



Binomial series

Fractional and negative indices

IB Mathematics: Analysis and Approaches HL | 50-question worksheet

Learning goal	Expand rational powers, find coefficients and use series to approximate values.
Syllabus link	AHL 1.10: fractional and negative indices only. Uses the binomial Maclaurin series linked to AHL 5.19.
Structure	15 short response; 15 extended response; 20 original IB-style questions.
Technology	Non-calculator throughout (Paper 1 style). Work on separate paper.

Instructions

Show your working and give exact answers unless an approximation is requested. Use

$$(1 + u)^n = 1 + nu + \frac{n(n-1)}{2!}u^2 + \frac{n(n-1)(n-2)}{3!}u^3 + \dots, \quad |u| < 1.$$

All expansions are about $x = 0$. For validity questions use the strict condition $|u| < 1$; endpoint testing is not required. Write $\approx$ when using a finite truncation to estimate a value.

1. Short response

- Expand $(1 + x)^{1/2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

- Expand $(1 - x)^{-1}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

- Expand $(1 + 2x)^{-2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

- Expand $(1 - 3x)^{1/3}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

- Expand $(1 + 4x)^{-1/2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

- Expand $(1 - 2x)^{3/2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .



7. Expand $(4 + x)^{1/2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

8. Expand $(8 - 4x)^{1/3}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

9. Expand $(9 + 6x)^{-1/2}$ in ascending powers of x , up to and including the term in x^3 . State the restriction on x .

10. Expand $(1 - 2x^2)^{-1/2}$ in ascending powers of x , up to and including the term in x^6 . State the restriction on x .

11. Find the coefficient of x^4 in the expansion of $(1 - 3x)^{-2}$.

12. State the restriction on x obtained from $|u| < 1$ when expanding each expression.

(a) $(1 + 5x)^{-1/3}$

(b) $(16 - 2x)^{1/4}$

(c) $(1 - 9x^2)^{-1}$

13. Write $(27 + 9x)^{2/3}$ in the form $A(1 + kx)^{2/3}$, giving A and k .

14. The Maclaurin series for $(1 - 4x)^{-1/2}$ is used. Which of $x = -\frac{1}{5}$, $x = \frac{1}{4}$ and $x = \frac{3}{10}$ satisfy the condition $|-4x| < 1$?

15. A student writes $(1 + x)^{-1/2} = 1 - \frac{1}{2}x - \frac{1}{8}x^2 + \dots$. Identify and correct the error.

**2. Extended response**

16. Expand $\frac{1+2x}{\sqrt{1-x}}$ up to and including the term in x^3 . State a restriction on x for which the series used are valid.

17. Expand $(1-x)(1+3x)^{-1/2}$ up to and including the term in x^3 . State a restriction on x for which the series used are valid.

18. Expand $\frac{\sqrt{4+x}}{1-x}$ up to and including the term in x^3 . State a restriction on x for which the series used are valid.

19. Expand $\sqrt{\frac{1+x}{1-x}}$ up to and including the term in x^3 . State a restriction on x for which the series used are valid.

20. Use the first three terms of the series for $(1+u)^{1/2}$ to find a rational approximation to $\sqrt{26}$. State your value of u .

21. Use a binomial series through the quadratic term to find a rational approximation to $\sqrt[3]{26}$.

22. Expand $(1-2x)^{-3/2}$ through x^2 . Hence approximate $(0.98)^{-3/2}$, giving a terminating decimal.

23. In $(1+kx)^{-1/2}$, the coefficient of x is 3. Find k , the coefficient of x^2 and the restriction on x .

24. The expansion of $(1+ax)^n$ starts $1+6x+24x^2+\dots$, where $a \neq 0$ and $n \in \mathbb{Q}$. Find a and n .

25. Find p so that the coefficient of x in $(1+px)(1-4x)^{-1/2}$ is zero. For this p , find the coefficients of x^2 and x^3 .

26. For $(1-4x^2)^{-1/2}$, find the coefficient of x^6 and explain why the coefficient of x^5 is zero.



27. Complete the table. In each case, use the condition $|u| < 1$.

Expression	Rewritten as $A(1+u)^n$	Restriction on x
$(25 - 10x)^{-1/2}$		
$(8 + x)^{2/3}$		
$(1 + 16x^2)^{-1/4}$		

28. Use the series for $(1+u)^{-2/3}$ with $u = -8x^3$ to expand $(1 - 8x^3)^{-2/3}$ through x^9 . State the restriction on x and explain why the coefficient of x^7 is zero.

29. Two proposed ways to approximate $\sqrt{10}$ are $\sqrt{1+9}$ and $3\sqrt{1+\frac{1}{9}}$. Explain which is suitable for the series of $(1+u)^{1/2}$. Use its first three terms to obtain a rational approximation.

30. Let $f(x) = \sqrt{1+2x}$. Its cubic approximation is $P_3(x) = 1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3$.

(a) At $x = \frac{1}{10}$, find $P_3(x)$ exactly.

(b) Find the x^4 term of the series for $f(x)$. Use its sign to predict whether $P_3(x)$ overestimates or underestimates $f(x)$ for small positive x .

**3. IB exam-style / Paper 1****31.** [*Maximum mark: 7*]

- (a) Find the first three terms of $(1+x)^{1/3}$. [3]
- (b) By a suitable substitution, obtain a rational approximation to $\sqrt[3]{65}$. [3]
- (c) Explain why your substitution is allowed. [1]

32. [*Maximum mark: 8*]Consider $f(x) = (16 - 8x)^{3/4}$.

- (a) Find the expansion of $f(x)$ through x^3 . [5]
- (b) State the restriction on x . [1]
- (c) Use your expansion to approximate $(15.84)^{3/4}$ as a terminating decimal. [2]

33. [*Maximum mark: 8*]Let $f(x) = (1 + 6x)^{-1/2}$.

- (a) Find the expansion of $f(x)$ through x^3 . [4]
- (b) Find a such that $(1 + ax)f(x)$ has no x^2 term. [2]
- (c) For this value of a , find the coefficient of x^3 in the product. [2]

34. [*Maximum mark: 8*]The expansion of $(1 + kx)^n$ begins $1 - 4x + 12x^2 + \dots$, where $k \neq 0$.

- (a) Find n and k . [5]
- (b) Find the coefficient of x^3 . [2]
- (c) State the restriction on x . [1]

35. [*Maximum mark: 8*]In the expansion of $(q - 2x^2)^{-1/2}$, where $q > 0$, the coefficient of x^6 is 320.

- (a) Show that this coefficient is $\frac{5}{2}q^{-7/2}$. [3]
- (b) Find q . [3]
- (c) State the restriction on x for the resulting expansion. [2]

**36.** [Maximum mark: 8]

Let $F(x) = (1 + ax)^{-1/2} - (1 - 2x)^{1/2}$, where $a \in \mathbb{Q}$. Its expansion through x^2 is $bx + bx^2$.

- (a) Find the possible values of a and the corresponding values of b . [6]
- (b) For each pair, state a restriction on x for which both component series are valid. [2]

37. [Maximum mark: 12]

Let $f(x) = (1 + 3x)^{2/3}$ and $R(x) = \frac{1 + px}{1 + qx}$.

- (a) Expand $f(x)$ through x^3 . [4]
- (b) Expand $R(x)$ through x^2 in terms of p and q . [3]
- (c) Find p and q so that the constant, x and x^2 terms agree. [3]
- (d) Use $R(\frac{1}{30})$ to find a rational approximation to $(\frac{11}{10})^{2/3}$. [2]

38. [Maximum mark: 8]

Let $f(x) = \sqrt{\frac{1 - 2x}{1 + x}}$.

- (a) Expand each factor of $(1 - 2x)^{1/2}(1 + x)^{-1/2}$ through x^3 . [4]
- (b) Hence find the expansion of $f(x)$ through x^3 . [3]
- (c) State a restriction on x for which both series are valid. [1]

39. [Maximum mark: 8]

Consider $f(x) = (4 - 3x)^{-1}$.

- (a) Find the first four terms of its Maclaurin series. [3]
- (b) Use these terms to obtain a rational approximation to $\frac{10}{37}$. [3]
- (c) Explain whether this approximation is an overestimate or an underestimate. [2]

40. [Maximum mark: 8]

The function $f(x) = (1 - 2x)^{-3}$ is expanded about $x = 0$.

- (a) Find the first four terms. [4]
- (b) Find the coefficient of x^5 . [3]
- (c) State the restriction on x . [1]

**41.** [Maximum mark: 8]

Let $S(x) = \sqrt{1+4x} + \sqrt{1-4x}$.

- (a) Explain why the odd powers of x cancel. [2]
- (b) Find the expansion of $S(x)$ through x^4 . [4]
- (c) Hence approximate $\sqrt{1.04} + \sqrt{0.96}$ as a terminating decimal. [2]

42. [Maximum mark: 7]

Let $D(x) = (1-3x)^{-1/2} - (1+3x)^{-1/2}$.

- (a) Find the expansion of $D(x)$ through x^3 . [4]
- (b) State the restriction on x . [1]
- (c) Use your result to approximate $\frac{1}{\sqrt{0.97}} - \frac{1}{\sqrt{1.03}}$ as a terminating decimal. [2]

43. [Maximum mark: 7]

The expansion of $(a+bx)^{-1/2}$, where $a > 0$, begins $\frac{1}{2} + \frac{3}{4}x + \dots$.

- (a) Find a and b . [4]
- (b) Find the coefficient of x^2 . [2]
- (c) State the restriction on x . [1]

44. [Maximum mark: 9]

Consider $F(x) = (1+ax+bx^2)(1-2x)^{-1/2}$.

- (a) Expand $(1-2x)^{-1/2}$ through x^3 . [3]
- (b) Find a and b such that the coefficients of x and x^2 in $F(x)$ are both zero. [4]
- (c) For these values, find the coefficient of x^3 in $F(x)$. [2]

45. [Maximum mark: 8]

An expansion is required for $f(x) = (1+4x^2)^{-3/2}$.

- (a) Find the expansion through x^6 . [4]
- (b) State the restriction on x . [2]
- (c) Find the coefficient of x^8 . [2]

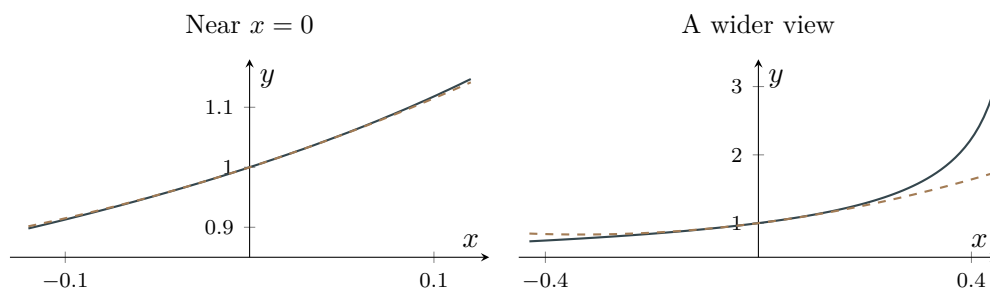
46. [Maximum mark: 7]

A student proposes $\sqrt{4+x} = 2 + \frac{1}{2}x - \frac{1}{8}x^2 + \dots$.

- (a) Explain the error and find the correct expansion through x^3 . [4]
- (b) Use the corrected expansion to approximate $\sqrt{4.08}$ as a terminating decimal. [2]
- (c) State the restriction on x for the series. [1]

47. [Maximum mark: 7]

Let $f(x) = (1-2x)^{-1/2}$. The diagrams show f (solid) and a polynomial approximation P (dashed) at two scales.



- (a) Find the quadratic Maclaurin approximation $P(x)$. [3]
- (b) Find $P(\frac{1}{10})$ and $P(\frac{2}{5})$ exactly. [2]
- (c) Use the diagrams and the substituted quantity to explain why $|2x| < 1$ does not ensure that this quadratic approximation is equally accurate throughout the interval. [2]

**48.** [Maximum mark: 10]

Let $f(x) = (1+x)^{-1/3}$ and $R(x) = \frac{1+px}{1+qx}$.

- (a) Find the expansion of $f(x)$ through x^3 . [4]
 - (b) Find p and q so that f and R have the same terms through x^2 . [4]
 - (c) Use $R(\frac{1}{8})$ to approximate $(\frac{8}{9})^{1/3}$ as a rational number. [2]
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49. [Maximum mark: 10]

Let $f(x) = (1-3x)^{-1/2}$.

- (a) Find the expansion of $f(x)$ through x^3 . [4]
 - (b) Find A and B so that $Af(x) + Bf(-x)$ has constant term 1 and coefficient of x equal to 0. [3]
 - (c) For these values, find the coefficient of x^4 in the combined series. [3]
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50. [Maximum mark: 14]

Let $f(x) = \sqrt{\frac{1+2x}{1-x}}$ and $R(x) = \frac{1+px}{1+qx}$.

- (a) Expand $f(x)$ through x^3 and state a restriction on x for which both component series are valid. [5]
- (b) Find p and q so that f and R have the same terms through x^2 . [4]
- (c) Find the coefficient of x^3 in $f(x) - R(x)$. [2]
- (d) Use $R(\frac{1}{10})$ to approximate $\frac{2}{\sqrt{3}}$. Use part (c) to predict the direction of the error for sufficiently small positive x . [3]

Answer key and solution outlines

The intervals below come from $|u| < 1$; no claim is made about convergence at endpoints. A series with $+\dots$ continues beyond the displayed terms.

1. Short response

1. $1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots$, valid for $-1 < x < 1$.
2. $1 + x + x^2 + x^3 + \dots$, valid for $-1 < x < 1$.
3. $1 - 4x + 12x^2 - 32x^3 + \dots$, valid for $-\frac{1}{2} < x < \frac{1}{2}$.
4. $1 - x - x^2 - \frac{5}{3}x^3 + \dots$, valid for $-\frac{1}{3} < x < \frac{1}{3}$.
5. $1 - 2x + 6x^2 - 20x^3 + \dots$, valid for $-\frac{1}{4} < x < \frac{1}{4}$.
6. $1 - 3x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \dots$, valid for $-\frac{1}{2} < x < \frac{1}{2}$.
7. $2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 + \dots$, valid for $-4 < x < 4$.
8. $2 - \frac{1}{3}x - \frac{1}{18}x^2 - \frac{5}{324}x^3 + \dots$, valid for $-2 < x < 2$.
9. $\frac{1}{3} - \frac{1}{9}x + \frac{1}{18}x^2 - \frac{5}{162}x^3 + \dots$, valid for $-\frac{3}{2} < x < \frac{3}{2}$.
10. $1 + x^2 + \frac{3}{2}x^4 + \frac{5}{2}x^6 + \dots$, valid for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$.
11. $\frac{(-2)(-3)(-4)(-5)}{4!}(-3)^4 = 5(81) = 405$.
12. (a) $|x| < \frac{1}{5}$; (b) $|x| < 8$; (c) $|x| < \frac{1}{3}$.
13. $27^{2/3}(1 + x/3)^{2/3} = 9(1 + x/3)^{2/3}$, so $A = 9$, $k = \frac{1}{3}$.
14. Only $x = -\frac{1}{5}$, since $\frac{4}{5} < 1$, whereas 1 and $\frac{6}{5}$ are not less than 1.
15. The x^2 coefficient is $\frac{(-1/2)(-3/2)}{2!} = \frac{3}{8}$, so it is positive: $1 - \frac{1}{2}x + \frac{3}{8}x^2 + \dots$.

2. Extended response

- 16.** Use $(1-x)^{-1/2} = 1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots$ and multiply by $1+2x$: $1 + \frac{5}{2}x + \frac{11}{8}x^2 + \frac{17}{16}x^3 + \dots$, for $|x| < 1$.
- 17.** Use $(1+3x)^{-1/2} = 1 - \frac{3}{2}x + \frac{27}{8}x^2 - \frac{135}{16}x^3 + \dots$ and multiply by $1-x$: $1 - \frac{5}{2}x + \frac{39}{8}x^2 - \frac{189}{16}x^3 + \dots$, for $|x| < \frac{1}{3}$.
- 18.** Multiply $(2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 + \dots)$ by $(1 + x + x^2 + x^3 + \dots)$: $2 + \frac{9}{4}x + \frac{143}{64}x^2 + \frac{1145}{512}x^3 + \dots$, for $|x| < 1$.
- 19.** Multiply $(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots)$ by $(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots)$: $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots$, for $|x| < 1$.
- 20.** $\sqrt{26} = 5(1 + \frac{1}{25})^{1/2}$, so $u = \frac{1}{25}$ and $\sqrt{26} \approx 5(1 + \frac{1}{50} - \frac{1}{5000}) = \frac{5099}{1000}$.
- 21.** $\sqrt[3]{26} = 3(1 - \frac{1}{27})^{1/3} \approx 3(1 - \frac{1}{81} - \frac{1}{6561}) = \frac{6479}{2187}$.
- 22.** $1 + 3x + \frac{15}{2}x^2 + \dots$. With $x = \frac{1}{100}$, the approximation is 1.03075.
- 23.** $-k/2 = 3 \Rightarrow k = -6$; the coefficient of x^2 is $3k^2/8 = \frac{27}{2}$; $|x| < \frac{1}{6}$.
- 24.** $na = 6$ and $n(n-1)a^2/2 = 24$. Hence $(n-1)/n = 4/3$, giving $n = -3$ and $a = -2$.
- 25.** Use $(1-4x)^{-1/2} = 1 + 2x + 6x^2 + 20x^3 + \dots$. Thus $p+2=0$, so $p=-2$; the required coefficients are $6+2p=2$ and $20+6p=8$.
- 26.** The u^3 term with $u = -4x^2$ gives $(-\frac{5}{16})(-4)^3x^6 = 20x^6$. Every power of x is even, so the x^5 coefficient is 0.
- 27.** $\frac{1}{5}(1 - \frac{2}{5}x)^{-1/2}$, $|x| < \frac{5}{2}$; $4(1 + \frac{1}{8}x)^{2/3}$, $|x| < 8$; $(1 + 16x^2)^{-1/4}$, $|x| < \frac{1}{4}$.
- 28.** $1 - \frac{2}{3}u + \frac{5}{9}u^2 - \frac{40}{81}u^3 + \dots$ gives $1 + \frac{16}{3}x^3 + \frac{320}{9}x^6 + \frac{20480}{81}x^9 + \dots$. The condition $|8x^3| < 1$ gives $|x| < \frac{1}{2}$. Only powers divisible by 3 occur, so the x^7 coefficient is 0.
- 29.** The second uses $u = \frac{1}{9}$, so $|u| < 1$; the first has $u = 9$. The approximation is $3(1 + \frac{1}{18} - \frac{1}{648}) = \frac{683}{216}$.
- 30.** (a) $\frac{2191}{2000}$. (b) The next term is $-\frac{5}{8}x^4$. It predicts an overestimate for small positive x ; this is a local prediction from the leading omitted term.

**3. IB exam-style / Paper 1: Questions 31–40**

- 31.** (a) $1 + \frac{1}{3}x - \frac{1}{9}x^2 + \dots$. (b) $\sqrt[3]{65} = 4(1 + \frac{1}{64})^{1/3} \approx 4(1 + \frac{1}{192} - \frac{1}{36864}) = \frac{37055}{9216}$. (c) $|x| = \frac{1}{64} < 1$.
- 32.** (a) $f(x) = 8(1 - \frac{1}{2}x)^{3/4} = 8 - 3x - \frac{3}{16}x^2 - \frac{5}{128}x^3 + \dots$. (b) $|x| < 2$. (c) $x = \frac{1}{50}$ gives 7.9399246875.
- 33.** (a) $1 - 3x + \frac{27}{2}x^2 - \frac{135}{2}x^3 + \dots$. (b) $\frac{27}{2} - 3a = 0$, so $a = \frac{9}{2}$. (c) $-\frac{135}{2} + \frac{27a}{2} = -\frac{27}{4}$.
- 34.** (a) $nk = -4$, $n(n-1)k^2/2 = 12$. Dividing gives $(n-1)/n = 3/2$, so $n = -2$, $k = 2$. (b) $n(n-1)(n-2)k^3/6 = -32$. (c) $|x| < \frac{1}{2}$.
- 35.** (a) Write $q^{-1/2}(1 - 2x^2/q)^{-1/2}$. The x^6 coefficient is $q^{-1/2}(-\frac{5}{16})(-2/q)^3 = \frac{5}{2}q^{-7/2}$. (b) $q^{-7/2} = 128 = 2^7$, so $q = \frac{1}{4}$. (c) $|8x^2| < 1$, giving $|x| < \frac{1}{2\sqrt{2}}$.
- 36.** (a) $b = 1 - a/2$ and $b = 3a^2/8 + 1/2$. Hence $3a^2 + 4a - 4 = 0$, so $(3a-2)(a+2) = 0$. The pairs are $(a, b) = (\frac{2}{3}, \frac{2}{3})$ and $(-2, 2)$. (b) For both pairs the common restriction is $|x| < \frac{1}{2}$.
- 37.** (a) $1 + 2x - x^2 + \frac{4}{3}x^3 + \dots$. (b) $1 + (p-q)x + (q^2 - pq)x^2 + \dots$. (c) $p - q = 2$, $-q(p-q) = -1$, so $q = \frac{1}{2}$, $p = \frac{5}{2}$. (d) $R(\frac{1}{30}) = \frac{65}{61}$.
- 38.** (a) $1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \dots$ and $1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \dots$. (b) $1 - \frac{3}{2}x + \frac{3}{8}x^2 - \frac{15}{16}x^3 + \dots$. (c) $|x| < \frac{1}{2}$.
- 39.** (a) $\frac{1}{4} + \frac{3}{16}x + \frac{9}{64}x^2 + \frac{27}{256}x^3 + \dots$. (b) $x = \frac{1}{10}$ gives $\frac{69187}{256000}$. (c) It is an underestimate: all omitted geometric-series terms are positive at this x .
- 40.** (a) $1 + 6x + 24x^2 + 80x^3 + \dots$. (b) $\frac{(-3)(-4)(-5)(-6)(-7)}{5!}(-2)^5 = 672$. (c) $|x| < \frac{1}{2}$.



3. IB exam-style / Paper 1: Questions 41–50

41. (a) Replacing x by $-x$ reverses every odd-power term; addition cancels them. (b) $S(x) = 2 - 4x^2 - 20x^4 + \dots$. (c) $x = \frac{1}{100}$ gives 1.9995998.
42. (a) The even powers cancel: $D(x) = 3x + \frac{135}{8}x^3 + \dots$. (b) $|x| < \frac{1}{3}$. (c) $x = \frac{1}{100}$ gives 0.030016875.
43. (a) $a^{-1/2} = \frac{1}{2} \Rightarrow a = 4$. The x coefficient is $-b/(2a^{3/2}) = -b/16 = \frac{3}{4}$, so $b = -12$. (b) $\frac{1}{2} \cdot \frac{3}{8}(-3)^2 = \frac{27}{16}$. (c) $|x| < \frac{1}{3}$.
44. (a) $1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \dots$. (b) $a + 1 = 0$, $b + a + \frac{3}{2} = 0$, so $a = -1$, $b = -\frac{1}{2}$. (c) $\frac{5}{2} + \frac{3}{2}a + b = \frac{1}{2}$.
45. (a) $1 - 6x^2 + 30x^4 - 140x^6 + \dots$. (b) $4x^2 < 1$, so $|x| < \frac{1}{2}$. (c) $\frac{(-3/2)(-5/2)(-7/2)(-9/2)}{4!}4^4 = 630$.
46. (a) The factorization is $2(1 + x/4)^{1/2}$: the substituted quantity is $x/4$. The series is $2 + \frac{1}{4}x - \frac{1}{64}x^2 + \frac{1}{512}x^3 + \dots$. (b) $x = \frac{2}{25}$ gives 2.019901. (c) $|x| < 4$.
47. (a) $P(x) = 1 + x + \frac{3}{2}x^2$. (b) $\frac{223}{200}$ and $\frac{41}{25}$. (c) The infinite series converges for $|2x| < 1$, but a quadratic retains only three terms. The diagrams show larger discrepancies farther from zero; at $x = \frac{2}{5}$, $|2x| = \frac{4}{5}$ and higher-power terms are more significant than at $x = \frac{1}{10}$.
48. (a) $1 - \frac{1}{3}x + \frac{2}{9}x^2 - \frac{14}{81}x^3 + \dots$. (b) $p - q = -\frac{1}{3}$ and $-q(p - q) = \frac{2}{9}$, so $q = \frac{2}{3}$, $p = \frac{1}{3}$. (c) $R(\frac{1}{8}) = \frac{25}{26}$.
49. (a) $1 + \frac{3}{2}x + \frac{27}{8}x^2 + \frac{135}{16}x^3 + \dots$. (b) $A + B = 1$, $\frac{3}{2}(A - B) = 0$, so $A = B = \frac{1}{2}$. (c) The even terms are unchanged; the coefficient is $\frac{35}{128}3^4 = \frac{2835}{128}$.
50. (a) Multiply $(1 + x - \frac{1}{2}x^2 + \frac{1}{2}x^3 + \dots)$ by $(1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3 + \dots)$: $f(x) = 1 + \frac{3}{2}x + \frac{3}{8}x^2 + \frac{15}{16}x^3 + \dots$, for $|x| < \frac{1}{2}$. (b) $p - q = \frac{3}{2}$, $-q(p - q) = \frac{3}{8}$, giving $q = -\frac{1}{4}$, $p = \frac{5}{4}$. (c) The x^3 coefficient of R is $q^2(p - q) = \frac{3}{32}$, so the difference has coefficient $\frac{27}{32}$. (d) $f(\frac{1}{10}) = \frac{2}{\sqrt{3}} \approx R(\frac{1}{10}) = \frac{15}{13}$. Since $f - R$ has positive leading term $\frac{27}{32}x^3$, R underestimates f for sufficiently small positive x .