Binomial series: quiz preparation

IB Mathematics: Analysis and Approaches HL | Rational exponents

Learning goal	Expand rational powers, approximate values and recover unknown parameters.
Syllabus link	AHL 1.10: binomial expansions with fractional and negative indices.
30-minute route	Q1–3: 11 min; Q4–5: 8 min; Q6: 7 min; check and correct: 4 min.
Technology	No calculator. Question 7 is optional if you finish early.

Instructions: Work on separate paper. Show your working and keep coefficients exact. For validity, give the standard open interval obtained from $|u| < 1$; endpoint analysis is not required. Worked answers begin on page 3.

We already know: first identify n and the *whole* expression u.

$$(1+u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \cdots$$

If the constant term inside the bracket is not 1, factor it out first. Apply $|u| < 1$ to the whole substituted expression.

1 Direct expansions — About 11 minutes

1. Complete the table without expanding.

Expression	n	u	Open interval for x
(a) $(1-5x)^{1/3}$			
(b) $(1+3x)^{-2/3}$			
(c) $(1-\frac{x}{2})^{3/2}$			

2. Expand each expression in ascending powers of x, up to and including the term in x^3. State the standard open interval of validity for each series.

(a) $(1-2x)^{1/2}$

(b) $(1+3x)^{1/3}$

3. Expand $(1+2x)^{-1/2}$ up to and including the term in x^2. State the standard open interval of validity.

2 Rewriting and approximation — About 8 minutes

4. For each expression, first rewrite it as a constant multiple of $(1+u)^n$. Then expand up to and including the term in x^2, and state the standard open interval of validity.

(a) $(9+6x)^{1/2}$

(b) $(9-6x)^{-1/2}$

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5. Use your cubic expansion from Question 2(a) to obtain an approximation for $\sqrt{0.8}$. State the value of x you use and give your estimate as a decimal.

3 Working backwards from coefficients — About 7 minutes

6. For sufficiently small $|x|$,

$$(1+ax)^n = 1 - 2x + 6x^2 + \cdots,$$

where a is a non-zero real constant and $n \in \mathbb{Q}$.

(a) Write down two equations by comparing the coefficients of x and x^2. [2]

(b) Hence find the values of a and n. [2]

(c) Hence find the coefficient of x^3. [2]

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7. If you finish early: spot and correct the errors

A student writes

$$(25-10x)^{-1/2} = 5\left(1-\frac{2x}{5}\right)^{-1/2} = 5 + x + \frac{3x^2}{10} + \cdots,$$

and gives the standard open interval of validity as $x < \frac{5}{2}$.

Identify the error in the factor outside the bracket and the error in the interval. Write the corrected expansion and interval.

Last 4 minutes: check and correct. Check your substitutions, signs and powers of the coefficient of x. Then compare with the worked answers. Correct one error and explain why it occurred.

Answer key and solution outlines

Check the method as well as the final coefficients

Section 1 Direct expansions

1. Identify n and u, then solve $|u| < 1$.

Part	n	u	Open interval for x
(a)	$\frac{1}{3}$	$-5x$	$-\frac{1}{5} < x < \frac{1}{5}$
(b)	$-\frac{2}{3}$	$3x$	$-\frac{1}{3} < x < \frac{1}{3}$
(c)	$\frac{3}{2}$	$-\frac{x}{2}$	$-2 < x < 2$

2(a) Here $n = \frac{1}{2}$ and $u = -2x$.

$$(1-2x)^{1/2} = 1 - x - \frac{1}{8}(-2x)^2 + \frac{1}{16}(-2x)^3 + \cdots = \boxed{1 - x - \frac{x^2}{2} - \frac{x^3}{2} + \cdots}.$$

Since $|-2x| < 1$, the standard open interval is $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$.

2(b) Here $n = \frac{1}{3}$ and $u = 3x$.

$$(1+3x)^{1/3} = 1 + x - \frac{1}{9}(3x)^2 + \frac{5}{81}(3x)^3 + \cdots = \boxed{1 + x - x^2 + \frac{5}{3}x^3 + \cdots}.$$

Since $|3x| < 1$, the standard open interval is $\boxed{-\frac{1}{3} < x < \frac{1}{3}}$.

3. Here $n = -\frac{1}{2}$ and $u = 2x$.

$$(1+2x)^{-1/2} = 1 - \frac{1}{2}(2x) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}(2x)^2 + \cdots = \boxed{1 - x + \frac{3}{2}x^2 + \cdots}.$$

Since $|2x| < 1$, the standard open interval is $\boxed{-\frac{1}{2} < x < \frac{1}{2}}$.

Section 2 Rewriting and approximation: Question 4

4(a) Factor out 9; the factor outside becomes $9^{1/2} = 3$:

$$(9+6x)^{1/2} = 3\left(1 + \frac{2x}{3}\right)^{1/2} = 3\left[1 + \frac{x}{3} - \frac{x^2}{18} + \cdots\right].$$

Therefore $\boxed{(9+6x)^{1/2} = 3 + x - \frac{x^2}{6} + \cdots}$.

4(b) Here the factor outside is $9^{-1/2} = \frac{1}{3}$:

$$(9-6x)^{-1/2} = \frac{1}{3}\left(1 - \frac{2x}{3}\right)^{-1/2} = \frac{1}{3}\left[1 + \frac{x}{3} + \frac{x^2}{6} + \cdots\right].$$

Therefore $\boxed{(9-6x)^{-1/2} = \frac{1}{3} + \frac{x}{9} + \frac{x^2}{18} + \cdots}$.

For both parts, $\left|\frac{2x}{3}\right| < 1$ gives the standard open interval $\boxed{-\frac{3}{2} < x < \frac{3}{2}}$.

Answers continued

Approximations, unknown parameters and checking errors

Section 2 Rewriting and approximation: Question 5

5. Set $1-2x=0.8$, giving $\boxed{x=0.1}$. This satisfies $|x|<\frac{1}{2}$.

$$\sqrt{0.8}\approx 1-0.1-\frac{(0.1)^2}{2}-\frac{(0.1)^3}{2}$$

$$=1-0.1-0.005-0.0005=\boxed{0.8945}.$$

This is the estimate from the cubic expansion. It is not a claim that the exact value rounds to 0.8945 to four decimal places.

Section 3 Working backwards from coefficients

6(a) The coefficients of x and x^2 give, respectively,

$$\boxed{na=-2}, \qquad \boxed{\frac{n(n-1)a^2}{2}=6}.$$

6(b) Because $na=-2$, neither n nor a is zero. Divide the second equation by the first:

$$\frac{(n-1)a}{2}=-3 \quad\Longrightarrow\quad (n-1)a=-6.$$

Using $na=-2$ gives $-2-a=-6$, so

$$\boxed{a=4}, \qquad n=\frac{-2}{4}=\boxed{-\frac{1}{2}}.$$

6(c) The coefficient of x^3 is

$$\frac{n(n-1)(n-2)a^3}{6}=\frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})\,4^3}{6}=\boxed{-20}.$$

Mark guide: (a) one mark for each equation; (b) one for each correct parameter; (c) one for the coefficient formula with substitution and one for -20. Total: 6 marks.

Optional Question 7 Spot and correct the errors

First error: $25^{-1/2}=\frac{1}{5}$, so the factor outside the bracket must be $\frac{1}{5}$, not 5.

$$(25-10x)^{-1/2}=\frac{1}{5}\left(1-\frac{2x}{5}\right)^{-1/2}$$

$$=\frac{1}{5}\left[1+\frac{x}{5}+\frac{3x^2}{50}+\cdots\right]$$

$$=\boxed{\frac{1}{5}+\frac{x}{25}+\frac{3x^2}{250}+\cdots}.$$

Second error: the interval needs both a lower and an upper bound:

$$\left|-\frac{2x}{5}\right|<1 \quad\Longrightarrow\quad \boxed{-\frac{5}{2}<x<\frac{5}{2}}.$$