Discrete random variables

IB Mathematics AA SL & HL ▪ Section 4.7

60 original questions: distributions, expected value and fair games

Learning goal	Read and construct discrete probability distributions, calculate means and analyse fair games.
Syllabus link	SL 4.7, shared by AA SL and AA HL. The HL challenges deepen the same content.
Structure	Q1–15 short response; Q16–30 extended response; Q31–50 exam-style; Q51–60 HL-exclusive challenge. Worked solutions begin on page 17.
Technology	Non-calculator practice throughout, including the original Paper 1-style questions. The IB formula booklet may be used.

Instructions. Work on separate paper and show your reasoning. Give exact answers unless stated otherwise; terminating decimal probabilities may be treated as exact. Define each random variable, including its units, when asked to do so. For games, distinguish prize money from net gain after the entry fee. Listed values may have probability zero unless stated otherwise.

1 Short response

Read carefully, select the correct values and show a short calculation where needed.

1. Define the random variable

A class has 28 students. Attendance is recorded at the beginning of one lesson.

(a) Define a discrete random variable X for this experiment.

(b) For your definition, list the possible values of X.

2. Values and probabilities

A player's net gain G, in euros, on one play has the distribution

g	-1.5	0	2
$\mathrm{P}(G=g)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{5}$

(a) Explain why G is discrete even though one value is not an integer.

(b) Find $\mathrm{P}(G=1)$ and $\mathrm{P}(G\geq 0)$.

3. Select the correct values

Let X be the number of messages received in one five-minute interval, with

x	0	1	2	3
$\mathrm{P}(X=x)$	0.10	0.25	0.40	0.25

(a) Find $\mathrm{P}(X>1)$.

(b) Find $\mathrm{P}(0<X\leq 2)$.

4. Check both conditions

Each row below is proposed as the complete list of probabilities for a discrete random variable. Explain why each row is invalid.

(a) $0.20,\ 0.35,\ 0.55$.

(b) $-0.10,\ 0.40,\ 0.70$.

5. A missing constant

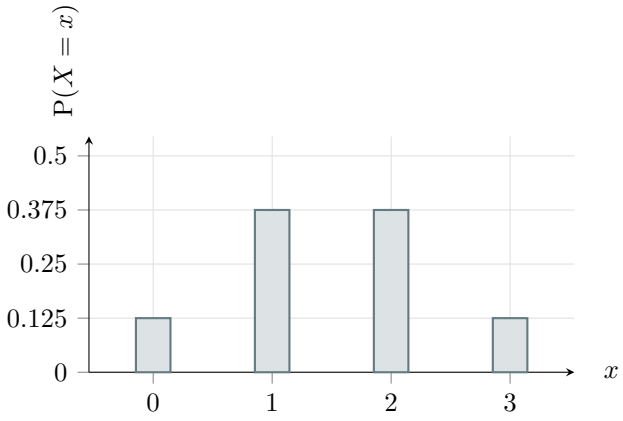
The number X of books borrowed on one visit has the distribution

x	0	1	3
$\mathrm{P}(X = x)$	$\frac{1}{4}$	k	$2k$

Find k and check that every probability is valid.

6. Read the bar heights

The chart gives the distribution of X, the number of damaged items in one small package.



(a) Write down $\mathrm{P}(X = 1)$.

(b) Find $\mathrm{P}(X \geq 2)$.

7. Use a probability formula

On one spin, the score X has probability function $\mathrm{P}(X = x) = k(5 - x)$ for $x \in \{1, 2, 3, 4\}$.

(a) Find k.

(b) Find $\mathrm{P}(X = 4)$.

8. A probability-weighted mean

The number X of customers arriving in one minute has the distribution

x	0	1	4
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

Calculate $\mathrm{E}(X)$.

9. Retain negative signs

The score X on one round of a game has the distribution

x	-3	0	2
$\mathrm{P}(X = x)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

Calculate $\mathrm{E}(X)$.

10. An expected total

The mean number of bicycles repaired per day is 2.4. Assume the same model applies on each of 20 working days.

(a) Find the expected total number repaired.

(b) Explain why this is not a prediction of the exact total.

11. Must the mean be possible?

Let X be the number of goals scored by a team in one match. A model gives $\mathrm{E}(X) = 1.8$. A student says, "This is impossible because a team cannot score 1.8 goals. Explain the error.

12. Prize or net gain?

A ticket costs €1. The prize R, in euros, is 6 with probability $\frac{1}{3}$ and 0 otherwise. Let G be the player's net gain on one ticket.

(a) Write down the two possible values of G.

(b) Calculate $\mathrm{E}(G)$ and decide whether the game is fair.

13. Fairness is about the mean

The net gain G, in euros, on one play is 8 with probability $\frac{1}{5}$ and -2 with probability $\frac{4}{5}$.

(a) Show that the game is fair.

(b) Find the probability of making a profit.

14. Allowed parameter values

A model for a count X is

x	0	1	2
$\mathrm{P}(X = x)$	a	$2a$	$1 - 3a$

Find all values of a for which this is a valid distribution.

15. An infinite list can be discrete

A fair die is rolled repeatedly until the first six appears. Let N be the number of rolls used, including the final roll.

(a) List the possible values of N and explain why it is discrete.

(b) Write down $\mathrm{P}(N = 0)$.

2 Extended response

Connect representations and explain your results in context.

16. Repeated labels on a spinner

A spinner has four equally likely sectors labelled $0, 0, 2, 5$. Let X be the score on one spin.

(a) Construct the probability distribution of X.

(b) Find $\mathrm{P}(X \geq 2)$.

(c) Calculate and interpret $\mathrm{E}(X)$.

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17. Equally likely cards

Five cards are labelled $-2, -2, 1, 3, 3$. One card is chosen at random, each card being equally likely. Let X be its number.

(a) Construct the probability distribution of X.

(b) Find $\mathrm{P}(X \leq 1)$ and $\mathrm{E}(X)$.

(c) Explain why giving each distinct value probability $\frac{1}{3}$ would be incorrect.

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18. Construct a distribution from outcomes

Two spinners are spun independently. Each has two equally likely sectors, labelled 1 and 3. Let S be the sum of the two scores.

	Second spinner	
First spinner	1	3
1		
3		

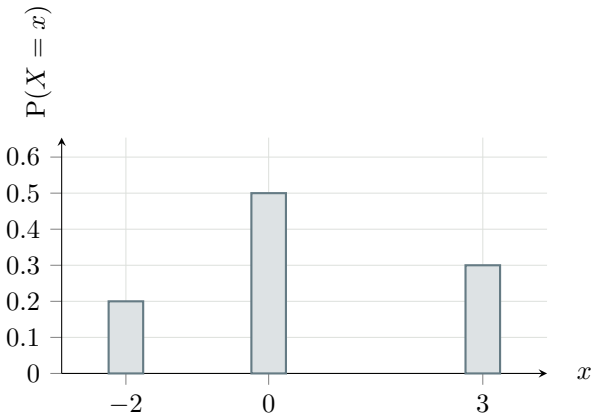
(a) Complete the outcome table, then construct the distribution of S.

(b) Find $\mathrm{P}(S \geq 4)$.

(c) Calculate $\mathrm{E}(S)$.

19. A chart with negative values

Let X be the score on one attempt. Its distribution is shown below.



(a) Construct a probability table.

(b) Find $\mathrm{P}(-2 < X \leq 3)$.

(c) Calculate $\mathrm{E}(X)$.

20. A mean determines missing probabilities

The number X of parcels collected by a customer is modelled by

x	0	2	5
$\mathrm{P}(X = x)$	a	b	$\frac{1}{5}$

Given that $\mathrm{E}(X) = 2$,

(a) find a and b, and check the completed distribution;

(b) find the probability that the customer collects at least one parcel.

21. From a formula to a mean

The number X of files sent in one transaction has distribution $\mathrm{P}(X = x) = k(x + 2)$ for $x \in \{1, 2, 3\}$.

(a) Find k and write the distribution as a table.

(b) Find $\mathrm{P}(X > 1)$.

(c) Calculate $\mathrm{E}(X)$.

22. An unknown value

The score X on one round has distribution

x	0	2	v
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

where $v > 2$ and $\mathrm{E}(X) = \frac{5}{2}$.

(a) Find v.

(b) Find $\mathrm{P}(X > \mathrm{E}(X))$.

23. Equal means, different chances

Two machines produce a score on one use. Machine A gives $0, 2, 4$ with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$. Machine B gives 0 or 4, each with probability $\frac{1}{2}$.

(a) Calculate the expected score for each machine.

(b) For each machine, find the probability of a score of at least 3.

(c) Explain why the mean alone does not answer the question in part (b).

24. Hourly counts and a total

Let X be the number of donations received in one hour. A model gives

x	0	1	2	3
$\mathrm{P}(X = x)$	0.20	0.30	0.40	0.10

(a) Calculate and interpret $\mathrm{E}(X)$.

(b) Assuming this distribution applies to each hour, find the expected total over 50 hours.

(c) Find the probability of receiving no donations in one hour.

25. Build the net-gain distribution

A game costs €3 per play. The prize R, in euros, has distribution

r	0	4	10
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$

Let G be the player's net gain on one play.

(a) Construct the distribution of G.

(b) Calculate $\mathrm{E}(G)$ and the probability of a profit.

(c) Find the entry fee that would make the game fair.

26. The organiser's target

The prize R on one play is €0, €5 or €20, with probabilities 0.70, 0.20 and 0.10. The entry fee is c euros. The organiser pays only the prize and has no other costs.

(a) Find the mean prize.

(b) Find c if the organiser wants an expected profit of €1 per play.

(c) At this fee, find the player's probability of a profit.

27. Diagnose an incorrect mean

A model for the number X of visitors in one interval is

x	1	3	6
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

A student calculates $\mathrm{E}(X) = \frac{1+3+6}{3} = \frac{10}{3}$.

(a) Explain the mistake.

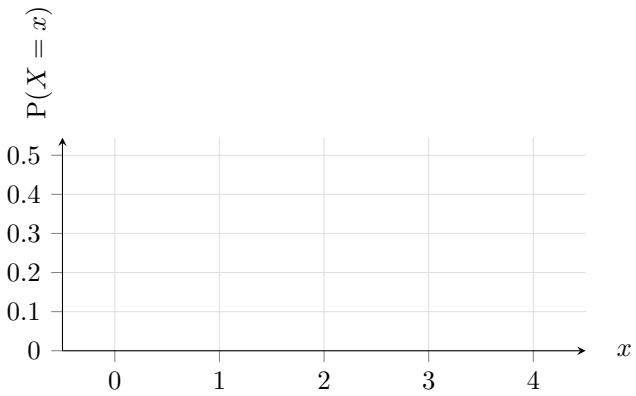
(b) Calculate the correct mean and interpret it.

28. Draw a probability bar chart

The number X of items sold in one interval has distribution

x	0	1	2	4
$\mathrm{P}(X = x)$	$\frac{1}{10}$	$\frac{3}{10}$	$\frac{2}{5}$	$\frac{1}{5}$

(a) Draw a probability bar chart on the supplied axes. Keep the numerical spacing of the values correct.



(b) Find $\mathrm{P}(X = 3)$ and $\mathrm{E}(X)$.

29. A parameter and the mean

The score X on one attempt has distribution

x	1	2	5
$\mathrm{P}(X = x)$	t	$\frac{1}{2} - t$	$\frac{1}{2}$

(a) Find the allowed values of t.

(b) Express $\mathrm{E}(X)$ in terms of t.

(c) Determine whether $\mathrm{E}(X) = \frac{11}{4}$ is possible. Justify your answer.

30. Two fair games

Both games cost €2 per play. Game A pays a prize of €12 with probability $\frac{1}{6}$ and nothing otherwise. Game B pays €4 with probability $\frac{1}{2}$ and nothing otherwise.

(a) Define a net-gain random variable for each game and show that both games are fair.

(b) Which game gives a greater probability of a profit?

(c) Does being fair guarantee that a player breaks even after ten plays? Explain.

3 IB exam-style / Paper 1

These are original practice questions. Mark allocations indicate the expected depth of working.

31. A symmetric distribution [Maximum mark: 7]

Let X be the number of requests received in one minute. Its distribution is

x	0	1	2	3	4
$\mathrm{P}(X = x)$	k	$2k$	$4k$	$2k$	k

(a) Find k. [2 marks]

(b) Find $\mathrm{P}(X > 2)$. [2 marks]

(c) Calculate $\mathrm{E}(X)$. [2 marks]

(d) Interpret your answer to part (c). [1 mark]

32. Use two equations [Maximum mark: 6]

The number X of items collected on one visit is modelled by

x	0	2	3
$\mathrm{P}(X = x)$	a	$\frac{1}{4}$	b

It is given that $\mathrm{E}(X) = \frac{7}{4}$.

(a) Write down two equations in a and b. [2 marks]

(b) Hence find a and b. [3 marks]

(c) Find $\mathrm{P}(X \geq 2)$. [1 mark]

33. A decreasing probability function [Maximum mark: 7]

The score X on one attempt has probability function $\mathrm{P}(X = x) = k(6 - x)$ for $x \in \{1, 2, 3, 4\}$.

(a) Find k. [2 marks]

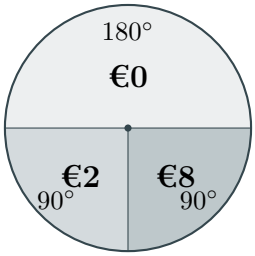
(b) Find $\mathrm{P}(2 \leq X < 4)$. [2 marks]

(c) Calculate $\mathrm{E}(X)$. [3 marks]

34. A spinner game

[Maximum mark: 8]

The spinner below is equally likely to stop in any direction. The three sectors show the prize amounts in euros; their angles are labelled. It costs €3 to spin once.



Let R be the prize, in euros, and G the player's net gain, in euros, on one spin.

(a) Construct the distribution of R. [2 marks]

(b) Construct the distribution of G. [2 marks]

(c) Calculate $\mathrm{E}(G)$ and state whether the game is fair. [3 marks]

(d) Find $\mathrm{P}(G > 0)$. [1 mark]

35. The larger of two numbers

[Maximum mark: 7]

A card numbered 0, 1 or 2 is chosen at random from three cards, replaced, and a second card is chosen independently. Let X be the larger of the two numbers (or their common number if they are equal).

(a) Using a table of the nine equally likely ordered outcomes, find the distribution of X. [4 marks]

(b) Find $\mathrm{P}(X = 1)$. [1 mark]

(c) Calculate $\mathrm{E}(X)$. [2 marks]

36. A raffle

[Maximum mark: 7]

A raffle sells all 100 tickets for €1 each. Exactly one ticket wins €40, exactly four other tickets win €10 each, and all remaining tickets win nothing. Each ticket is equally likely to be selected by a buyer. Let R be the prize on one ticket, in euros.

(a) Construct the distribution of R. [2 marks]

(b) Calculate the player's expected net gain on one ticket. [2 marks]

(c) Find the organiser's total profit after all prizes are paid, ignoring other costs. [2 marks]

(d) Find the probability that one ticket makes a profit. [1 mark]

37. A compensation model

[Maximum mark: 7]

For one delivery, a company pays compensation R of €0, €5 or €25 with probabilities 0.80, 0.15 and 0.05. It charges a separate €3 protection fee for each delivery. Ignore all other income and costs.

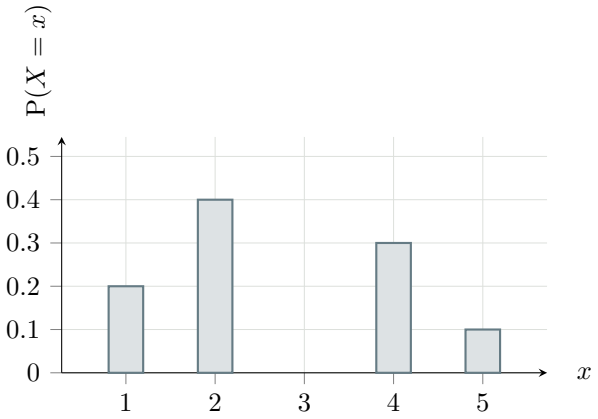
(a) Calculate the expected compensation per delivery. [3 marks]

(b) Find the company's expected net income from the protection fee and compensation on one delivery. [2 marks]

(c) Assuming the model remains unchanged, find the expected total compensation for 1000 deliveries. [2 marks]

38. A probability chart [Maximum mark: 6]

The chart shows the number X of orders received in one interval.



(a) Write down $\mathrm{P}(X = 3)$. [1 mark]

(b) Find $\mathrm{P}(1 < X \leq 4)$. [2 marks]

(c) Calculate $\mathrm{E}(X)$. [2 marks]

(d) Find the expected total over 50 intervals governed by this model. [1 mark]

39. A gain with an unknown probability [Maximum mark: 6]

The net gain G, in euros, on one play has distribution

g	-1	1	4
$\mathrm{P}(G = g)$	$\frac{1}{2}$	p	$\frac{1}{2} - p$

It is given that $\mathrm{E}(G) = 1$.

(a) Find p and verify that the probabilities are valid. [3 marks]

(b) Find $\mathrm{P}(G > 0)$. [2 marks]

(c) State whether the game is fair, giving a reason. [1 mark]

40. Set a fair prize [Maximum mark: 8]

A game costs €3 per play. Its prize R is €0, €6 or w euros, with probabilities $\frac{7}{10}$, $\frac{1}{5}$ and $\frac{1}{10}$ respectively, where $w > 6$. Let G be the player's net gain.

(a) Write down the distribution of G. [2 marks]

(b) Find w so that the game is fair. [3 marks]

(c) For this value of w, find the probability of a profit. [1 mark]

(d) For 75 plays under the same model, find the expected total prize money. [2 marks]

41. Compare a value with the mean

[Maximum mark: 5]

The score X on one attempt has distribution

x	-2	1	m
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

where $m > 1$ and $\mathrm{E}(X) = 1$.

(a) Find m. [2 marks]

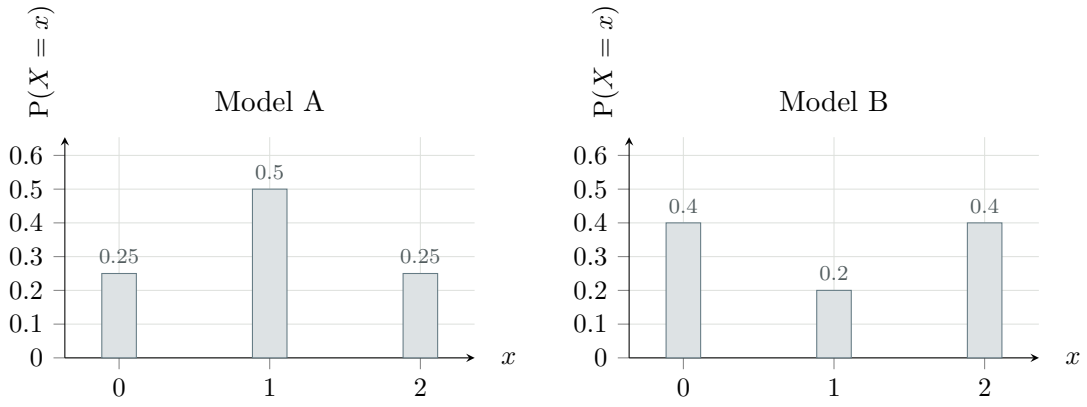
(b) Find $\mathrm{P}(X \geq \mathrm{E}(X))$. [2 marks]

(c) Find $\mathrm{P}(X = \mathrm{E}(X))$. [1 mark]

42. Compare two graphical models

[Maximum mark: 6]

The charts give the number of seats booked in one transaction under two models, A and B.



(a) Calculate the expected number of seats under each model. [2 marks]

(b) Find the probability of at least one seat being booked under each model. [2 marks]

(c) A student claims that equal expected values imply identical distributions. Use these models to explain why this is false. [2 marks]

43. A die-based game

[Maximum mark: 8]

A fair six-sided die is rolled once. The prize is €0 for a score of 1 or 2, €2 for a score of 3, 4 or 5, and €6 for a score of 6. The entry fee is €2. Let R be the prize and G the player's net gain, both in euros.

(a) Construct the distribution of R. [2 marks]

(b) Calculate $\mathrm{E}(R)$. [2 marks]

(c) Construct the distribution of G and decide whether the game is fair. [3 marks]

(d) Find $\mathrm{P}(G > 0)$. [1 mark]

44. Two parameters in a table [Maximum mark: 6]

The number X of documents printed in one job has distribution

x	0	1	2	3
$\mathrm{P}(X = x)$	a	b	$2b$	a

It is given that $\mathrm{E}(X) = \frac{8}{5}$.

(a) Write down two equations in a and b. [2 marks]

(b) Find a and b. [3 marks]

(c) Find $\mathrm{P}(X \leq 1)$. [1 mark]

45. A fixed or a random bonus [Maximum mark: 7]

For each completed delivery, a rider chooses either a guaranteed €2 bonus or a random bonus R. The random bonus is €0, €2 or €6 with probabilities $\frac{1}{2}$, $\frac{3}{10}$ and $\frac{1}{5}$.

(a) Calculate $\mathrm{E}(R)$. [3 marks]

(b) Which option has the higher expected bonus? [1 mark]

(c) Find the probability that the random bonus is strictly greater than the fixed bonus. [1 mark]

(d) If the same choice is used for 100 deliveries, find the expected total bonus for each option. [2 marks]

46. Use the outcomes of two tosses [Maximum mark: 8]

A coin has probability p of landing heads, where $0 < p < 1$. It is tossed twice independently. Let X be the number of heads. It is given that $\mathrm{P}(X = 0) = \frac{4}{9}$. Use the individual outcomes; no named distribution formula is needed.

(a) Find p. [3 marks]

(b) Construct the distribution of X. [3 marks]

(c) Calculate $\mathrm{E}(X)$. [2 marks]

47. A family of games [Maximum mark: 7]

The player's net gain G, in euros, has distribution

g	-1	0	2
$\mathrm{P}(G = g)$	a	$\frac{1}{2}$	$\frac{1}{2} - a$

(a) Find the allowed values of a. [2 marks]

(b) Express $\mathrm{E}(G)$ in terms of a. [2 marks]

(c) Find a for a fair game. [2 marks]

(d) For the fair game, find the probability of a profit. [1 mark]

48. Linked prize amounts [Maximum mark: 7]

A prize R, in euros, has distribution

r	0	a	b
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

where $a > 0$, $b = 2a + 3$ and $\mathrm{E}(R) = 3$.

(a) Find a and b. [3 marks]

(b) State the fair entry fee. [1 mark]

(c) If the entry fee is €4, calculate the expected net gain. [2 marks]

(d) At this fee, find the probability of a profit. [1 mark]

49. Build a model from frequencies [Maximum mark: 7]

Over 100 days, a repair shop records the number of returns each day.

Returns in one day	0	1	2	3
Number of days	20	25	35	20

Let X be the number of returns on a future day. Use the relative frequencies as a probability model.

(a) Write down the proposed distribution of X. [2 marks]

(b) Calculate $\mathrm{E}(X)$. [2 marks]

(c) Find the expected total over 200 future days. [2 marks]

(d) State one assumption needed for this forecast. [1 mark]

50. A claim about fairness [Maximum mark: 9]

A game has prize R of €0, €5 or €15 with probabilities $\frac{1}{2}$, $\frac{2}{5}$ and $\frac{1}{10}$.

(a) Calculate $\mathrm{E}(R)$. [2 marks]

(b) Find the fair entry fee. [2 marks]

(c) When the fee is €4, find the player's expected net gain and probability of a profit. [3 marks]

(d) A student says, "A game is fair whenever the probability of a profit is one half. Evaluate this claim using your results. [2 marks]

4 AA HL exclusive challenge

The shared SL/HL worksheet ends at Question 50. This section is intended for HL students and uses more demanding reasoning within SL 4.7.

51. A family of distributions [Maximum mark: 9]

The score X on one attempt has distribution

x	0	1	3	5
$\mathrm{P}(X=x)$	a	b	b	$2a$

(a) Express b in terms of a, and find the full allowed interval for a. [3 marks]

(b) Show that $\mathrm{E}(X) = 2 + 4a$. Hence find all attainable mean values. [3 marks]

(c) Find the distribution for which $\mathrm{E}(X) = 3$. [2 marks]

(d) For this distribution, find $\mathrm{P}(X \geq 3)$. [1 mark]

52. Are the conditions compatible? [Maximum mark: 9]

The only possible scores of X are $-2, 1, 4$. It is known that $\mathrm{P}(X > 0) = 3\,\mathrm{P}(X < 0)$.

(a) Given additionally that $\mathrm{E}(X) = 1$, determine the distribution. [5 marks]

(b) Could the probability condition instead hold together with $\mathrm{E}(X) = 3$? Justify your answer. [4 marks]

53. A parameter in a probability function [Maximum mark: 9]

The score X has distribution $\mathrm{P}(X = x) = k(a + x)$ for $x \in \{0, 1, 2\}$, where $a \geq 0$ and $k > 0$.

(a) Find k in terms of a and show that $\mathrm{E}(X) = 1 + \frac{2}{3(a+1)}$. [4 marks]

(b) Find all attainable mean values, carefully distinguishing a bound from an attained endpoint. [3 marks]

(c) Find a and the distribution when $\mathrm{E}(X) = \frac{4}{3}$. [2 marks]

54. A game based on the gap [Maximum mark: 9]

Two independent spinners each have three equally likely sectors labelled $0, 1, 2$. Let X be the absolute difference between the two scores. A game costs €1. It pays €0 if $X = 0$, €1 if $X = 1$, and w euros if $X = 2$, where $w > 1$.

(a) Construct the distribution of X by listing the nine equally likely ordered outcomes. [4 marks]

(b) Find w for a fair game, clearly defining the net-gain variable. [3 marks]

(c) For this fair game, find the probabilities of a profit and of breaking even. [2 marks]

55. Bounds on a probability [Maximum mark: 8]

A score X takes values $0, 2, 6$ with probabilities p, q, r, respectively. It is given that $\mathrm{E}(X) = 3$.

(a) Express p and q in terms of r, and determine the full allowed interval for r. [4 marks]

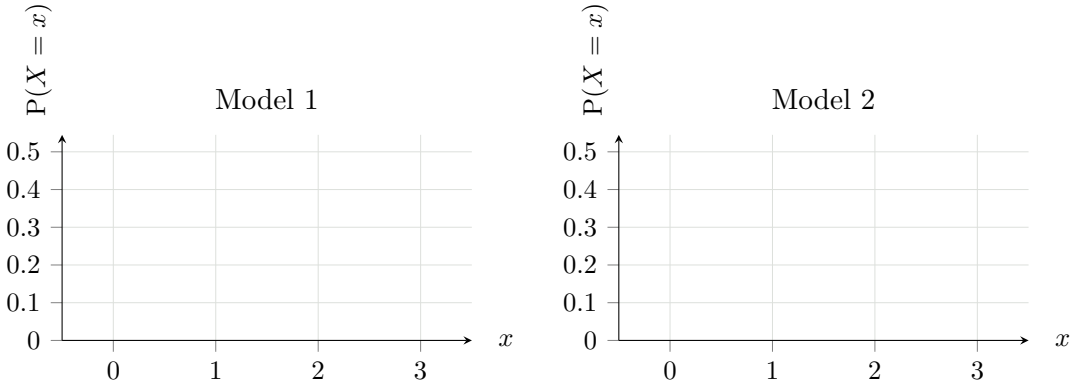
(b) Find the smallest and largest possible values of $\mathrm{P}(X \geq 2)$. Give a distribution attaining each value. [4 marks]

56. Design more than one model [Maximum mark: 10]

A distribution is to be defined on the values $0, 1, 2, 3$, with mean $\frac{3}{2}$ and $\mathrm{P}(X \geq 2) = \frac{1}{2}$. Let $p_i = \mathrm{P}(X = i)$.

(a) Show that every such distribution has $p_0 = p_3 = a$ and $p_1 = p_2 = \frac{1}{2} - a$. State the allowed interval for a. [4 marks]

(b) Choose two different values of a that make all four probabilities positive. Give both distributions and draw their probability bar charts on the axes below.



[4 marks]

(c) Explain what your examples show about specifying a distribution using a mean and one event probability. [2 marks]

57. A refund changes the calculation [Maximum mark: 9]

A game costs c euros, where $0 < c < 8$. After paying, a player receives €8 with probability $\frac{1}{4}$, receives an exact refund of the fee c with probability $\frac{1}{4}$, and receives nothing with probability $\frac{1}{2}$. These are mutually exclusive outcomes. Let G be the player's net gain on one play.

(a) Construct the distribution of G in terms of c. [2 marks]

(b) Find the fair fee. [3 marks]

(c) For the fair fee, find the probabilities of a profit and of breaking even. [2 marks]

(d) Find the fee giving the organiser an expected profit of €1 per play, ignoring other costs. [2 marks]

58. A quadratic fairness condition [Maximum mark: 9]

A coin with head probability p, where $0 < p < 1$, is tossed twice independently. A player pays €4. The prize is €0 for no heads, €3 for exactly one head and €12 for two heads.

(a) Construct the prize distribution in terms of p, using the individual outcomes. [3 marks]

(b) Find the exact value of p that makes the game fair. [4 marks]

(c) For that value of p, find the exact probability of a profit. [2 marks]

59. Unknown gains and losses [Maximum mark: 8]

The net gain G, in euros, on one play takes values $-a, 0, b$ with respective probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, where $a, b > 0$. It is given that $ab = 12$ and $\mathrm{E}(G) = 1$.

(a) Show that $b - a = 4$, then find a and b. [5 marks]

(b) Find $\mathrm{P}(G > \mathrm{E}(G))$. [1 mark]

(c) Keeping the probabilities and positive gain unchanged, replace the losing outcome by $-L$ euros. Find L for a fair game. [2 marks]

60. Fair games with different winning chances [Maximum mark: 10]

The prize R, in euros, takes values $0, 5, 20$ and has mean 4. Write their respective probabilities as p, q, r. The entry fee is €4.

(a) Express p and q in terms of r, and determine all allowed values of r. [4 marks]

(b) Find the smallest and largest possible probabilities of a profit. [2 marks]

(c) Give a distribution attaining each bound. [2 marks]

(d) Explain why every game in this family is fair despite these different winning probabilities. [2 marks]

Answer key and worked solutions

Try the questions before reading the solutions. The question numbers and sections below match the worksheet.

Section 1: Short response — Questions 1–15

1. Solution

(a) For example, let X be the number of students absent at the beginning of this lesson.

(b) $X \in \{0, 1, 2, \ldots, 28\}$. An equally clear definition using the number present is also valid, with the same set of possible values.

2. Solution

(a) The possible values form a finite list. Discrete values need not be whole numbers.

(b) 1 is not a possible gain, so $\mathrm{P}(G = 1) = 0$. Also, $\mathrm{P}(G \geq 0) = \frac{3}{10} + \frac{1}{5} = \frac{1}{2}$.

3. Solution

(a) Include $2, 3$: $\mathrm{P}(X > 1) = 0.40 + 0.25 = 0.65$.

(b) Include $1, 2$: $\mathrm{P}(0 < X \leq 2) = 0.25 + 0.40 = 0.65$. The events differ even though their probabilities are equal.

4. Solution

(a) The total is $1.10 \neq 1$, although every entry is between 0 and 1.

(b) The entries sum to 1, but -0.10 is negative. Every probability must lie in $[0, 1]$, and all probabilities must sum to 1.

5. Solution

$\frac{1}{4} + k + 2k = 1$, so $3k = \frac{3}{4}$ and $\boxed{k = \frac{1}{4}}$. The probabilities are $\frac{1}{4}, \frac{1}{4}, \frac{1}{2}$: they lie in $[0, 1]$ and sum to 1.

6. Solution

(a) $\mathrm{P}(X = 1) = 0.375 = \frac{3}{8}$.

(b) $\mathrm{P}(X \geq 2) = 0.375 + 0.125 = 0.5 = \frac{1}{2}$.

7. Solution

(a) The probabilities are $4k, 3k, 2k, k$. Thus $10k = 1$ and $k = \frac{1}{10}$.

(b) $\mathrm{P}(X = 4) = k(5 - 4) = \frac{1}{10}$.

8. Solution

$\mathrm{E}(X) = 0(\frac{1}{2}) + 1(\frac{1}{4}) + 4(\frac{1}{4}) = \boxed{\frac{5}{4}}$ customers per minute.

9. Solution

$\mathrm{E}(X) = (-3)(\frac{1}{5}) + 0(\frac{1}{2}) + 2(\frac{3}{10}) = -\frac{3}{5} + \frac{3}{5} = \boxed{0}$.

10. Solution

(a) The expected total is $20(2.4) = \boxed{48}$ bicycles.

(b) An expected total is an average across repeated 20-day periods under the model. Individual totals can be different.

Section 1: Short response — Questions 1–15 (continued)

11. Solution

The mean is a long-run average, not a required outcome of one match. Whole-number goal counts can have a mean of 1.8.

12. Solution

(a) $G = R - 1$, so the net gains are 5 and -1 euros.

(b) $\mathrm{E}(G) = 5(\frac{1}{3}) + (-1)(\frac{2}{3}) = \boxed{€1}$. It is not fair: the player has a positive expected net gain.

13. Solution

(a) $\mathrm{E}(G) = 8(\frac{1}{5}) - 2(\frac{4}{5}) = 0$, so the game is fair.

(b) A profit means $G > 0$, so its probability is $\frac{1}{5}$, not $\frac{1}{2}$.

14. Solution

The entries always sum to 1. Non-negativity requires $a \geq 0$ and $1 - 3a \geq 0$, so $\boxed{0 \leq a \leq \frac{1}{3}}$. On this interval all three probabilities are at most 1.

15. Solution

(a) $N \in \{1, 2, 3, \ldots\}$. Its values can be listed, even though the list has no final value.

(b) $\mathrm{P}(N = 0) = 0$: at least one roll is required.

Section 2: Extended response — Questions 16–30

16. Solution

(a)

x	0	2	5
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(b) $\mathrm{P}(X \geq 2) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

(c) $\mathrm{E}(X) = 0(\frac{1}{2}) + 2(\frac{1}{4}) + 5(\frac{1}{4}) = \frac{7}{4}$. The long-run mean score is 1.75 per spin.

17. Solution

(a)

x	-2	1	3
$\mathrm{P}(X = x)$	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{2}{5}$

(b) $\mathrm{P}(X \leq 1) = \frac{2}{5} + \frac{1}{5} = \frac{3}{5}$. Also $\mathrm{E}(X) = (-2)(\frac{2}{5}) + 1(\frac{1}{5}) + 3(\frac{2}{5}) = \frac{3}{5}$.

(c) The cards are equally likely, but the values are not: two cards show -2, one shows 1 and two show 3.

18. Solution

(a) The four equally likely sums are $2, 4, 4, 6$.

s	2	4	6
$\mathrm{P}(S = s)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(b) $\mathrm{P}(S \geq 4) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$.

(c) $\mathrm{E}(S) = 2(\frac{1}{4}) + 4(\frac{1}{2}) + 6(\frac{1}{4}) = 4$.

Section 2: Extended response — Questions 16–30 (continued)

19. Solution

(a)

x	-2	0	3
$\mathrm{P}(X=x)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

(b) Include $0, 3$ but not -2: $\mathrm{P}(-2 < X \leq 3) = \frac{1}{2} + \frac{3}{10} = \frac{4}{5}$.

(c) $\mathrm{E}(X) = (-2)(\frac{1}{5}) + 0(\frac{1}{2}) + 3(\frac{3}{10}) = \frac{1}{2}$.

20. Solution

(a) $a + b = \frac{4}{5}$ and $2b + 5(\frac{1}{5}) = 2$. Hence $b = \frac{1}{2}$ and $a = \frac{3}{10}$. All entries lie in $[0, 1]$ and their sum is 1.

(b) $\mathrm{P}(X \geq 1) = b + \frac{1}{5} = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.

21. Solution

(a) $3k + 4k + 5k = 1$, so $k = \frac{1}{12}$.

x	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{5}{12}$

(b) $\mathrm{P}(X > 1) = \frac{1}{3} + \frac{5}{12} = \frac{3}{4}$.

(c) $\mathrm{E}(X) = 1(\frac{1}{4}) + 2(\frac{1}{3}) + 3(\frac{5}{12}) = \frac{3+8+15}{12} = \frac{13}{6}$.

22. Solution

(a) $0(\frac{1}{4}) + 2(\frac{1}{2}) + v(\frac{1}{4}) = \frac{5}{2}$, so $1 + \frac{v}{4} = \frac{5}{2}$ and $v = 6$.

(b) Only 6 exceeds $\frac{5}{2}$, so the probability is $\frac{1}{4}$.

23. Solution

(a) A: $0(\frac{1}{4}) + 2(\frac{1}{2}) + 4(\frac{1}{4}) = 2$. B: $0(\frac{1}{2}) + 4(\frac{1}{2}) = 2$.

(b) A: $\frac{1}{4}$. B: $\frac{1}{2}$.

(c) The same probability-weighted mean can arise from different distributions, with different probabilities for a specified event.

24. Solution

(a) $\mathrm{E}(X) = 0(0.20) + 1(0.30) + 2(0.40) + 3(0.10) = 1.4$. The mean over many hours is 1.4 donations per hour.

(b) $50(1.4) = 70$ donations.

(c) $\mathrm{P}(X = 0) = 0.20 = \frac{1}{5}$.

25. Solution

(a) $G = R - 3$.

g	-3	1	7
$\mathrm{P}(G=g)$	$\frac{1}{2}$	$\frac{2}{5}$	$\frac{1}{10}$

(b) $\mathrm{E}(G) = -3(\frac{1}{2}) + 1(\frac{2}{5}) + 7(\frac{1}{10}) = -\frac{2}{5}$ euros. Also $\mathrm{P}(G > 0) = \frac{2}{5} + \frac{1}{10} = \frac{1}{2}$.

(c) The mean prize is $4(\frac{2}{5}) + 10(\frac{1}{10}) = \frac{13}{5}$ euros. A fee of €2.60 gives expected net gain 0.

Section 2: Extended response — Questions 16–30 (continued)

26. Solution

(a) $\mathrm{E}(R) = 0(0.70) + 5(0.20) + 20(0.10) = €3$.

(b) The organiser's gains are $c, c-5, c-20$. Their mean is $0.7c + 0.2(c-5) + 0.1(c-20) = c - 3$. Thus $c - 3 = 1$, so $c = 4$.

(c) Both prizes €5 and €20 exceed €4. The probability is $0.20 + 0.10 = 0.30$.

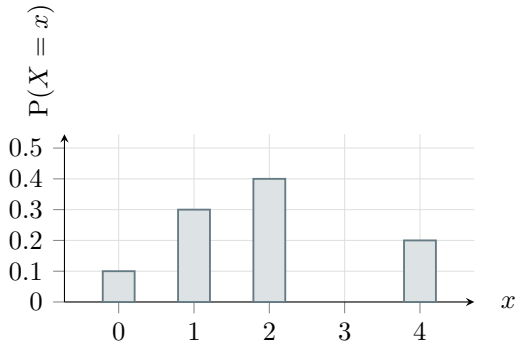
27. Solution

(a) This unweighted average treats the three values as equally likely. The actual probabilities are unequal.

(b) $\mathrm{E}(X) = 1(\frac{1}{2}) + 3(\frac{1}{3}) + 6(\frac{1}{6}) = \frac{5}{2}$. Over many intervals the mean is 2.5 visitors per interval.

28. Solution

(a) Bars have heights $0.1, 0.3, 0.4, 0.2$ at $0, 1, 2, 4$ respectively; there is no positive bar at 3.



(b) $\mathrm{P}(X = 3) = 0$. Also $\mathrm{E}(X) = 1(\frac{3}{10}) + 2(\frac{2}{5}) + 4(\frac{1}{5}) = \frac{19}{10}$.

29. Solution

(a) Non-negative probabilities require $0 \le t \le \frac{1}{2}$; the sum is already 1.

(b) $\mathrm{E}(X) = t + 2(\frac{1}{2} - t) + 5(\frac{1}{2}) = \frac{7}{2} - t$.

(c) $\frac{7}{2} - t = \frac{11}{4}$ gives $t = \frac{3}{4}$, outside the allowed interval. Therefore **no valid distribution in this family has that mean**.

30. Solution

(a) Let G_A, G_B be the respective net gains, in euros, on one play. $\mathrm{E}(G_A) = 10(\frac{1}{6}) - 2(\frac{5}{6}) = 0$ and $\mathrm{E}(G_B) = 2(\frac{1}{2}) - 2(\frac{1}{2}) = 0$.

(b) Game B: probability $\frac{1}{2}$, compared with $\frac{1}{6}$ for A.

(c) No. Fairness concerns expected net gain, not a guaranteed outcome over a finite number of plays.

Section 3: IB exam-style / Paper 1 — Questions 31–50

31. Solution

(a) $10k = 1$, so $k = \frac{1}{10}$.

(b) $\mathrm{P}(X > 2) = 2k + k = \frac{3}{10}$.

(c) $\mathrm{E}(X) = 0(\frac{1}{10}) + 1(\frac{2}{10}) + 2(\frac{4}{10}) + 3(\frac{2}{10}) + 4(\frac{1}{10}) = 2$.

(d) The mean number of requests over many one-minute intervals is 2 per interval.

32. Solution

(a) $a + b = \frac{3}{4}$ and $2(\frac{1}{4}) + 3b = \frac{7}{4}$.

(b) $3b = \frac{5}{4}$, so $b = \frac{5}{12}$ and $a = \frac{3}{4} - \frac{5}{12} = \frac{1}{3}$. All probabilities are valid.

(c) $\mathrm{P}(X \geq 2) = \frac{1}{4} + \frac{5}{12} = \frac{2}{3}$.

33. Solution

(a) $(5 + 4 + 3 + 2)k = 1$, so $k = \frac{1}{14}$.

(b) Include $2, 3$: $\mathrm{P}(2 \leq X < 4) = \frac{4}{14} + \frac{3}{14} = \frac{1}{2}$.

(c) $\mathrm{E}(X) = \frac{1(5)+2(4)+3(3)+4(2)}{14} = \frac{30}{14} = \frac{15}{7}$.

34. Solution

(a) Use each sector angle divided by $360°$.

r	0	2	8
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(b) $G = R - 3$.

g	-3	-1	5
$\mathrm{P}(G = g)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(c) $\mathrm{E}(G) = -3(\frac{1}{2}) - 1(\frac{1}{4}) + 5(\frac{1}{4}) = -\frac{1}{2}$ euros. The game is not fair; the expected loss is €0.50 per spin.

(d) Only the €8 prize makes a profit, so $\mathrm{P}(G > 0) = \frac{1}{4}$.

Section 3: IB exam-style / Paper 1 — Questions 31–50 (continued)

35. Solution

(a) The larger values in the 3×3 outcome table are

	Second card		
First card	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

There is one 0, three 1s and five 2s.

x	0	1	2
$\mathrm{P}(X = x)$	$\frac{1}{9}$	$\frac{3}{9}$	$\frac{5}{9}$

(b) $\mathrm{P}(X = 1) = \frac{3}{9} = \frac{1}{3}$.

(c) $\mathrm{E}(X) = 0(\frac{1}{9}) + 1(\frac{3}{9}) + 2(\frac{5}{9}) = \frac{13}{9}$.

36. Solution

(a)

r	0	10	40
$\mathrm{P}(R = r)$	$\frac{95}{100}$	$\frac{4}{100}$	$\frac{1}{100}$

(b) Net gains are $-1, 9, 39$. Their mean is $\frac{-95+36+39}{100} = -\frac{1}{5}$ euros, an expected loss of €0.20.

(c) Income is €100 and prize cost is $40 + 4(10) =$ €80. Total profit is €20.

(d) All five winning tickets make a profit: $\frac{5}{100} = \frac{1}{20}$.

37. Solution

(a) $\mathrm{E}(R) = 0(0.80) + 5(0.15) + 25(0.05) =$ €2.

(b) The company's net incomes are $3, -2, -22$ euros. Their mean is $3(0.80) - 2(0.15) - 22(0.05) =$ €1.

(c) $1000(2) =$ €2000. This is an expected total, not a guaranteed payment.

38. Solution

(a) $\mathrm{P}(X = 3) = 0$.

(b) $\mathrm{P}(1 < X \leq 4) = 0.40 + 0.30 = 0.70$.

(c) $\mathrm{E}(X) = 1(0.20) + 2(0.40) + 4(0.30) + 5(0.10) = 2.7 = \frac{27}{10}$.

(d) $50(2.7) = 135$ orders.

39. Solution

(a) $-\frac{1}{2} + p + 4(\frac{1}{2} - p) = 1$, so $\frac{3}{2} - 3p = 1$ and $p = \frac{1}{6}$. The probabilities are $\frac{1}{2}, \frac{1}{6}, \frac{1}{3}$, all in $[0, 1]$, with sum 1.

(b) $\mathrm{P}(G > 0) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2}$.

(c) Not fair: $\mathrm{E}(G) =$ €$1 \neq 0$, despite a profit probability of $\frac{1}{2}$.

Section 3: IB exam-style / Paper 1 — Questions 31–50 (continued)

40. Solution

(a)

g	-3	3	$w-3$
$\mathrm{P}(G=g)$	$\frac{7}{10}$	$\frac{1}{5}$	$\frac{1}{10}$

(b) $-3(\frac{7}{10})+3(\frac{1}{5})+(w-3)(\frac{1}{10})=0$. Multiplying by 10 gives $-21+6+w-3=0$, so $w=18$.

(c) $\mathrm{P}(G>0)=\frac{1}{5}+\frac{1}{10}=\frac{3}{10}$.

(d) The fair mean prize is €3, so the expected total is $75(3)=$ €225.

41. Solution

(a) $-2(\frac{1}{4})+1(\frac{1}{2})+\frac{m}{4}=1$, so $\frac{m}{4}=1$ and $m=4$.

(b) Include 1 and 4: the probability is $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$.

(c) Only $X=1$ equals the mean, so the probability is $\frac{1}{2}$.

42. Solution

(a) A: $0(\frac{1}{4})+1(\frac{1}{2})+2(\frac{1}{4})=1$. B: $0(\frac{2}{5})+1(\frac{1}{5})+2(\frac{2}{5})=1$.

(b) A: $\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$. B: $\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$.

(c) Both means are 1, but, for example, the probabilities of no booking are $\frac{1}{4}$ and $\frac{2}{5}$. The distributions differ.

43. Solution

(a)

r	0	2	6
$\mathrm{P}(R=r)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

(b) $\mathrm{E}(R)=0(\frac{1}{3})+2(\frac{1}{2})+6(\frac{1}{6})=$ €2.

(c) $G=R-2$ has values $-2,0,4$ with the same respective probabilities. $\mathrm{E}(G)=-2(\frac{1}{3})+4(\frac{1}{6})=0$, so the game is fair.

(d) Only a six gives a positive net gain: $\mathrm{P}(G>0)=\frac{1}{6}$.

44. Solution

(a) $2a+3b=1$ and $3a+5b=\frac{8}{5}$.

(b) Multiply the first equation by 3 and the second by 2: $6a+9b=3$, $6a+10b=\frac{16}{5}$. Thus $b=\frac{1}{5}$ and then $a=\frac{1}{5}$.

(c) $\mathrm{P}(X\le 1)=a+b=\frac{2}{5}$.

45. Solution

(a) $\mathrm{E}(R)=0(\frac{1}{2})+2(\frac{3}{10})+6(\frac{1}{5})=\frac{9}{5}=$ €1.80.

(b) The guaranteed €2 bonus has the higher expected value.

(c) Only €6 is strictly greater than €2: probability $\frac{1}{5}$.

(d) Fixed: $100(2)=$ €200. Random: $100(1.80)=$ €180.

Section 3: IB exam-style / Paper 1 — Questions 31–50 (continued)

46. Solution

(a) $\mathrm{P(TT)} = (1-p)^2 = \frac{4}{9}$. Since $0 < 1-p < 1$, $1-p = \frac{2}{3}$, giving $p = \frac{1}{3}$.

(b) $\mathrm{P}(X=1) = \mathrm{P(HT)} + \mathrm{P(TH)} = 2(\frac{1}{3})(\frac{2}{3}) = \frac{4}{9}$ and $\mathrm{P}(X=2) = (\frac{1}{3})^2 = \frac{1}{9}$.

x	0	1	2
$\mathrm{P}(X=x)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

(c) $\mathrm{E}(X) = 0(\frac{4}{9}) + 1(\frac{4}{9}) + 2(\frac{1}{9}) = \frac{2}{3}$.

47. Solution

(a) $0 \le a \le \frac{1}{2}$. The entries sum to 1 throughout this interval.

(b) $\mathrm{E}(G) = -a + 2(\frac{1}{2} - a) = 1 - 3a$ euros.

(c) $1 - 3a = 0$, so $a = \frac{1}{3}$.

(d) $\mathrm{P}(G > 0) = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

48. Solution

(a) $\frac{a}{3} + \frac{2a+3}{6} = 3$. Hence $4a + 3 = 18$, so $a = \frac{15}{4} = 3.75$ and $b = \frac{21}{2} = 10.50$.

(b) The fair fee equals the mean prize: €3.

(c) Net gains are $-4, -\frac{1}{4}, \frac{13}{2}$. Their mean is $-4(\frac{1}{2}) - \frac{1}{4}(\frac{1}{3}) + \frac{13}{2}(\frac{1}{6}) = -1$ euro.

(d) Only €10.50 exceeds the fee, so the probability is $\frac{1}{6}$.

49. Solution

(a)

x	0	1	2	3
$\mathrm{P}(X=x)$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{7}{20}$	$\frac{1}{5}$

(b) $\mathrm{E}(X) = 0(\frac{1}{5}) + 1(\frac{1}{4}) + 2(\frac{7}{20}) + 3(\frac{1}{5}) = \frac{31}{20} = 1.55$.

(c) $200(\frac{31}{20}) = 310$ returns.

(d) The past relative frequencies should be representative of future days; for example, the relevant operating conditions should remain similar.

50. Solution

(a) $\mathrm{E}(R) = 5(\frac{2}{5}) + 15(\frac{1}{10}) = \frac{7}{2} = €3.50$.

(b) An entry fee of €3.50 makes the expected net gain zero.

(c) Let G be the player's net gain, in euros, on one play. The net gains are $-4, 1, 11$. Thus $\mathrm{E}(G) = -4(\frac{1}{2}) + 1(\frac{2}{5}) + 11(\frac{1}{10}) = -\frac{1}{2}$ euros. The profit probability is $\frac{2}{5} + \frac{1}{10} = \frac{1}{2}$.

(d) The claim is false. Here the profit probability is $\frac{1}{2}$, but the expected net gain is $-€0.50$, not zero. The sizes of gains and losses matter.

Section 4: AA HL exclusive challenge — Questions 51–60

51. Solution

(a) $3a + 2b = 1$, so $b = \frac{1-3a}{2}$. Non-negativity gives $0 \le a \le \frac{1}{3}$. These conditions also ensure no entry exceeds 1.

(b) $\mathrm{E}(X) = b + 3b + 10a = 4(\frac{1-3a}{2}) + 10a = 2 + 4a$. Thus $2 \le \mathrm{E}(X) \le \frac{10}{3}$, with both endpoints attainable.

(c) $2 + 4a = 3$ gives $a = \frac{1}{4}$, $b = \frac{1}{8}$. The probabilities for $0, 1, 3, 5$ are $\frac{1}{4}, \frac{1}{8}, \frac{1}{8}, \frac{1}{2}$.

(d) $\mathrm{P}(X \ge 3) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$.

52. Solution

(a) Write the probabilities as p, q, r. From $q + r = 3p$ and $p + q + r = 1$, $p = \frac{1}{4}$ and $q + r = \frac{3}{4}$. The mean condition gives $-\frac{1}{2} + q + 4r = 1$. Subtracting $q + r = \frac{3}{4}$ yields $3r = \frac{3}{4}$, so $r = \frac{1}{4}$ and $q = \frac{1}{2}$. These entries are valid.

(b) The probability condition still forces $p = \frac{1}{4}$. Since $q+r = \frac{3}{4}$ and $r \le \frac{3}{4}$, the mean is $-\frac{1}{2}+(\frac{3}{4}-r)+4r = \frac{1}{4} + 3r \le \frac{5}{2}$. Therefore mean 3 is impossible.

53. Solution

(a) $k[a + (a+1) + (a+2)] = 1$, so $k = \frac{1}{3(a+1)}$. Then $\mathrm{E}(X) = k[(a+1) + 2(a+2)] = \frac{3a+5}{3a+3} = 1 + \frac{2}{3(a+1)}$.

(b) Since $a + 1 \ge 1$, the positive fraction is at most $\frac{2}{3}$. Hence $1 < \mathrm{E}(X) \le \frac{5}{3}$. The upper value occurs at $a = 0$. The mean approaches 1 as a grows but never equals 1 for finite a.

(c) $\frac{2}{3(a+1)} = \frac{1}{3}$, so $a = 1$, $k = \frac{1}{6}$. The probabilities at $0, 1, 2$ are $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$.

54. Solution

(a) The gaps are $0, 1, 2$ in the first row of the outcome table, $1, 0, 1$ in the second and $2, 1, 0$ in the third. Thus the probabilities are $\frac{1}{3}, \frac{4}{9}, \frac{2}{9}$.

(b) Let G be the player's net gain in euros on one play. Its values are $-1, 0, w - 1$. Fairness gives $-1(\frac{1}{3}) + 0(\frac{4}{9}) + (w-1)(\frac{2}{9}) = 0$, so $-3 + 2w - 2 = 0$ and $w = \frac{5}{2}$.

(c) A profit occurs only when $X = 2$: probability $\frac{2}{9}$. Breaking even occurs when $X = 1$: probability $\frac{4}{9}$.

55. Solution

(a) $2q + 6r = 3$ gives $q = \frac{3}{2} - 3r$, and $p = 1 - q - r = 2r - \frac{1}{2}$. From $p, q, r \ge 0$, $\frac{1}{4} \le r \le \frac{1}{2}$. This interval makes all entries valid.

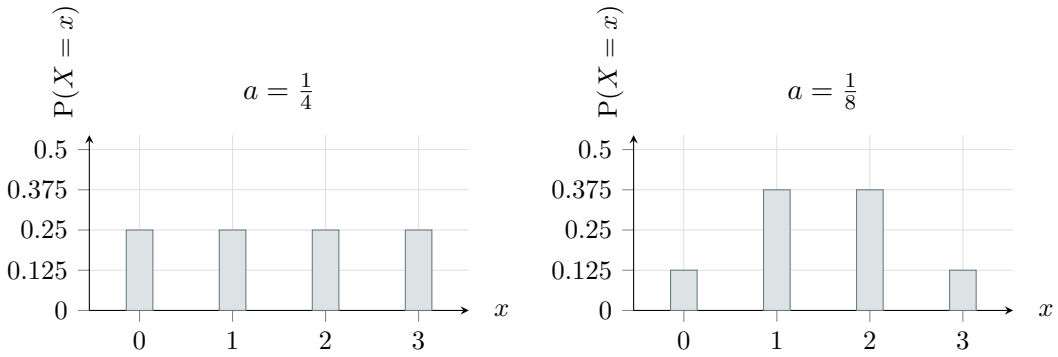
(b) $\mathrm{P}(X \ge 2) = q + r = \frac{3}{2} - 2r$. Its minimum is $\frac{1}{2}$ at $r = \frac{1}{2}$, with $(p, q, r) = (\frac{1}{2}, 0, \frac{1}{2})$. Its maximum is 1 at $r = \frac{1}{4}$, with $(p, q, r) = (0, \frac{3}{4}, \frac{1}{4})$.

Section 4: AA HL exclusive challenge — Questions 51–60 (continued)

56. Solution

(a) $p_0 + p_1 = \frac{1}{2}$ and $p_2 + p_3 = \frac{1}{2}$. Substitution into $p_1 + 2p_2 + 3p_3 = \frac{3}{2}$ gives $\frac{3}{2} - p_0 + p_3 = \frac{3}{2}$, hence $p_0 = p_3 = a$. Therefore $p_1 = p_2 = \frac{1}{2} - a$, with $0 \leq a \leq \frac{1}{2}$.

(b) For example, $a = \frac{1}{4}$ gives $(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$; $a = \frac{1}{8}$ gives $(\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8})$. Both have mean $\frac{3}{2}$ and upper-half probability $\frac{1}{2}$. The charts are:



(c) Those two numerical conditions need not determine a unique distribution. Here there is a whole family.

57. Solution

(a) The refund gives net gain 0.

g	$8-c$	0	$-c$
$\mathrm{P}(G=g)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$

(b) $\mathrm{E}(G) = \frac{8-c}{4} - \frac{c}{2} = 2 - \frac{3c}{4}$. Thus $c = \frac{8}{3}$ euros for fairness.

(c) Since $0 < \frac{8}{3} < 8$, the probabilities are $\frac{1}{4}$ for a profit and $\frac{1}{4}$ for breaking even.

(d) The organiser's mean is $\frac{3c}{4} - 2$. Setting this equal to 1 gives $c = 4$ euros.

58. Solution

(a) Let R be the prize in euros. For prizes $0, 3, 12$, the probabilities are $(1-p)^2, 2p(1-p), p^2$, since either HT or TH gives one head.

(b) The mean prize must be 4, so $3[2p(1-p)] + 12p^2 = 4$. Thus $3p^2 + 3p - 2 = 0$ and $p = \frac{-3 \pm \sqrt{33}}{6}$. Only $\boxed{p = \dfrac{\sqrt{33} - 3}{6}}$ lies in $(0, 1)$.

(c) The €3 prize still gives a loss. Only two heads give a profit, with probability $p^2 = \frac{(\sqrt{33}-3)^2}{36} = \boxed{\dfrac{7 - \sqrt{33}}{6}}$.

59. Solution

(a) $\frac{-a+b}{4} = 1$, so $b = a + 4$. Then $a(a+4) = 12$, giving $(a-2)(a+6) = 0$. Positivity requires $a = 2$ and $b = 6$.

(b) Only 6 exceeds the mean 1, so the probability is $\frac{1}{4}$.

(c) Fairness requires $-L(\frac{1}{4}) + 6(\frac{1}{4}) = 0$, hence $L = 6$.

Section 4: AA HL exclusive challenge — Questions 51–60 (continued)

60. Solution

(a) $5q + 20r = 4$, so $q = \frac{4}{5} - 4r$ and $p = 1 - q - r = \frac{1}{5} + 3r$. Non-negativity gives $0 \leq r \leq \frac{1}{5}$, and all entries then lie in $[0, 1]$.

(b) A profit means $R > 4$. Thus $\mathrm{P}(\text{profit}) = q + r = \frac{4}{5} - 3r$, with minimum $\frac{1}{5}$ and maximum $\frac{4}{5}$.

(c) Minimum: $(p, q, r) = (\frac{4}{5}, 0, \frac{1}{5})$. Maximum: $(p, q, r) = (\frac{1}{5}, \frac{4}{5}, 0)$.

(d) For $G = R - 4$, the gains are $-4, 1, 16$. Their mean is $-4p + q + 16r = 0$ after substitution. Equivalently the mean prize and fee are both €4.

Optional tutorial support

Distribution tables and bar charts
youtu.be/pC2p2yXxgYQ

Mean and mode of a discrete random variable
youtu.be/63glO_YdrYo

Scope and provenance. Syllabus: IB *Mathematics: analysis and approaches guide*, SL 4.7, p. 57. All questions and solutions are original. Project AA SL and AA HL Paper 1 references were consulted for command terms, subpart structure and mark-allocation style. The HL challenges do not require variance, continuous distributions or other AHL 4.14 content.