

Limits and higher derivatives

IB Mathematics AA HL 5.12

50 questions | Progressive practice and worked solution outlines

Learning goal	Understand limiting behaviour; derive polynomial derivatives from first principles; work with higher derivatives.
Syllabus link	AHL 5.12. Continuity and differentiability are explored informally; formal testing is not an examination requirement. First-principles differentiation is restricted to polynomials.
Structure	15 short + 15 extended + 20 original IB-style questions.
Technology	Non-calculator unless marked  GDC. Use radians; give exact answers unless decimals are requested. GDC answers: 3 significant figures.

Instructions. Show your method and use correct mathematical notation. State units in context. Retain unrounded values during GDC calculations. Work on separate paper. In first-principles questions, simplify the difference quotient for $h \neq 0$ before taking the limit.

1. Short response

1. The table gives values of f near $x = 2$. Estimate $\lim_{x \rightarrow 2} f(x)$. Does the table determine $f(2)$?

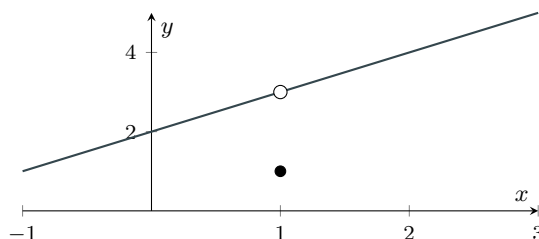
x	1.9	1.99	2.01	2.1
y	3.61	3.9601	4.0401	4.41

2. State the limits as $n \rightarrow \infty$ of (a) $(0.3)^n$; (b) $(-0.5)^n$.

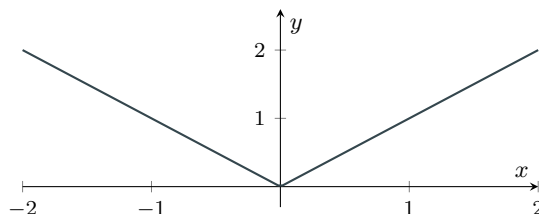
3. Describe the long-term behaviour of (a) 2^n ; (b) $(-1)^n$. Distinguish unbounded growth from oscillation.

4. For $f(x) = 1/x$, describe what happens as x approaches 0 from the right and from the left. Does a finite two-sided limit exist?

5. *Concept exploration.* The graph is $y = x + 2$ except that the point $(1, 3)$ is removed and the filled point is $(1, 1)$. State the limiting height as $x \rightarrow 1$ and the actual value at $x = 1$. Explain the visible break informally.



6. *Concept exploration.* The graph $y = |x|$ has no break at the origin. Explain why there is nevertheless no unique tangent gradient there.



7. Write the definition of $f'(x)$ from first principles. Explain why h cannot be set equal to zero before simplifying the difference quotient.

8. For $f(x) = x^2$, simplify $[f(x+h) - f(x)]/h$ for $h \neq 0$, and hence find $f'(x)$.

9. Use first principles to differentiate $f(x) = -3x + 5$.

10. Use first principles at $x = 1$ to find the gradient of $y = x^2 + 2x$.

11. Find $f^{(3)}(x)$ for $f(x) = 2x^5 - x^3 + 4$.

12. Find d^4y/dx^4 for $y = x^4 - 2x^3$.

13. Find the sixth derivative of $f(x) = 7x^5 - x^2$. Explain your answer.

14. Find the third derivative of $y = \sin(2x)$.

15. Write a formula for the n th derivative of $f(x) = e^{-3x}$, for positive integers n .

2. Extended response

16. Use first principles to find the derivative of $f(x) = 3x^2 - 2x + 4$. Hence find its tangent at $x = 1$.

17. Use first principles to differentiate $f(x) = x^3$. Hence find the normal to the curve at $(1, 1)$.

18. For $f(x) = x^3 - 2x$, derive the difference quotient in terms of x, h and use its limit to find $f'(2)$.

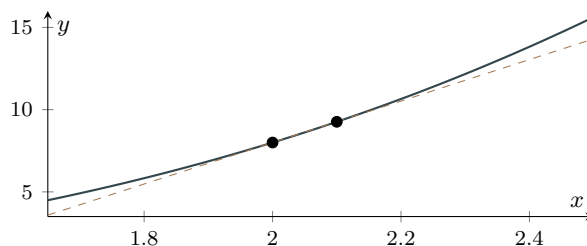
19. Let $f(x) = ax^2 + bx + c$. Use first principles to obtain $f'(x)$. Given $f'(1) = 7$, $f'(3) = 15$ and $f(0) = -1$, find a, b, c .

20. For $f(x) = x^3$, the table gives secant gradients between $x = 2$ and $x = 2 + h$.

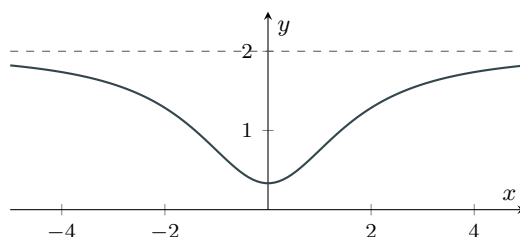
h	-0.01	0.01	0.1
Gradient	11.9401	12.0601	12.61

- (a) Estimate the tangent gradient at $x = 2$.
 (b) Show that the secant gradient is $12 + 6h + h^2$.
 (c) Find the exact error in the secant estimate when $h = 0.1$.

The dashed line illustrates the secant for $h = 0.1$.



21. Find $\lim_{x \rightarrow \infty} \frac{2x^2 + 1}{x^2 + 3}$ by dividing numerator and denominator by x^2 . State the corresponding horizontal asymptote.



22. Compare the sequences $u_n = 5 + (-3/4)^n$ and $v_n = 5 + (-4/3)^n$ as $n \rightarrow \infty$. Describe both the magnitudes and the alternating behaviour.

23. Let $f(x) = x^6 - 5x^4 + 3x^2$.

- (a) Find $f^{(3)}(x)$ and $f^{(4)}(x)$.
 (b) Solve $f^{(3)}(x) = 0$.
 (c) Interpret the sign of $f^{(4)}$ as information about whether $f^{(3)}$ increases or decreases.

24. For $y = e^{2x} \cos x$, find y'' and $y^{(3)}$, showing each use of the product rule.

25. For $y = \ln x$, $x > 0$, find the second, third and fourth derivatives. Conjecture a formula for $y^{(n)}$, where $n \geq 1$.

26. For $f(x) = 1/(x + 2)$, $x \neq -2$, find f' , f'' , $f^{(3)}$ and state a formula for $f^{(n)}$.

27. A function satisfies $f^{(3)}(x) = 24x - 12$, $f''(0) = 6$, $f'(0) = -2$ and $f(0) = 5$. Find $f(x)$.

28. A cubic polynomial has $f(0) = 1$, $f'(0) = 2$, $f''(0) = -6$ and $f^{(3)}(x) = 12$. Find the polynomial and verify all four conditions.

29. Use the repeating derivative patterns to find (a) the 101st derivative of $\sin x$; (b) the 202nd derivative of $\sin x$; (c) the 17th derivative of $\cos x$.

30.  *GDC* Let $f(x) = e^{x^2}$.

- (a) Find $f''(x)$ and $f^{(3)}(x)$ analytically.
 (b) Use your GDC to evaluate $f^{(3)}(0.7)$ to 3 significant figures.
 (c) Explain why $f^{(3)}(0) = 0$.

3. IB exam-style practice

31. Let $f(x) = 2x^2 - 5x + 3$.

- (a) Use first principles to show that $f'(x) = 4x - 5$. [5 marks]
 (b) Find the equation of the tangent parallel to $y = 3x + 7$. [4 marks]

32. Let $f(x) = x^3 - 3x$.

- (a) Find $f'(x)$ from first principles. [5 marks]
 (b) Find and classify the stationary points. [4 marks]
 (c) State $f^{(3)}(x)$. [1 mark]

33. Let $f(x) = (x - 2)^2 + 1$.

- (a) Use first principles to find $f'(x)$. [5 marks]
 (b) Find the normal at $x = 3$. [3 marks]

34. For constants a, b , let $g(x) = ax^3 + b$. It is given that $g'(2) = 24$ and $g(1) = 5$.

- (a) Use first principles to obtain $g'(x)$ in terms of a . [5 marks]
 (b) Find a, b and $g^{(3)}(x)$. [4 marks]

35. Let $f(x) = x^4 + ax^3 + bx^2$. It is given that $f^{(3)}(2) = 0$ and $f''(1) = 0$.

- (a) Find a and b . [5 marks]
 (b) Find all solutions of $f''(x) = 0$. [3 marks]
 (c) State $f^{(4)}(x)$. [1 mark]

36. A polynomial satisfies $f^{(3)}(x) = 48x + 6$, $f''(1) = 12$, $f'(0) = -1$ and $f(0) = 2$.

- (a) Find $f''(x)$. [3 marks]
 (b) Hence find $f(x)$. [5 marks]
 (c) State the degree and the fourth derivative of f . [2 marks]

37. Let $f(x) = 2x^6 - 3x^5 + x^4$.

- (a) Find $f^{(4)}(x)$. [4 marks]
 (b) Find $f^{(5)}(x)$ and $f^{(6)}(x)$. [3 marks]
 (c) Find the least positive integer n for which $f^{(n)}(x) = 0$ for every x . [2 marks]

38. Consider $f(x) = x^8$.

- (a) Find $f^{(5)}(x)$. [3 marks]
- (b) Write a formula for $f^{(n)}(x)$ when $1 \leq n \leq 8$. [2 marks]
- (c) Find $f^{(8)}(0)$ and $f^{(9)}(0)$. [2 marks]

39. Let $f(x) = (2x - 1)^5$.

- (a) Find $f^{(3)}(x)$ and $f^{(4)}(x)$. [4 marks]
- (b) Find the fifth and sixth derivatives. [2 marks]
- (c) Explain why a factor of 2 is introduced at every differentiation. [1 mark]

40. Let $f(x) = e^{-2x} \sin(3x)$.

- (a) Find f' and f'' . [5 marks]
- (b) Show that $f'' + 4f' + 13f = 0$. [2 marks]
- (c) Find $f^{(3)}(0)$. [3 marks]

41. Let $f(x) = xe^x$.

- (a) Find the first three derivatives. [4 marks]
- (b) Conjecture a formula for $f^{(n)}(x)$. [2 marks]
- (c) Differentiate your proposed formula to show that it produces the same pattern for $n + 1$. [2 marks]

42. Let $f(x) = x^2e^x$.

- (a) Find $f', f'', f^{(3)}$. [5 marks]
- (b) Find $f^{(4)}(0)$. [2 marks]
- (c) State a conjecture for $f^{(n)}(x)$. [2 marks]

43. Let $f(x) = \sin^2 x$.

- (a) Express f using $\cos(2x)$. [1 mark]
- (b) Find $f^{(4)}(x)$. [4 marks]
- (c) Find $f^{(5)}(\pi/4)$. [2 marks]

44. A sequence is defined by $u_n = 2 - 3/n$, where $n \geq 1$.

- (a) Find its limit as $n \rightarrow \infty$. [1 mark]
- (b) Show that $u_{n+1} > u_n$. [3 marks]
- (c) Find the least n such that $|u_n - 2| < 0.01$. [3 marks]

45. Let $f(x) = (5x^2 - x)/(2x^2 + 3)$.

- (a) Find the limits as $x \rightarrow +\infty$ and $x \rightarrow -\infty$. [3 marks]
- (b) State the horizontal asymptote. [1 mark]
- (c) Find where the curve crosses this asymptote, showing that crossing does not contradict the limiting behaviour. [3 marks]

46. Let $f(x) = 2x^3 - 2x^2 + 7$.

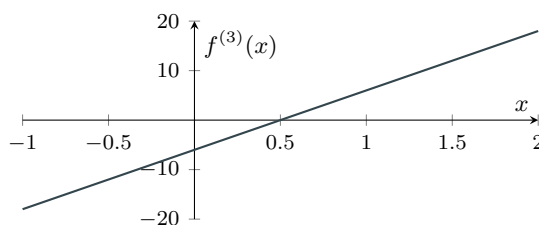
- (a) Use first principles to find $f'(a)$ in terms of a . [6 marks]
- (b) Find all values of a at which the tangent is horizontal. [2 marks]

47.  GDC Let $f(x) = x^4 - 2x^3 + x$. Define $D(h) = [f(1+h) - f(1)]/h$ for $h \neq 0$.

- (a) Use your GDC to find $D(0.1)$ and $D(-0.1)$ to 4 decimal places. [2 marks]
- (b) Show algebraically that $D(h) = -1 + 2h^2 + h^3$. [4 marks]
- (c) Hence find $f'(1)$ and explain why a small nonzero h gives an estimate rather than the exact derivative. [2 marks]

48. A function has $f^{(3)}(x) = 12x - 6$, as shown, with $f''(0) = 2$.

- (a) Find $f''(x)$. [3 marks]
- (b) Find and classify the stationary point of the graph of f'' . [4 marks]
- (c) Explain why f is concave up for every real x . [2 marks]



49. Let $f(x) = x^5 + ax^3 + bx$, with $f^{(3)}(1) = 0$ and $f'(1) = 0$.

- (a) Find a and b . [5 marks]
- (b) Classify the stationary point at $x = 1$. [3 marks]
- (c) Find $f^{(4)}(1)$ and $f^{(5)}(1)$. [2 marks]

50. Let $f(x) = \ln(1 + 2x)$ for $x > -1/2$.

- (a) Find the first three derivatives. [4 marks]
- (b) Write a formula for $f^{(n)}(x)$, $n \geq 1$. [3 marks]
- (c) Find $f^{(4)}(0)$. [2 marks]

Answer key and solution outlines

Equivalent forms and correct alternative methods are accepted. The examination-style questions are original practice items, not official IB questions.

1. Short response

- The values suggest a limit of 4. The value $f(2)$ is not supplied; a limit concerns nearby values.
- Both limits are 0, because the magnitude of each common ratio is less than 1. In (b) the sign alternates while the magnitude tends to zero.
- (a) $2^n \rightarrow +\infty$: unbounded divergence. (b) The values alternate between -1 and 1 : bounded oscillation with no limit.
- $f(x) \rightarrow +\infty$ from the right and $f(x) \rightarrow -\infty$ from the left. There is no finite two-sided limit.
- The limiting height is 3, while $f(1) = 1$. Nearby graph points approach a different height from the filled point, creating a removable break.
- The left branch has gradient -1 and the right branch gradient 1 . The corner prevents a unique derivative even though the graph is continuous.
- $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h)-f(x)}{h}$. The quotient is defined for $h \neq 0$; substituting 0 immediately gives the undefined expression $0/0$.
- $[(x+h)^2 - x^2]/h = (2xh + h^2)/h = 2x + h$. Taking $h \rightarrow 0$ gives $f'(x) = 2x$.
- $\frac{-3(x+h)+5-(-3x+5)}{h} = -3$. Therefore $f'(x) = \lim_{h \rightarrow 0} (-3) = -3$.
- $f(1) = 3$, $f(1+h) = 3 + 4h + h^2$. Thus $f'(1) = \lim_{h \rightarrow 0} (4h + h^2)/h = 4$.
- $f' = 10x^4 - 3x^2$, $f'' = 40x^3 - 6x$, so $f^{(3)} = 120x^2 - 6$.
- The successive derivatives are $4x^3 - 6x^2$, $12x^2 - 12x$, $24x - 12$, 24 .
- $f^{(6)} = 0$: each differentiation lowers the degree, and a polynomial of degree 5 has constant fifth derivative and zero sixth derivative.
- $y' = 2 \cos 2x$, $y'' = -4 \sin 2x$, $y^{(3)} = -8 \cos 2x$.
- $f^{(n)}(x) = (-3)^n e^{-3x}$. Each differentiation multiplies the existing expression by -3 .

2. Extended response

- The difference quotient is $(6xh + 3h^2 - 2h)/h = 6x + 3h - 2$. Taking the limit gives $f' = 6x - 2$. At $x = 1$, $f(1) = 5$ and $f'(1) = 4$, so $y - 5 = 4(x - 1)$, or $y = 4x + 1$.
- $[(x+h)^3 - x^3]/h = 3x^2 + 3xh + h^2 \rightarrow 3x^2$. At $x = 1$ the tangent gradient is 3, so the normal is $y - 1 = -\frac{1}{3}(x - 1)$.
- $[f(x+h) - f(x)]/h = 3x^2 + 3xh + h^2 - 2$. Hence $f'(x) = 3x^2 - 2$ and $f'(2) = 10$.
- The quotient is $2ax + ah + b$, giving $f' = 2ax + b$. Then $2a + b = 7$ and $6a + b = 15$, so $a = 2$, $b = 3$. The value at zero gives $c = -1$.
- The table suggests 12. The quotient is $[(2+h)^3 - 8]/h = 12 + 6h + h^2$. At $h = 0.1$, the estimate exceeds the true gradient by $0.6 + 0.01 = 0.61$.
- The ratio is $\frac{2+1/x^2}{1+3/x^2} \rightarrow 2$. The horizontal asymptote is $y = 2$.
- $u_n \rightarrow 5$ because $(3/4)^n \rightarrow 0$; it approaches from alternating sides. For v_n , the distance from 5 is $(4/3)^n \rightarrow \infty$ and the sign alternates, so it has neither a finite limit nor a single infinite limit.
- $f^{(3)} = 120x^3 - 120x = 120x(x-1)(x+1)$ and $f^{(4)} = 360x^2 - 120$. Zeros of $f^{(3)}$: $-1, 0, 1$. It increases where $|x| > 1/\sqrt{3}$ and decreases where $|x| < 1/\sqrt{3}$.
- $y' = e^{2x}(2 \cos x - \sin x)$. Then $y'' = e^{2x}[2(2 \cos x - \sin x) - 2 \sin x - \cos x] = e^{2x}(3 \cos x - 4 \sin x)$. Differentiating again gives $y^{(3)} = e^{2x}(2 \cos x - 11 \sin x)$.
- $y' = 1/x$, $y'' = -1/x^2$, $y^{(3)} = 2/x^3$, $y^{(4)} = -6/x^4$. The pattern is $y^{(n)} = (-1)^{n-1}(n-1)!/x^n$.
- $f' = -(x+2)^{-2}$, $f'' = 2(x+2)^{-3}$, $f^{(3)} = -6(x+2)^{-4}$. Hence $f^{(n)} = (-1)^n n!/(x+2)^{n+1}$.
- Integrating successively: $f'' = 12x^2 - 12x + 6$, $f' = 4x^3 - 6x^2 + 6x - 2$, and $f = x^4 - 2x^3 + 3x^2 - 2x + 5$. Each given value fixes the next integration constant.

- 28.** $f'' = 12x - 6$, $f' = 6x^2 - 6x + 2$, $f = 2x^3 - 3x^2 + 2x + 1$. Substitution at zero gives 1, 2, -6 respectively, and the third derivative is 12.
- 29.** The cycle has length 4. Since $101 \equiv 1$, $202 \equiv 2$ and $17 \equiv 1 \pmod{4}$, the answers are (a) $\cos x$; (b) $-\sin x$; (c) $-\sin x$.
- 30.** $f' = 2xe^{x^2}$, $f'' = (2 + 4x^2)e^{x^2}$, and $f^{(3)} = (12x + 8x^3)e^{x^2}$. At 0.7, this is 18.2. The third derivative is an odd function and contains the factor x , so its value at zero is 0.

3. IB exam-style practice

- 31.** $[f(x+h) - f(x)]/h = (4xh + 2h^2 - 5h)/h = 4x + 2h - 5 \rightarrow 4x - 5$. Parallel tangents have gradient 3, so $4x - 5 = 3$ gives $x = 2$, $f(2) = 1$. The tangent is $y - 1 = 3(x - 2)$, or $y = 3x - 5$.
- 32.** The quotient is $3x^2 + 3xh + h^2 - 3$, so $f' = 3x^2 - 3$. Stationary points are $(-1, 2)$ and $(1, -2)$. Since $f'' = 6x$, they are a local maximum and minimum respectively. The third derivative is 6.
- 33.** The quotient is $\{(x+h-2)^2 - (x-2)^2\}/h = 2(x-2) + h$, giving $f' = 2x - 4$. At $x = 3$, $f = 2$ and gradient 2. The normal is $y - 2 = -\frac{1}{2}(x - 3)$.
- 34.** The quotient is $a(3x^2 + 3xh + h^2)$, so $g' = 3ax^2$. Thus $12a = 24 \Rightarrow a = 2$. Then $2 + b = 5 \Rightarrow b = 3$. The higher derivatives are $g'' = 12x$ and $g^{(3)} = 12$.
- 35.** $f'' = 12x^2 + 6ax + 2b$ and $f^{(3)} = 24x + 6a$. At 2, $48 + 6a = 0$, so $a = -8$. Then $12 - 48 + 2b = 0$, so $b = 18$. Thus $f'' = 12(x-1)(x-3)$, with zeros 1, 3. The fourth derivative is 24.
- 36.** $f'' = 24x^2 + 6x + C$; $30 + C = 12$, so $C = -18$. Then $f' = 8x^3 + 3x^2 - 18x - 1$, and $f = 2x^4 + x^3 - 9x^2 - x + 2$. Its degree is 4 and $f^{(4)} = 48$.
- 37.** $f^{(4)} = 720x^2 - 360x + 24$, $f^{(5)} = 1440x - 360$, $f^{(6)} = 1440$. The first derivative that is identically zero is the seventh, so $n = 7$.
- 38.** $f^{(5)} = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4x^3 = 6720x^3$. Generally $f^{(n)} = \frac{8!}{(8-n)!}x^{8-n}$. Hence $f^{(8)}(0) = 8! = 40320$ and $f^{(9)}(0) = 0$.
- 39.** $f' = 10(2x-1)^4$, $f'' = 80(2x-1)^3$, $f^{(3)} = 480(2x-1)^2$, $f^{(4)} = 1920(2x-1)$, $f^{(5)} = 3840$, $f^{(6)} = 0$. The chain rule supplies the derivative of $2x-1$, which is 2.
- 40.** $f' = e^{-2x}(3\cos 3x - 2\sin 3x)$ and $f'' = e^{-2x}(-5\sin 3x - 12\cos 3x)$. In the stated sum, the sine coefficients are $-5 - 8 + 13 = 0$ and cosine coefficients $-12 + 12 = 0$. Also $f^{(3)} = e^{-2x}(46\sin 3x + 9\cos 3x)$, so $f^{(3)}(0) = 9$.
- 41.** $f' = e^x(x+1)$, $f'' = e^x(x+2)$, $f^{(3)} = e^x(x+3)$. Conjecture $f^{(n)} = e^x(x+n)$. Differentiating gives $e^x(x+n) + e^x = e^x[x + (n+1)]$, as required.
- 42.** $f' = e^x(x^2 + 2x)$, $f'' = e^x(x^2 + 4x + 2)$, $f^{(3)} = e^x(x^2 + 6x + 6)$. Thus $f^{(4)} = e^x(x^2 + 8x + 12)$ and $f^{(4)}(0) = 12$. The pattern is $f^{(n)} = e^x[x^2 + 2nx + n(n-1)]$.
- 43.** $f = (1 - \cos 2x)/2$. The successive derivatives are $\sin 2x$, $2\cos 2x$, $-4\sin 2x$, $-8\cos 2x$. Hence $f^{(5)} = 16\sin 2x$, giving $f^{(5)}(\pi/4) = 16$.
- 44.** The limit is 2. The difference $u_{n+1} - u_n = 3/n - 3/(n+1) = 3/[n(n+1)] > 0$. For the tolerance, $3/n < 0.01$ requires $n > 300$, so the least integer is 301.
- 45.** Dividing by x^2 gives $(5-1/x)/(2+3/x^2) \rightarrow 5/2$ at both ends. The asymptote is $y = 5/2$. Solve $2(5x^2-x) = 5(2x^2+3)$ to obtain $x = -15/2$. An asymptote describes end behaviour and may be crossed at a finite x .
- 46.** For $h \neq 0$, $[f(a+h) - f(a)]/h = 6a^2 + 6ah + 2h^2 - 4a - 2h$. Taking the limit gives $f'(a) = 6a^2 - 4a = 2a(3a-2)$. Thus $a = 0$ or $a = 2/3$.
- 47.** $D(0.1) = -0.9790$ and $D(-0.1) = -0.9810$. Expanding $f(1+h) - f(1)$ gives $-h + 2h^3 + h^4$, so $D(h) = -1 + 2h^2 + h^3$. The limit is -1. For nonzero h , the extra error term $2h^2 + h^3$ generally remains.
- 48.** $f'' = 6x^2 - 6x + 2$. Its derivative is $f^{(3)} = 12x - 6$, which changes $- \rightarrow +$ at $x = 1/2$, giving minimum $(1/2, 1/2)$. Hence $f''(x) \geq 1/2 > 0$, so f is concave up everywhere.
- 49.** $f^{(3)} = 60x^2 + 6a$, so $a = -10$. Then $f'(1) = 5 + 3a + b = 0$ gives $b = 25$. Now $f''(1) = 20 + 6a = -40 < 0$, so $(1, 16)$ is a local maximum. Finally $f^{(4)}(1) = 120$ and $f^{(5)}(1) = 120$.
- 50.** $f' = 2/(1+2x)$, $f'' = -4/(1+2x)^2$, $f^{(3)} = 16/(1+2x)^3$. The pattern is $f^{(n)} = (-1)^{n-1}(n-1)!2^n/(1+2x)^n$. Thus $f^{(4)}(0) = -96$.