2, 3 and 4. Two cards are drawn at random without replacement. Let X be the sum of their numbers. Each unordered pair of cards is equally likely.

(a) Construct the probability distribution of X. [3 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(c) A game pays $R = 2X$ euros and costs €10 to play. Define the player's net gain G, show that the game is fair, and find $\mathrm{Var}(G)$. [3 marks]

37. Keep the larger number [Maximum mark: 9]

Two independent fair spinners each have the numbers 1, 2 and 3 on equal sectors. Let X be the larger of the two numbers obtained. If the numbers match, X is that common number. The nine ordered outcomes are shown below.

	Second spinner		
First spinner	1	2	3
1	$(1,1)$	$(1,2)$	$(1,3)$
2	$(2,1)$	$(2,2)$	$(2,3)$
3	$(3,1)$	$(3,2)$	$(3,3)$

(a) Construct the distribution of X. [3 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(c) Find $\mathrm{P}(X > \mathrm{E}(X))$. [2 marks]

38. Find a scale from a standard deviation [Maximum mark: 9]

A random variable X takes the values $-a, 0, a$ with equal probabilities, where $a > 0$.

(a) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X) = \frac{2a^2}{3}$. [3 marks]

(b) Given that the standard deviation of X is 4, find a exactly. [3 marks]

(c) Let $Y = 7 - 3X$. Find its mean, variance and standard deviation. [3 marks]

39. Find an outcome and a transformation [Maximum mark: 9]

x	-2	0	r
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

It is given that $r > 0$ and $\mathrm{E}(X) = 1$.

(a) Find r and $\mathrm{Var}(X)$. [4 marks]

(b) The variable $Y = aX + b$ has mean 9 and standard deviation 6. Given $a < 0$, find a and b. [3 marks]

(c) Find $\mathrm{P}(Y < 0)$. [2 marks]

40. Reconstruct a game model [Maximum mark: 10]

x	0	1	3
$\mathrm{P}(X = x)$	p	q	r

It is given that $\mathrm{E}(X) = 1$ and $\mathrm{Var}(X) = 1$. A game pays $R = 5X + 2$ euros and charges a fixed entry fee c.

(a) Find $\mathrm{E}(X^2)$. [1 mark]

(b) Find p, q, r. [4 marks]

(c) Find the fair entry fee. [2 marks]

(d) At this fee, find the probability of a positive net gain and the probability of breaking even. [3 marks]

41. Work backwards from transformed moments [Maximum mark: 9]

A discrete random variable X has mean -2 and variance 5. For $Y = aX + b$, it is known that $\mathrm{E}(Y) = 7$, $\mathrm{Var}(Y) = 80$ and $a < 0$.

(a) Find a and b. [4 marks]

(b) Find $\mathrm{E}(Y^2)$. [2 marks]

(c) Find $\mathrm{E}((2X-1)^2)$. [3 marks]

42. A probability function with negative outcomes [Maximum mark: 10]

The probability function of X is $\mathrm{P}(X = x) = kx^2$ for $x = -2, -1, 1, 2$.

(a) Find k. [2 marks]

(b) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(c) Let $Y = 3 - 5X$. Find its mean and variance. [2 marks]

(d) Find $\mathrm{P}(Y < 0)$. [2 marks]

43. Change a prize rather than the entry fee [Maximum mark: 12]

A game pays R euros according to

r	0	4	10
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{5}$

(a) Find $\mathrm{E}(R)$ and $\mathrm{Var}(R)$. [4 marks]

(b) The entry fee is €3. Define the player's net gain G and find its expected value. Is the game fair? [2 marks]

(c) The organiser sets the fee to €4 and replaces the €10 prize by €v, keeping all probabilities unchanged. Find v so that this revised game is fair. [3 marks]

(d) Compare the variance of net gain in the revised game with that in the original game. [3 marks]

44. Revenue, cost and a discount [Maximum mark: 11]

The revenue R, in euros, from one stall session is modelled by

r	50	80	140
$\mathrm{P}(R = r)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

The fixed cost per session is €60, so the profit is $P = R - 60$.

(a) Find the mean and variance of R. [4 marks]

(b) Find the mean and standard deviation of P. [2 marks]

(c) A discount multiplies every possible revenue by $\frac{9}{10}$, with the same probabilities and fixed cost. Define the new profit Q in terms of R, and find its mean and standard deviation. [3 marks]

(d) Find the probability of a loss under each pricing model. [2 marks]

45. Fairness with a spread constraint

[Maximum mark: 10]

x	0	2	4
$\mathrm{P}(X = x)$	p	$1 - 2p$	p

A game pays $R = 3X + 1$ euros and charges a fixed fee c. The player's net gain is $G = R - c$.

(a) State the allowable range of p. [1 mark]

(b) Show that $\mathrm{E}(X) = 2$ and $\mathrm{Var}(X) = 8p$. [3 marks]

(c) Find the fee that makes the game fair for every allowable p. [2 marks]

(d) The standard deviation of net gain must be at most €3. Find the corresponding range of p. [3 marks]

(e) At the fair fee, express $\mathrm{P}(G > 0)$ in terms of p. [1 mark]

46. Maximise a variance

[Maximum mark: 11]

x	0	1	2
$\mathrm{P}(X = x)$	p	$\frac{1}{2}$	$\frac{1}{2} - p$

(a) State the allowable range of p. [1 mark]

(b) Show that $\mathrm{Var}(X) = \frac{1}{4} + 2p - 4p^2$. [3 marks]

(c) By completing the square, find the maximum possible variance and the value of p at which it occurs. [3 marks]

(d) Find both possible values of p when $\mathrm{Var}(X) = \frac{7}{16}$, and state the corresponding means. [4 marks]

47. Matching means and variances

[Maximum mark: 10]

Two discrete random variables have distributions

x	0	2	6
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

y	1	3	5
$\mathrm{P}(Y = y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Define $U = \dfrac{3X - 5}{\sqrt{41}}$ and $V = \dfrac{Y - 3}{\sqrt{2}}$.

(a) Find the mean and variance of X and of Y. [4 marks]

(b) Show that U and V each have mean 0 and variance 1. [3 marks]

(c) Find $\mathrm{P}(U > 0)$ and $\mathrm{P}(V > 0)$. [2 marks]

(d) Do equal means and variances determine the whole distribution? Explain using your results. [1 mark]

48. A conditional average in a fair game

[Maximum mark: 12]

The payout R, in euros, in one game has distribution

r	0	2	5	10
$\mathrm{P}(R=r)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{20}$

The entry fee is €2 and the player's net gain is $G = R - 2$.

(a) Find $\mathrm{E}(R)$ and $\mathrm{Var}(R)$, and show that the game is fair. [4 marks]

(b) Given that the payout is nonzero, construct the conditional distribution of R. [3 marks]

(c) Calculate the mean and variance of net gain for this conditional distribution. [3 marks]

(d) Explain why the conditional mean gain does not contradict the game's fairness. [2 marks]

49. Two models with the same moments

[Maximum mark: 12]

x	0	a	6
$\mathrm{P}(X=x)$	p	$\frac{1}{2}$	$\frac{1}{2}-p$

All three probabilities are positive and $0 < a < 6$. It is known that $\mathrm{E}(X) = 3$ and $\mathrm{Var}(X) = \frac{13}{2}$.

(a) Show that $p = \frac{a}{12}$. [2 marks]

(b) Find $\mathrm{E}(X^2)$ and hence show that $a^2 - 6a + 5 = 0$. [4 marks]

(c) Find the two possible pairs (a, p), checking their admissibility. [3 marks]

(d) It is also known that $\mathrm{P}(X > 3) > \frac{1}{2}$. Determine which pair is correct. [3 marks]

50. Design a fair game with less variation

[Maximum mark: 12]

A game has payout X, in euros, with distribution

x	0	r	$3r$
$\mathrm{P}(X=x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Here $r > 0$ and the entry fee is €5. The player's net gain is $G = X - 5$.

(a) Find r so that the game is fair. [3 marks]

(b) For this value of r, find $\mathrm{Var}(G)$ and the standard deviation of G. [3 marks]

(c) The payout is changed to $Y = aX + b$, where $a > 0$, while the fee stays at €5. Find a, b so that the new game is fair and its net-gain standard deviation is half the original value. [4 marks]

(d) Could the standard deviation be halved by changing only the entry fee? Explain. [2 marks]

Tutorial: variance and standard deviation

Use this AHL 4.14 tutorial to revisit $\mathrm{E}(X^2)$, variance and its square root.

youtu.be/mQrY17mmFi0

Answer key and worked solutions

For a discrete variable, $\mathrm{E}(X) = \sum xp(x)$ and $\mathrm{E}(X^2) = \sum x^2 p(x)$, where $p(x) = \mathrm{P}(X = x)$. The **second moment about zero** is another name for $\mathrm{E}(X^2)$: the expected value of the squared outcomes. It generally differs from $[\mathrm{E}(X)]^2$.

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2, \qquad \sigma_X = \sqrt{\mathrm{Var}(X)}.$$

For $Y = aX + b$, $\mathrm{E}(Y) = a\,\mathrm{E}(X) + b$, $\mathrm{Var}(Y) = a^2\,\mathrm{Var}(X)$ and $\sigma_Y = |a|\sigma_X$. Variance is also the probability-weighted mean of $(X - \mathrm{E}(X))^2$, sometimes called the *mean squared deviation from the mean.* Standard deviation is the square root of the variance.

Section 1: Short response

1. Solution

(a) Every probability must lie between 0 and 1, inclusive, and the probabilities of all possible values must sum to 1.

(b) $k = 1 - \frac{1}{5} - \frac{3}{10} = \frac{1}{2}$, which lies in $[0, 1]$. Hence $\mathrm{P}(X > 0) = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$.

2. Solution

(a) $\mathrm{E}(X^2)$ is the expected value of the squared outcomes: square each possible value of X, multiply by its probability, then add. It is also called the *second moment about zero.*

(b) $\mathrm{Var}(X) = 14 - 3^2 = 5$, so $\sigma_X = \sqrt{5}$.

3. Solution

$$\mathrm{E}(X) = 0\left(\frac{3}{4}\right) + 2\left(\frac{1}{4}\right) = \frac{1}{2}, \qquad \mathrm{E}(X^2) = 0^2\left(\frac{3}{4}\right) + 2^2\left(\frac{1}{4}\right) = 1.$$

Thus $\mathrm{Var}(X) = 1 - (\frac{1}{2})^2 = \frac{3}{4}$ and $\sigma_X = \frac{\sqrt{3}}{2}$.

4. Solution

$$\mathrm{E}(X) = (-2)\left(\frac{1}{4}\right) + 0\left(\frac{1}{4}\right) + 1\left(\frac{1}{2}\right) = 0.$$

$$\mathrm{E}(X^2) = (-2)^2\left(\frac{1}{4}\right) + 0^2\left(\frac{1}{4}\right) + 1^2\left(\frac{1}{2}\right) = \frac{3}{2}, \qquad \mathrm{Var}(X) = \frac{3}{2}.$$

Negative and positive contributions cancel in the mean. Here X can be -2 or 1, and the positive variance confirms that X is not constant.

5. Solution

$\sigma_H = \sqrt{\frac{9}{4}} = \frac{3}{2}$ hours. The variance is $\frac{9}{4}$ hours squared (h^2); the standard deviation has the same units as H.

6. Solution

$\mathrm{E}(Y) = 7 - 3 = 4$, $\mathrm{Var}(Y) = 4$ and $\sigma_Y = \sqrt{4} = 2$. Subtracting a constant shifts every value equally, so the spread is unchanged.

7. Solution

$\mathrm{E}(Y) = 4(-2) = -8$, $\mathrm{Var}(Y) = 4^2(3) = 48$ and $\sigma_Y = \sqrt{48} = 4\sqrt{3}$.

8. Solution

$\mathrm{Var}(X) = 3^2 = 9$. Then $\mathrm{E}(Y) = 11 - 2(5) = 1$, $\mathrm{Var}(Y) = (-2)^2(9) = 36$ and $\sigma_Y = \sqrt{36} = 6$. Standard deviation is non-negative.

Section 1: Short response (continued)

9. Solution

Rearrange $\text{Var}(X) = \text{E}(X^2) - [\text{E}(X)]^2$:

$$\text{E}(X^2) = 8 + (-1)^2 = 9.$$

10. Solution

The multiplier must be squared when calculating variance, and adding a constant does not change variance. Thus $\text{Var}(Y) = 3^2(2) = 18$ and $\sigma_Y = \sqrt{18} = 3\sqrt{2}$.

11. Solution

No: these values would give $\text{Var}(X) = 13 - 4^2 = -3$. Variance cannot be negative, so the pair of stated values is inconsistent.

12. Solution

$\text{E}(X) = 6$, $\text{E}(X^2) = 6^2 = 36$ and $\text{Var}(X) = 36 - 6^2 = 0$. There is no spread: the outcome is always 6.

13. Solution

$G = R - c$. The expected payout is

$$\text{E}(R) = 0\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right) + 12\left(\frac{1}{6}\right) = \frac{10}{3}.$$

Fairness requires $\text{E}(G) = \frac{10}{3} - c = 0$, so $c = \frac{10}{3}$ euros.

14. Solution

$H = G - 2$. Since the original game is fair, $\text{E}(G) = 0$. Hence $\text{E}(H) = -2$ euros and $\text{Var}(H) = 36$ euros squared. The higher fee changes the mean, not the spread.

15. Solution

$\text{E}(X) = (-1)(\frac{1}{4}) + 0(\frac{1}{2}) + 1(\frac{1}{4}) = 0$.

$$\text{E}(X^2) = (-1)^2\left(\frac{1}{4}\right) + 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{1}{4}\right) = \frac{1}{2}.$$

Therefore $\text{Var}(X) = \frac{1}{2}$ and $\text{P}(|X| = 1) = \text{P}(X = -1) + \text{P}(X = 1) = \frac{1}{2}$.

Section 2: Extended response

16. Solution

The $xp(x)$ row is $-\frac{2}{3}, \frac{1}{2}, \frac{2}{3}$. The $x^2p(x)$ row is $\frac{4}{3}, \frac{1}{2}, \frac{8}{3}$; the probabilities themselves are not squared.

(a) $\text{E}(X) = \frac{1}{2}$, $\text{E}(X^2) = \frac{4}{3} + \frac{1}{2} + \frac{8}{3} = \frac{9}{2}$.

(b) $\text{Var}(X) = \frac{9}{2} - (\frac{1}{2})^2 = \frac{17}{4}$; $\sigma_X = \frac{\sqrt{17}}{2}$.

(c) $X > \frac{1}{2}$ means $X = 1$ or 4, so the probability is $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$.

Section 2: Extended response (continued)

17. Solution

(a) The distinct values are $0, 1, 2, 5$, with probabilities $\frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{8}$ respectively.

(b)
$$\mathrm{E}(X) = 0\left(\frac{1}{4}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{1}{4}\right) + 5\left(\frac{1}{8}\right) = \frac{3}{2}.$$
$$\mathrm{E}(X^2) = 0^2\left(\frac{1}{4}\right) + 1^2\left(\frac{3}{8}\right) + 2^2\left(\frac{1}{4}\right) + 5^2\left(\frac{1}{8}\right) = \frac{9}{2}.$$
Thus $\mathrm{Var}(X) = \frac{9}{2} - \frac{9}{4} = \frac{9}{4}$ and $\sigma_X = \frac{3}{2}$.

(c) $\mathrm{E}(R) = 3(\frac{3}{2}) + 2 = \frac{13}{2}$ euros; $\mathrm{Var}(R) = 3^2(\frac{9}{4}) = \frac{81}{4}$ euros squared.

18. Solution

(a) $k(4 + 3 + 2 + 1) = 1$, so $k = \frac{1}{10}$.

(b) $\mathrm{E}(X) = \frac{0(4)+1(3)+2(2)+3(1)}{10} = 1$ and $\mathrm{E}(X^2) = \frac{0^2(4)+1^2(3)+2^2(2)+3^2(1)}{10} = 2$. Hence $\mathrm{Var}(X) = 2 - 1^2 = 1$.

(c)

z	-1	1	3	5
$\mathrm{P}(Z = z)$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{3}{10}$	$\frac{2}{5}$

The probabilities follow their corresponding transformed values.

(d) $\mathrm{E}(Z) = 5 - 2(1) = 3$, $\mathrm{Var}(Z) = (-2)^2(1) = 4$, and $\mathrm{P}(Z \geq 3) = \frac{3}{10} + \frac{2}{5} = \frac{7}{10}$.

19. Solution

(a) $0 \leq p \leq \frac{1}{2}$.

(b) $\mathrm{E}(X) = 0p + 1p + 3(1 - 2p) = 3 - 5p = 2$, so $p = \frac{1}{5}$.

(c) $\mathrm{E}(X^2) = 0^2(\frac{1}{5}) + 1^2(\frac{1}{5}) + 3^2(\frac{3}{5}) = \frac{28}{5}$. Thus $\mathrm{Var}(X) = \frac{28}{5} - 2^2 = \frac{8}{5}$.

20. Solution

(a) $0(\frac{1}{4}) + 2(\frac{1}{2}) + \frac{a}{4} = 3$, so $a = 8 > 2$.

(b) $\mathrm{E}(X^2) = 0^2(\frac{1}{4}) + 2^2(\frac{1}{2}) + 8^2(\frac{1}{4}) = 18$. Hence $\mathrm{Var}(X) = 18 - 9 = 9$ and $\sigma_X = 3$.

(c) $\mathrm{E}(X - 3) = 3 - 3 = 0$, but $\mathrm{Var}(X - 3) = 9$. The outcomes become $-3, -1, 5$; they still vary.

21. Solution

First $\mathrm{E}(X^2) = \frac{5}{4} + (\frac{1}{2})^2 = \frac{3}{2}$. The equations are $p + q + r = 1$, $-p + 2r = \frac{1}{2}$ and $p + 4r = \frac{3}{2}$. Adding the last two gives $6r = 2$, so $r = \frac{1}{3}$. Then $p = \frac{1}{6}$ and $q = \frac{1}{2}$. All probabilities are in $[0, 1]$ and sum to 1.

22. Solution

(a) $\mathrm{E}(X) = 0(\frac{1}{6}) + 3(\frac{2}{3}) + 6(\frac{1}{6}) = 3$ and $\mathrm{E}(Y) = 1(\frac{1}{4}) + 3(\frac{1}{2}) + 5(\frac{1}{4}) = 3$.

(b) $\mathrm{E}(X^2) = 0^2(\frac{1}{6}) + 3^2(\frac{2}{3}) + 6^2(\frac{1}{6}) = 12$, so $\mathrm{Var}(X) = 3$. Also $\mathrm{E}(Y^2) = 1^2(\frac{1}{4}) + 3^2(\frac{1}{2}) + 5^2(\frac{1}{4}) = 11$, so $\mathrm{Var}(Y) = 2$. Hence Y has the smaller standard deviation.

(c) Only the value 3 lies within distance 1 in either model. The probabilities are $\frac{2}{3}$ for X and $\frac{1}{2}$ for Y. No: this example has smaller variance for Y but a smaller probability in that particular interval.

Section 2: Extended response (continued)

23. Solution

(a) $\mathrm{P}(X=0)=(\frac{1}{4})^2=\frac{1}{16}$; $\mathrm{P}(X=1)=2(\frac{3}{4})(\frac{1}{4})=\frac{3}{8}$; $\mathrm{P}(X=2)=(\frac{3}{4})^2=\frac{9}{16}$.

(b) $\mathrm{E}(X)=0(\frac{1}{16})+1(\frac{3}{8})+2(\frac{9}{16})=\frac{3}{2}$.

$$\mathrm{E}(X^2)=0^2\left(\frac{1}{16}\right)+1^2\left(\frac{3}{8}\right)+2^2\left(\frac{9}{16}\right)=\frac{21}{8}.$$

So $\mathrm{Var}(X)=\frac{21}{8}-\frac{9}{4}=\frac{3}{8}$.

(c) $\mathrm{E}(S)=4(\frac{3}{2})-3=3$, $\mathrm{Var}(S)=4^2(\frac{3}{8})=6$, so $\sigma_S=\sqrt{6}$.

24. Solution

(a) $\mathrm{P}(X=0)=\frac{2}{5}\frac{1}{4}=\frac{1}{10}$; $\mathrm{P}(X=2)=\frac{3}{5}\frac{1}{2}=\frac{3}{10}$; $\mathrm{P}(X=1)=\frac{3}{5}\frac{1}{2}+\frac{2}{5}\frac{3}{4}=\frac{3}{5}$.

(b) $\mathrm{E}(X)=0(\frac{1}{10})+1(\frac{3}{5})+2(\frac{3}{10})=\frac{6}{5}$; $\mathrm{E}(X^2)=0^2(\frac{1}{10})+1^2(\frac{3}{5})+2^2(\frac{3}{10})=\frac{9}{5}$. Therefore $\mathrm{Var}(X)=\frac{9}{5}-\frac{36}{25}=\frac{9}{25}$ and $\sigma_X=\frac{3}{5}$ counters.

25. Solution

(a) $\mathrm{E}(R)=0(\frac{3}{5})+5(\frac{3}{10})+25(\frac{1}{10})=4$. $\mathrm{E}(R^2)=0^2(\frac{3}{5})+5^2(\frac{3}{10})+25^2(\frac{1}{10})=70$; hence $\mathrm{Var}(R)=70-16=54$.

(b) $\mathrm{E}(G)=4-c=0$ gives $c=4$ euros.

(c) The gains are $-4, 1, 21$, so $\mathrm{P}(G>0)=\frac{2}{5}$ and $\mathrm{P}(G<0)=\frac{3}{5}$. Fairness concerns the probability-weighted *amounts*: $(-4)(\frac{3}{5})+1(\frac{3}{10})+21(\frac{1}{10})=0$. It does not require equal chances of gain and loss.

26. Solution

(a) $\mathrm{E}(C)=100(\frac{3}{20})+500(\frac{1}{20})=40$. $\mathrm{E}(C^2)=100^2(\frac{3}{20})+500^2(\frac{1}{20})=14000$. Thus $\mathrm{Var}(C)=14000-40^2=12400$.

(b) $\mathrm{E}(P)=50-40=10$ euros; $\mathrm{Var}(P)=(-1)^2(12400)=12400$, so $\sigma_P=20\sqrt{31}$ euros.

(c) A premium d must satisfy $d-40=30$, so $d=70$ euros. Only a constant changes; the standard deviation remains $20\sqrt{31}$ euros.

27. Solution

(a) Since $\sigma_Y=|a|\sigma_X$, $6=2|a|$ gives $a=3$ or -3. From $20=8a+b$, the pairs are $(3,-4)$ and $(-3,44)$.

(b) With $Y=44-3X$, multiplying $X\le 9$ by -3 reverses the inequality: $Y\ge 44-27=17$.

28. Solution

(a) $\mathrm{E}(X)=2(\frac{1}{3})+4(\frac{1}{6})=\frac{4}{3}$ and $\mathrm{E}(X^2)=2^2(\frac{1}{3})+4^2(\frac{1}{6})=4$. Thus $\mathrm{Var}(X)=4-\frac{16}{9}=\frac{20}{9}$.

(b) $\mathrm{P}(X>0)=\frac{1}{2}$. Dividing each retained probability by $\frac{1}{2}$ gives probabilities $\frac{2}{3}$ at 2 and $\frac{1}{3}$ at 4.

(c) The conditional mean is $2(\frac{2}{3})+4(\frac{1}{3})=\frac{8}{3}$. The conditional expected square is $2^2(\frac{2}{3})+4^2(\frac{1}{3})=8$; the variance is $8-(\frac{8}{3})^2=\frac{8}{9}$.

29. Solution

(a) $\mathrm{E}(R_A)=8(\frac{1}{4})=2$ and $\mathrm{E}(R_B)=4(\frac{1}{2})=2$, so both fair fees are €2.

(b) $G_A=R_A-2$ and $G_B=R_B-2$. $\mathrm{Var}(G_A)=8^2(\frac{1}{4})-2^2=12$; $\mathrm{Var}(G_B)=4^2(\frac{1}{2})-2^2=4$.

(c) Game B: its standard deviation is 2 euros, compared with $2\sqrt{3}$ euros for A.

(d) A positive net gain occurs exactly when the nonzero payout is obtained: the probabilities are $\frac{1}{4}$ for A and $\frac{1}{2}$ for B.

Section 2: Extended response (continued)

30. Solution

(a) $0 \le p \le \frac{1}{2}$.

(b) $\mathrm{E}(X) = -p + p = 0$; $\mathrm{E}(X^2) = (-1)^2 p + 0^2(1-2p) + 1^2 p = 2p$. Therefore $\mathrm{Var}(X) = 2p$.

(c) $2p = (\frac{1}{2})^2 = \frac{1}{4}$, so $p = \frac{1}{8}$.

(d) $2p$ increases as p increases, so the maximum variance is 1, attained at $p = \frac{1}{2}$.

Section 3: Paper 1-style questions

31. Solution

(a) $10k = 1$, so $k = \frac{1}{10}$.

(b)

$$\mathrm{E}(X) = \frac{0(1) + 1(2) + 2(3) + 4(4)}{10} = \frac{12}{5}, \quad \mathrm{E}(X^2) = \frac{0^2(1) + 1^2(2) + 2^2(3) + 4^2(4)}{10} = \frac{39}{5}.$$

(c) $\mathrm{Var}(X) = \frac{39}{5} - (\frac{12}{5})^2 = \frac{51}{25}$ and $\sigma_X = \frac{\sqrt{51}}{5}$.

(d) $\mathrm{E}(Y) = 5(\frac{12}{5}) - 7 = 5$; $\mathrm{Var}(Y) = 5^2(\frac{51}{25}) = 51$.

32. Solution

(a) $0 \le p \le \frac{1}{2}$.

(b) $\mathrm{E}(X) = -2p + (1-2p) + 3p = 1 - p$ and $\mathrm{E}(X^2) = 4p + (1-2p) + 9p = 1 + 11p$. Thus $\mathrm{Var}(X) = 1 + 11p - (1-p)^2 = 13p - p^2$.

(c)

$$p^2 - 13p + \frac{51}{16} = 0 \quad \Longrightarrow \quad \left(p - \frac{1}{4}\right)\left(p - \frac{51}{4}\right) = 0.$$

Only $p = \frac{1}{4}$ lies in $[0, \frac{1}{2}]$.

(d) $\mathrm{E}(X) = \frac{3}{4}$, so $\mathrm{E}(Y) = 8 - 4(\frac{3}{4}) = 5$. $\mathrm{Var}(Y) = 16(\frac{51}{16}) = 51$ and $\sigma_Y = \sqrt{51}$.

33. Solution

(a) $2a + 2b = 1$, so $a + b = \frac{1}{2}$.

(b) $\mathrm{E}(X) = b + 2b + 3a = 3(a+b) = \frac{3}{2}$. Also $\mathrm{E}(X^2) = b + 4b + 9a = 5(\frac{1}{2} - a) + 9a = \frac{5}{2} + 4a$. Hence $\mathrm{Var}(X) = \frac{5}{2} + 4a - \frac{9}{4} = \frac{1}{4} + 4a$.

(c) $4a = \frac{17}{20} - \frac{1}{4} = \frac{3}{5}$, so $a = \frac{3}{20}$ and $b = \frac{7}{20}$.

(d) $\mathrm{P}(X \ge 2) = a + b = \frac{1}{2}$ and $\sigma_X = \sqrt{\frac{17}{20}} = \frac{\sqrt{85}}{10}$.

34. Solution

(a) $\mathrm{E}(X) = 1(\frac{3}{10}) + 2(\frac{1}{5}) + 5(\frac{1}{10}) = \frac{6}{5}$. $\mathrm{E}(X^2) = 1^2(\frac{3}{10}) + 2^2(\frac{1}{5}) + 5^2(\frac{1}{10}) = \frac{18}{5}$. Thus $\mathrm{Var}(X) = \frac{18}{5} - \frac{36}{25} = \frac{54}{25}$.

(b) $G = 3X - 4$; $\mathrm{E}(G) = 3(\frac{6}{5}) - 4 = -\frac{2}{5}$ euros and $\mathrm{Var}(G) = 9(\frac{54}{25}) = \frac{486}{25}$ euros squared.

(c) $G > 0$ means $X > \frac{4}{3}$, so $X = 2$ or 5. The probability is $\frac{1}{5} + \frac{1}{10} = \frac{3}{10}$.

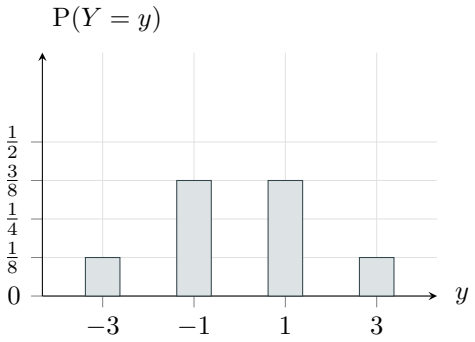
(d) The fair fee is $\mathrm{E}(R) = 3(\frac{6}{5}) + 1 = \frac{23}{5}$ euros.

Section 3: Paper 1-style questions (continued)

35. Solution

(a) $\mathrm{E}(X) = \frac{0+3+6+3}{8} = \frac{3}{2}$ and $\mathrm{E}(X^2) = \frac{0^2+3(1^2)+3(2^2)+3^2}{8} = 3$. Thus $\mathrm{Var}(X) = 3 - \frac{9}{4} = \frac{3}{4}$.

(b) The bars have heights $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$ at $y = -3, -1, 1, 3$ respectively.



(c) $\mathrm{E}(Y) = 2(\frac{3}{2}) - 3 = 0$ and $\mathrm{Var}(Y) = 2^2(\frac{3}{4}) = 3$.

36. Solution

(a) The pairs $(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)$ give sums $3, 4, 5, 5, 6, 7$.

x	3	4	5	6	7
$\mathrm{P}(X = x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$

(b) $\mathrm{E}(X) = \frac{3+4+2(5)+6+7}{6} = 5$; $\mathrm{E}(X^2) = \frac{3^2+4^2+2(5^2)+6^2+7^2}{6} = \frac{80}{3}$. Thus $\mathrm{Var}(X) = \frac{80}{3} - 25 = \frac{5}{3}$.

(c) $G = 2X - 10$. Since $\mathrm{E}(G) = 2(5) - 10 = 0$, the game is fair. $\mathrm{Var}(G) = 4(\frac{5}{3}) = \frac{20}{3}$ euros squared.

37. Solution

(a) There is one outcome with maximum 1, three with maximum 2 and five with maximum 3. Thus the probabilities at $1, 2, 3$ are $\frac{1}{9}, \frac{3}{9}, \frac{5}{9}$.

(b)
$$\mathrm{E}(X) = \frac{1(1) + 2(3) + 3(5)}{9} = \frac{22}{9}, \qquad \mathrm{E}(X^2) = \frac{1^2(1) + 2^2(3) + 3^2(5)}{9} = \frac{58}{9}.$$
Therefore $\mathrm{Var}(X) = \frac{58}{9} - (\frac{22}{9})^2 = \frac{38}{81}$.

(c) Since $2 < \frac{22}{9} < 3$, only $X = 3$ satisfies the inequality. The probability is $\frac{5}{9}$.

38. Solution

(a) $\mathrm{E}(X) = \frac{-a+0+a}{3} = 0$ and $\mathrm{E}(X^2) = \frac{(-a)^2+0^2+a^2}{3} = \frac{2a^2}{3}$. Subtracting the square of the mean leaves $\mathrm{Var}(X) = \frac{2a^2}{3}$.

(b) $\frac{2a^2}{3} = 4^2 = 16$, so $a^2 = 24$. Since $a > 0$, $a = 2\sqrt{6}$.

(c) $\mathrm{E}(Y) = 7$, $\mathrm{Var}(Y) = (-3)^2(16) = 144$ and $\sigma_Y = 12$.

39. Solution

(a) $-\frac{1}{2} + \frac{r}{4} = 1$ gives $r = 6$. $\mathrm{E}(X^2) = (-2)^2(\frac{1}{4}) + 0^2(\frac{1}{2}) + 6^2(\frac{1}{4}) = 10$, so $\mathrm{Var}(X) = 10 - 1^2 = 9$.

(b) $\sigma_X = 3$ and $6 = 3|a|$. Since $a < 0$, $a = -2$; then $9 = -2(1) + b$ gives $b = 11$.

(c) $Y = 11 - 2X < 0$ means $X > \frac{11}{2}$. Only $X = 6$ qualifies, so $\mathrm{P}(Y < 0) = \frac{1}{4}$.

Section 3: Paper 1-style questions (continued)

40. Solution

(a) $\mathrm{E}(X^2) = 1 + 1^2 = 2$.

(b) $q + 3r = 1$ and $q + 9r = 2$, so $6r = 1$. Thus $r = \frac{1}{6}$, $q = \frac{1}{2}$ and $p = \frac{1}{3}$.

(c) With net gain $G = R - c$, $\mathrm{E}(G) = 5(1) + 2 - c = 0$ gives $c = 7$ euros.

(d) $G = 5X - 5$. A positive gain requires $X = 3$, with probability $\frac{1}{6}$. Breaking even requires $X = 1$, with probability $\frac{1}{2}$.

41. Solution

(a) $80 = 5a^2$ gives $a = \pm 4$. Hence $a = -4$, and $7 = (-4)(-2) + b$ gives $b = -1$.

(b) $\mathrm{E}(Y^2) = \mathrm{Var}(Y) + [\mathrm{E}(Y)]^2 = 80 + 49 = 129$.

(c) Let $W = 2X - 1$. Then $\mathrm{E}(W) = 2(-2) - 1 = -5$ and $\mathrm{Var}(W) = 2^2(5) = 20$. Therefore $\mathrm{E}(W^2) = 20 + (-5)^2 = 45$.

42. Solution

(a) $k(4 + 1 + 1 + 4) = 1$, so $k = \frac{1}{10}$.

(b)

$$\mathrm{E}(X) = \frac{(-2)(4) + (-1)(1) + 1(1) + 2(4)}{10} = 0.$$

$$\mathrm{E}(X^2) = \frac{(-2)^2(4) + (-1)^2(1) + 1^2(1) + 2^2(4)}{10} = \frac{17}{5}.$$

Hence $\mathrm{Var}(X) = \frac{17}{5}$.

(c) $\mathrm{E}(Y) = 3$ and $\mathrm{Var}(Y) = 25(\frac{17}{5}) = 85$.

(d) $Y < 0$ means $X > \frac{3}{5}$. Therefore $\mathrm{P}(Y < 0) = \mathrm{P}(X = 1) + \mathrm{P}(X = 2) = \frac{1}{10} + \frac{4}{10} = \frac{1}{2}$.

43. Solution

(a) $\mathrm{E}(R) = 4(\frac{3}{10}) + 10(\frac{1}{5}) = \frac{16}{5}$. $\mathrm{E}(R^2) = 4^2(\frac{3}{10}) + 10^2(\frac{1}{5}) = \frac{124}{5}$, so $\mathrm{Var}(R) = \frac{124}{5} - (\frac{16}{5})^2 = \frac{364}{25}$.

(b) $G = R - 3$; $\mathrm{E}(G) = \frac{16}{5} - 3 = \frac{1}{5} > 0$. The original game is favourable to the player, not fair.

(c) Let R' be the revised payout and $G' = R' - 4$. Fairness gives $\frac{6}{5} + \frac{v}{5} = 4$, so $v = 14$.

(d) $\mathrm{E}((R')^2) = 4^2(\frac{3}{10}) + 14^2(\frac{1}{5}) = 44$, hence $\mathrm{Var}(G') = 44 - 4^2 = 28$. This is greater than $\mathrm{Var}(G) = \frac{364}{25}$. The changed prize increases the spread; changing the fee alone would not.

44. Solution

(a) $\mathrm{E}(R) = 50(\frac{1}{5}) + 80(\frac{1}{2}) + 140(\frac{3}{10}) = 92$. $\mathrm{E}(R^2) = 50^2(\frac{1}{5}) + 80^2(\frac{1}{2}) + 140^2(\frac{3}{10}) = 9580$. Therefore $\mathrm{Var}(R) = 9580 - 92^2 = 1116$.

(b) $\mathrm{E}(P) = 32$ euros and $\sigma_P = \sqrt{1116} = 6\sqrt{31}$ euros.

(c) $Q = \frac{9}{10}R - 60$, so $\mathrm{E}(Q) = \frac{9}{10}(92) - 60 = \frac{114}{5}$ euros and $\sigma_Q = \frac{9}{10}(6\sqrt{31}) = \frac{27\sqrt{31}}{5}$ euros.

(d) $P < 0$ requires $R = 50$. The possible discounted revenues are 45,72,126, so $Q < 0$ also requires the first outcome. Both loss probabilities are $\frac{1}{5}$.

Section 3: Paper 1-style questions (continued)

45. Solution

(a) $0 \le p \le \frac{1}{2}$.

(b) $\mathrm{E}(X) = 2(1-2p) + 4p = 2$. $\mathrm{E}(X^2) = 2^2(1-2p) + 4^2 p = 4 + 8p$, so $\mathrm{Var}(X) = 4 + 8p - 2^2 = 8p$.

(c) $\mathrm{E}(G) = 3(2) + 1 - c = 7 - c$, so $c = 7$ euros.

(d) $\mathrm{Var}(G) = 3^2(8p) = 72p$. Since standard deviation is non-negative, $\sqrt{72p} \le 3$ is equivalent to $72p \le 9$. Thus $0 \le p \le \frac{1}{8}$.

(e) With $c = 7$, $G = 3X - 6 > 0$ only when $X = 4$, so $\mathrm{P}(G > 0) = p$.

46. Solution

(a) $0 \le p \le \frac{1}{2}$.

(b) $\mathrm{E}(X) = \frac{1}{2} + 2(\frac{1}{2} - p) = \frac{3}{2} - 2p$ and $\mathrm{E}(X^2) = \frac{1}{2} + 4(\frac{1}{2} - p) = \frac{5}{2} - 4p$. Subtracting the square of the mean gives $\mathrm{Var}(X) = \frac{1}{4} + 2p - 4p^2$.

(c)

$$\mathrm{Var}(X) = \frac{1}{2} - 4\left(p - \frac{1}{4}\right)^2.$$

The squared term is non-negative, and $p = \frac{1}{4}$ is allowable. The maximum variance is $\frac{1}{2}$.

(d) $4(p - \frac{1}{4})^2 = \frac{1}{2} - \frac{7}{16} = \frac{1}{16}$, so $p = \frac{1}{4} \pm \frac{1}{8}$. Thus $p = \frac{1}{8}$ gives mean $\frac{5}{4}$, while $p = \frac{3}{8}$ gives mean $\frac{3}{4}$. Both distributions are valid.

47. Solution

(a) $\mathrm{E}(X) = 2(\frac{1}{3}) + 6(\frac{1}{6}) = \frac{5}{3}$; $\mathrm{E}(X^2) = 2^2(\frac{1}{3}) + 6^2(\frac{1}{6}) = \frac{22}{3}$, so $\mathrm{Var}(X) = \frac{22}{3} - \frac{25}{9} = \frac{41}{9}$. For Y, $\mathrm{E}(Y) = 3$ and $\mathrm{E}(Y^2) = 11$, so $\mathrm{Var}(Y) = 2$.

(b)

$$\mathrm{E}(U) = \frac{3(5/3) - 5}{\sqrt{41}} = 0, \quad \mathrm{Var}(U) = \frac{9}{41}\frac{41}{9} = 1.$$

$$\mathrm{E}(V) = \frac{3 - 3}{\sqrt{2}} = 0, \quad \mathrm{Var}(V) = \frac{1}{2}(2) = 1.$$

(c) $U > 0$ means $X > \frac{5}{3}$, so the probability is $\frac{1}{3} + \frac{1}{6} = \frac{1}{2}$. $V > 0$ means $Y > 3$, so the probability is $\frac{1}{4}$.

(d) No. U and V have the same mean and variance but different probabilities of being positive.

48. Solution

(a) $\mathrm{E}(R) = 2(\frac{1}{4}) + 5(\frac{1}{5}) + 10(\frac{1}{20}) = 2$; $\mathrm{E}(R^2) = 2^2(\frac{1}{4}) + 5^2(\frac{1}{5}) + 10^2(\frac{1}{20}) = 11$. Thus $\mathrm{Var}(R) = 11 - 4 = 7$. Since $\mathrm{E}(G) = 2 - 2 = 0$, the game is fair.

(b) $\mathrm{P}(R > 0) = \frac{1}{2}$. At payouts $2, 5, 10$, the conditional probabilities are $\frac{1}{2}, \frac{2}{5}, \frac{1}{10}$.

(c) The conditional payout mean is $2(\frac{1}{2}) + 5(\frac{2}{5}) + 10(\frac{1}{10}) = 4$; the expected square is $2^2(\frac{1}{2}) + 5^2(\frac{2}{5}) + 10^2(\frac{1}{10}) = 22$. The conditional mean gain is $4 - 2 = 2$ euros and its variance is $22 - 4^2 = 6$ euros squared.

(d) Fairness averages *all* plays, including the zero-payout plays that lose €2. Conditioning on a nonzero payout excludes those losses, so it describes a different group of outcomes.

Section 3: Paper 1-style questions (continued)

49. Solution

(a) $\mathrm{E}(X) = \frac{a}{2} + 6(\frac{1}{2} - p) = 3$ gives $\frac{a}{2} = 6p$, so $p = \frac{a}{12}$.

(b) $\mathrm{E}(X^2) = \frac{13}{2} + 3^2 = \frac{31}{2}$. Also $\mathrm{E}(X^2) = \frac{a^2}{2} + 36(\frac{1}{2} - \frac{a}{12}) = \frac{a^2}{2} + 18 - 3a$. Equating the expressions and multiplying by 2 gives $a^2 - 6a + 5 = 0$.

(c) $(a-1)(a-5) = 0$, giving $(a, p) = (1, \frac{1}{12})$ or $(5, \frac{5}{12})$. Both have $0 < a < 6$ and $0 < p < \frac{1}{2}$, so all three probabilities are positive.

(d) For $a = 1$, only $X = 6$ exceeds 3, giving probability $\frac{5}{12} < \frac{1}{2}$. For $a = 5$, the values 5 and 6 qualify, giving probability $\frac{1}{2} + \frac{1}{12} = \frac{7}{12} > \frac{1}{2}$. Thus $(a, p) = (5, \frac{5}{12})$.

50. Solution

(a) $\mathrm{E}(X) = r(\frac{1}{3}) + 3r(\frac{1}{6}) = \frac{5r}{6}$. Fairness requires $\mathrm{E}(G) = \frac{5r}{6} - 5 = 0$, so $r = 6$.

(b) The payouts are $0, 6, 18$, with mean 5.

$$\mathrm{E}(X^2) = 0^2\left(\frac{1}{2}\right) + 6^2\left(\frac{1}{3}\right) + 18^2\left(\frac{1}{6}\right) = 66.$$

Thus $\mathrm{Var}(G) = \mathrm{Var}(X) = 66 - 5^2 = 41$ euros squared and $\sigma_G = \sqrt{41}$ euros.

(c) Let $H = Y - 5 = aX + b - 5$ be the new net gain. Then $\sigma_H = a\sqrt{41}$ since $a > 0$. The target gives $a = \frac{1}{2}$. Fairness requires $\mathrm{E}(H) = \frac{1}{2}(5) + b - 5 = 0$, so $b = \frac{5}{2}$ euros.

(d) No. Changing only a fixed fee subtracts a constant from every payout and leaves variance and standard deviation unchanged.