



Limits and l'Hôpital's Rule

IB Mathematics AA HL | AHL 5.13

50 questions: indeterminate forms, repeated differentiation and series

Learning goal	Evaluate limits and justify the choice of method.
Syllabus link	AHL 5.13: l'Hôpital's rule, repeated use, and limits using Maclaurin series; links to horizontal asymptotes and continuity.
Structure	15 short-response, 15 extended-response and 20 IB-style questions.
Technology	Non-calculator throughout. All trigonometric arguments are in radians.

Instructions. Work on separate paper. Give exact values and show sufficient working. Before each use of l'Hôpital's rule, check for a quotient of type $0/0$ or ∞/∞ (including signed infinities). Use the requested method where specified. Standard Maclaurin series from the formula booklet may be used; $\dots$ denotes higher-order terms.

1. Short response | Questions 1–15

1. State the indeterminate form obtained as $x \rightarrow 0$, then evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x}$.

2. Use l'Hôpital's rule to evaluate $\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{x}$.

3. Evaluate $\lim_{x \rightarrow 0} \frac{\ln(1 + 3x)}{x}$.

4. Evaluate $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3}$.

5. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2}$.

6. Evaluate $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x}$.

7. Evaluate $\lim_{x \rightarrow \infty} \frac{x}{e^{2x}}$.



1. Short response | continued

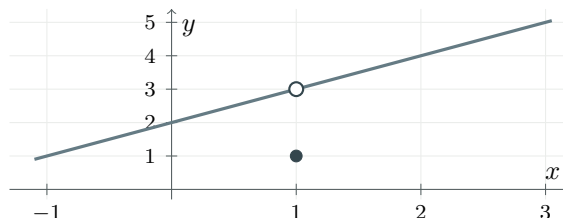
8. Evaluate $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2}$.

9. Evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2}$.

10. Evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x}{x^3}$.

11. Find $\lim_{x \rightarrow 1} \frac{x^2 + 4}{x + 1}$. Explain why l'Hôpital's rule cannot be applied directly.

12. The graph of $y = f(x)$ is shown. An open circle is an excluded point; a filled circle is included. Find $\lim_{x \rightarrow 1} f(x)$ and $f(1)$.



13. Using $e^x = 1 + x + \frac{x^2}{2} + \dots$ and $\sin 3x = 3x - \frac{(3x)^3}{6} + \dots$, evaluate $\lim_{x \rightarrow 0} \frac{e^x - 1}{\sin 3x}$.

14. Using $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \dots$, evaluate $\lim_{x \rightarrow 0} \frac{\cos x - 1 + \frac{1}{2}x^2}{x^4}$.

15. Evaluate $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}$.

**2. Extended response | Questions 16–30**

16. Consider $L = \lim_{x \rightarrow 0} \frac{\sqrt{1+4x} - 1}{\sin 3x}$.

(a) Use l'Hôpital's rule to find L .

(b) Verify your result by rationalizing the numerator and using $\lim_{u \rightarrow 0} \frac{\sin u}{u} = 1$.

17. Consider $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1}$.

(a) Check the indeterminate form and evaluate the limit.

(b) For $x > 0$ close to, but different from, 1, determine the sign of the quotient on each side of 1. Explain why this is consistent with your answer.

18. Evaluate $\lim_{x \rightarrow 0} \frac{e^{2x} - 1 - 2x}{x^2}$ by repeated use of l'Hôpital's rule. Show the form after the first differentiation and state the number of applications needed.

19. Evaluate $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$ in two ways:

(a) by repeated use of l'Hôpital's rule;

(b) using $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \dots$.

20. Let $L = \lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2}$.

(a) Evaluate L using l'Hôpital's rule.

(b) Verify using $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$.

(c) Hence determine whether $\ln(1+x)$ lies above or below x for sufficiently small nonzero x .

21. Evaluate $\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{1 - \cos 2x}$.

(a) Use l'Hôpital's rule, checking the form before each application.

(b) Verify your answer using the first nonzero terms of the numerator and denominator.

22. Let $f(x) = \frac{x^2}{e^x}$, for $x > 0$.

(a) Find $\lim_{x \rightarrow \infty} f(x)$ by repeated use of l'Hôpital's rule.

(b) State the corresponding horizontal asymptote and interpret the result as a comparison of growth rates.

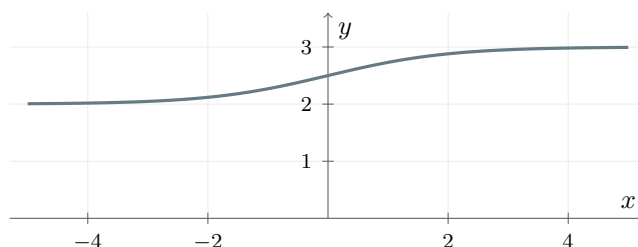
23. Evaluate $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x}$. After one application of l'Hôpital's rule, rewrite the resulting expression before deciding whether a further application is valid.



24. Consider $\lim_{x \rightarrow 0^+} x \ln x$.

- (a) Explain why l'Hôpital's rule cannot be applied directly to the product.
(b) Rewrite the product as a quotient of indeterminate type, then evaluate the limit.

25. The graph of $f(x) = \frac{3e^x + 2}{e^x + 1}$ is shown.



- (a) Find $\lim_{x \rightarrow \infty} f(x)$ and $\lim_{x \rightarrow -\infty} f(x)$.
(b) Explain at which end l'Hôpital's rule applies directly.
(c) Add both horizontal asymptotes to the graph and label their equations.

26. The function h is defined by $h(x) = \frac{\sin(ax)}{x}$ for $x \neq 0$, and $h(0) = 4$, where $a \in \mathbb{R}$.

- (a) Find a so that h is continuous at 0.
(b) With this value of a , suppose that $h(0)$ is changed to 7. State the limit as $x \rightarrow 0$ and explain whether h remains continuous.

27. For constants $a, b \in \mathbb{R}$, the limit $\lim_{x \rightarrow 0} \frac{e^{2x} - a - bx}{x^2}$ is finite.

- (a) Find a and b , explaining each condition.
(b) Evaluate the resulting limit.

28. You are given $\tan x = x + \frac{x^3}{3} + \dots$ and $\sin x = x - \frac{x^3}{6} + \dots$.

- (a) Find the first nonzero term of $\tan x - \sin x$.
(b) Hence evaluate $\lim_{x \rightarrow 0} \frac{\tan x - \sin x}{x^3}$.
(c) Explain why retaining only the linear terms would be insufficient.

29. The functions f and g have continuous second derivatives near $x = 2$. Their values are shown below.

	Value at 2	First derivative at 2	Second derivative at 2
f	0	0	-12
g	0	0	4

Assume the hypotheses for l'Hôpital's rule hold in a punctured neighbourhood of 2.

- (a) Explain why one differentiation does not resolve the limit.
(b) Find $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$.



30. A student writes

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \lim_{x \rightarrow 0} \frac{2x \cos 2x - \sin 2x}{x^2} = 0.$$

- (a) Identify the error in the first equality.
- (b) Give a correct solution using l'Hôpital's rule.
- (c) Explain the difference between differentiating a quotient and applying l'Hôpital's rule.

3. IB exam-style / Paper 1 | Questions 31–50

31. [Maximum mark: 6]

Consider limits involving $e^{3x} - 1$.

- (a) Use l'Hôpital's rule to find $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x}$. [3 marks]
- (b) Hence, or otherwise, find $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{\sin 2x}$. [3 marks]

32. [Maximum mark: 6]

Let $L = \lim_{x \rightarrow 0} \frac{1 - \cos 4x}{x^2}$.

- (a) Find L by repeated use of l'Hôpital's rule. [4 marks]
- (b) Verify the result using the Maclaurin series of $\cos 4x$. [2 marks]

33. [Maximum mark: 6]

The limit $L = \lim_{x \rightarrow 0} \frac{\ln(1 + 2x) - mx}{x^2}$ is finite, where $m \in \mathbb{R}$.

- (a) Find m . [2 marks]
- (b) For this value of m , use l'Hôpital's rule to determine L . [4 marks]

34. [Maximum mark: 7]

A function is defined by

$$f(x) = \begin{cases} \frac{e^x - 1 - x}{x^2}, & x \neq 0, \\ c, & x = 0. \end{cases}$$

- (a) Find c so that f is continuous at 0. [3 marks]
- (b) For this value of c , use a Maclaurin series to evaluate $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x}$. State what your answer represents. [4 marks]

35. [Maximum mark: 7]

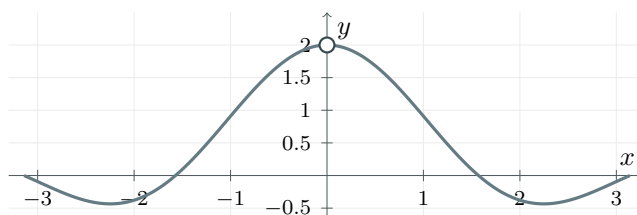
Consider the expression $\sin 3x - 3x$.

- (a) Use l'Hôpital's rule to evaluate $\lim_{x \rightarrow 0} \frac{\sin 3x - 3x}{x^3}$. [4 marks]
- (b) Hence, or otherwise, find $\lim_{x \rightarrow 0} \frac{\sin 3x - 3x}{x(1 - \cos x)}$. [3 marks]



36. [Maximum mark: 8]

The curve $y = \frac{\sin 2x}{x}$, for $x \neq 0$, is shown with its point at $x = 0$ omitted.



Define $h(x) = \frac{\sin 2x}{x}$ for $x \neq 0$ and $h(0) = k$.

- (a) Use l'Hôpital's rule to find k so that h is continuous at 0. [3 marks]
- (b) Describe the change to the graph when this value is assigned. [2 marks]
- (c) Use the Maclaurin series of $\sin 2x$ to find $\lim_{x \rightarrow 0} \frac{h(x) - k}{x^2}$. [3 marks]

37. [Maximum mark: 6]

Let $f(x) = \frac{2x^2 - 3x + 1}{x^2 + x - 2}$, where $x \neq -2, 1$.

- (a) Use l'Hôpital's rule to find $\lim_{x \rightarrow 1} f(x)$. [3 marks]
- (b) Verify your answer by factorizing and simplifying $f(x)$. [2 marks]
- (c) State the coordinates of the removable hole at $x = 1$. [1 marks]

38. [Maximum mark: 7]

Consider the function $\sqrt{9+x}$ near $x = 0$.

- (a) Evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3}{x}$. [3 marks]
- (b) By repeated use of l'Hôpital's rule, evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3 - \frac{x}{6}}{x^2}$. [4 marks]

39. [Maximum mark: 6]

Let $f(x) = \frac{x^2 + 3x}{e^x}$, for $x > 0$.

- (a) Find $\lim_{x \rightarrow \infty} f(x)$ using l'Hôpital's rule. [4 marks]
- (b) State the horizontal asymptote of the graph as $x \rightarrow \infty$. [1 marks]
- (c) Deduce $\lim_{x \rightarrow \infty} \frac{e^x}{x^2 + 3x}$. [1 marks]

40. [Maximum mark: 7]

Let $x > 1$.

- (a) Find $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}}$. [3 marks]
- (b) Hence, or otherwise, find $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{\sqrt{x}}$. [4 marks]



41. [Maximum mark: 6]

The constants $a, b \in \mathbb{R}$ satisfy

$$\lim_{x \rightarrow 0} \frac{\sin(ax) - bx}{x^3} = -\frac{4}{3}.$$

(a) Show that $b = a$.

[3 marks]

(b) Find a and b .

[3 marks]

42. [Maximum mark: 6]

For $k \in \mathbb{R}$, define $L(k) = \lim_{x \rightarrow 0} \frac{e^{kx} - 1 - kx}{1 - \cos 2x}$.

(a) Show that $L(k) = \frac{k^2}{4}$.

[4 marks]

(b) Find all values of k for which $L(k) = 9$.

[2 marks]

43. [Maximum mark: 7]

You are given

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + \cdots, \quad \cos x = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots.$$

(a) Find the terms in x^3 and x^5 in $\sin x - x \cos x$.

[4 marks]

(b) Hence evaluate $\lim_{x \rightarrow 0} \frac{\sin x - x \cos x - \frac{x^3}{3}}{x^5}$.

[3 marks]

44. [Maximum mark: 7]

A model for the mass collected during the first t seconds is $M(t) = 80(1 - e^{-t/4})$ grams, where $t \geq 0$. The average collection rate is $A(t) = \frac{M(t)}{t}$ for $t > 0$.

(a) Use l'Hôpital's rule to find $\lim_{t \rightarrow 0^+} A(t)$, including units.

[3 marks]

(b) Find $M'(0)$ and explain its connection with part (a).

[2 marks]

(c) Find $\lim_{t \rightarrow \infty} A(t)$. Explain why l'Hôpital's rule is not directly applicable to $M(t)/t$ in this limit.

[2 marks]

45. [Maximum mark: 7]

A function is given by $R(x) = \frac{4e^{2x} + 1}{2e^{2x} + 3}$ for $x \in \mathbb{R}$.

(a) Find $\lim_{x \rightarrow \infty} R(x)$ and $\lim_{x \rightarrow -\infty} R(x)$.

[4 marks]

(b) State both horizontal asymptotes.

[2 marks]

(c) A student applies l'Hôpital's rule at both ends and obtains the same limit. Explain the error.

[1 marks]

46. [Maximum mark: 8]

The constants $a, b \in \mathbb{R}$ are chosen so that $L = \lim_{x \rightarrow 0} \frac{e^x - \cos x - ax - bx^2}{x^3}$ is finite.

(a) Use Maclaurin series to find a and b .

[4 marks]

(b) Find L .

[2 marks]

(c) State how many successive applications of l'Hôpital's rule would be needed to obtain L for these values.

[2 marks]



47. [Maximum mark: 8]

The limit $\lim_{x \rightarrow 0} \frac{\ln(1+ax) - bx}{x^2} = -8$, where $a, b \in \mathbb{R}$.

- (a) Show that $b = a$. [3 marks]
(b) Find all ordered pairs (a, b) . [3 marks]
(c) For each pair, state the real domain of $\ln(1+ax)$. [2 marks]
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48. [Maximum mark: 7]

You are given the expansion $\sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} + \dots$ near 0.

- (a) Find the terms up to and including x^3 in $\sqrt{1+2x}$. [3 marks]
(b) Hence evaluate $\lim_{x \rightarrow 0} \frac{\sqrt{1+2x} - 1 - x + \frac{x^2}{2}}{\sin^3 x}$. [4 marks]
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49. [Maximum mark: 8]

Let $p \in \mathbb{R}$ and $F_p(x) = \frac{e^x - 1 - x}{x^p}$ for $x > 0$.

- (a) Show that $\lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{x^2} = \frac{1}{2}$. [3 marks]
(b) Hence determine $\lim_{x \rightarrow 0^+} F_p(x)$ in each case: $p < 2$, $p = 2$, and $p > 2$. [3 marks]
(c) State all values of p for which F_p has a continuous extension to $x = 0$, and give the required value at 0. [2 marks]
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50. [Maximum mark: 10]

The function f is defined for $x \neq 0$ by $f(x) = \frac{\ln(1+x) - x}{1 - \cos x}$, with $-1 < x < 1$.

- (a) Use l'Hôpital's rule to find $\lim_{x \rightarrow 0} f(x)$. [4 marks]
(b) Extend f continuously to 0. Using Maclaurin series, show that near 0, $f(x) = -1 + \frac{2}{3}x +$
terms of degree at least 2. [4 marks]
(c) Find the equation of the tangent to the extended curve at $x = 0$. [2 marks]



Answer key and solution outlines

The symbol $\stackrel{H}{=}$ indicates one application of l'Hôpital's rule. Equalities relate limits, not the original functions. In each use below, differentiation is valid near the limiting point and the derivative quotient has the stated limit.

Section 1: Short response

1. The form is $0/0$. Then $\lim_{x \rightarrow 0} \frac{\sin 3x}{5x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \cos 3x}{5} = \boxed{\frac{3}{5}}$.
2. The form is $0/0$. $\lim_{x \rightarrow 0} \frac{e^{4x} - 1}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} 4e^{4x} = \boxed{4}$.
3. The form is $0/0$. $\lim_{x \rightarrow 0} \frac{\ln(1+3x)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3}{1+3x} = \boxed{3}$.
4. The form is $0/0$. $\lim_{x \rightarrow 3} \frac{x^2 - 9}{x - 3} \stackrel{H}{=} \lim_{x \rightarrow 3} 2x = \boxed{6}$. Alternatively, cancel $x - 3$ for $x \neq 3$.
5. Both successive quotients have form $0/0$: $\lim_{x \rightarrow 0} \frac{1 - \cos 2x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2 \sin 2x}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{4 \cos 2x}{2} = \boxed{2}$.
6. The form is $0/0$. $\lim_{x \rightarrow 0} \frac{\sin 5x}{\sin 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{5 \cos 5x}{2 \cos 2x} = \boxed{\frac{5}{2}}$.
7. The form is ∞/∞ . $\lim_{x \rightarrow \infty} \frac{x}{e^{2x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1}{2e^{2x}} = \boxed{0}$.
8. The form is ∞/∞ . $\lim_{x \rightarrow \infty} \frac{\ln x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{2x} = \lim_{x \rightarrow \infty} \frac{1}{2x^2} = \boxed{0}$.
9. The initial form and the form after one application are $0/0$: $\lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$.
10. Using $\sin x = x - x^3/6 + \dots$, the numerator is $-x^3/6 + \dots$. After division by x^3 , the limit is $\boxed{-\frac{1}{6}}$.
11. Direct substitution gives $\boxed{\frac{5}{2}}$. Neither $0/0$ nor ∞/∞ occurs, so the hypotheses for direct use of l'Hôpital's rule are not met.
12. From both sides, the graph approaches the open point $(1, 3)$, so $\boxed{\lim_{x \rightarrow 1} f(x) = 3}$. The filled point gives $\boxed{f(1) = 1}$. The limit need not equal the function value.
13. Using the supplied series, $\frac{e^x - 1}{\sin 3x} = \frac{x + x^2/2 + \dots}{3x - (27/6)x^3 + \dots} = \frac{1 + x/2 + \dots}{3 - (9/2)x^2 + \dots} \rightarrow \boxed{\frac{1}{3}}$.
14. The constant and quadratic terms cancel: $\frac{\cos x - 1 + x^2/2}{x^4} = \frac{x^4/24 + \dots}{x^4} \rightarrow \boxed{\frac{1}{24}}$.
15. The form is $0/0$. $\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1}{x} = \boxed{1}$.



Section 2: Extended response

16. (a) The form is $0/0$:

$$L \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2/\sqrt{1+4x}}{3 \cos 3x} = \boxed{\frac{2}{3}}.$$

(b) Rationalizing gives $\frac{4x}{(\sqrt{1+4x}+1)\sin 3x}$. Since $x/\sin 3x \rightarrow 1/3$, the limit is $4/(2 \cdot 3) = \boxed{2/3}$.

17. (a) The form is $0/0$: $\lim_{x \rightarrow 1} \frac{\ln x}{x^2 - 1} \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{1/x}{2x} = \boxed{\frac{1}{2}}$. (b) On $0 < x < 1$, both numerator and denominator are negative; on $x > 1$, both are positive. The quotient is positive on both sides, consistent with a positive limit.

18. The form is initially $0/0$. After one application it remains $0/0$:

$$\lim_{x \rightarrow 0} \frac{e^{2x} - 1 - 2x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2e^{2x} - 2}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{4e^{2x}}{2} = \boxed{2}.$$

Two applications are needed.

19. (a) Each quotient before differentiation has form $0/0$:

$$\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1 - \cos x}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\sin x}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{\cos x}{6} = \boxed{\frac{1}{6}}.$$

(b) $x - \sin x = x^3/6 - x^5/120 + \dots$. Divide by x^3 and let $x \rightarrow 0$ to obtain $\boxed{1/6}$.

20. (a) The initial form is $0/0$. One application, then simplification for $x \neq 0$, gives

$$L \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/(1+x) - 1}{2x} = \lim_{x \rightarrow 0} \frac{-1}{2(1+x)} = \boxed{-\frac{1}{2}}.$$

(b) $\ln(1+x) - x = -x^2/2 + x^3/3 - \dots$, so the same limit follows. (c) The quotient tends to a negative number, and $x^2 > 0$, so $\ln(1+x) < x$ sufficiently close to 0, with $x \neq 0$.

21. (a) The first two quotients have form $0/0$:

$$\lim_{x \rightarrow 0} \frac{1 - \cos 3x}{1 - \cos 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \sin 3x}{2 \sin 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{9 \cos 3x}{4 \cos 2x} = \boxed{\frac{9}{4}}.$$

(b) The leading terms are $9x^2/2$ and $2x^2$; their ratio tends to $\boxed{9/4}$.

22. (a) The first two quotients have form ∞/∞ : $\lim_{x \rightarrow \infty} \frac{x^2}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2x}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = \boxed{0}$. (b) The asymptote is $\boxed{y = 0}$. Exponential growth dominates quadratic growth: the ratio x^2/e^x tends to zero.

23. The original form is ∞/∞ :

$$\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2 \ln x}{x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{x} = \boxed{0}.$$

The rewritten quotient $2 \ln x/x$ also has form ∞/∞ , allowing the second application.

24. (a) The expression is a product of type $0 \cdot (-\infty)$; the rule applies to suitable quotients. (b) Write $x \ln x = \frac{\ln x}{1/x}$, of type $-\infty/\infty$ as $x \rightarrow 0^+$. Then $\lim_{x \rightarrow 0^+} \frac{\ln x}{1/x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2} = \lim_{x \rightarrow 0^+} (-x) = \boxed{0}$.

25. (a) At $+\infty$ the form is ∞/∞ , so $\lim_{x \rightarrow \infty} f(x) \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{3e^x}{e^x} = \boxed{3}$. At $-\infty$, $e^x \rightarrow 0$, so $f(x) \rightarrow 2/1 = \boxed{2}$. (b) The rule applies directly only at $+\infty$. (c) Draw horizontal dashed lines at $\boxed{y = 3}$ and $\boxed{y = 2}$.

26. (a) The form is $0/0$: $\lim_{x \rightarrow 0} \frac{\sin(ax)}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} a \cos(ax) = a$. Continuity requires this limit to equal $h(0) = 4$, hence $\boxed{a = 4}$. (b) The limit remains $\boxed{4}$, but $h(0) = 7 \neq 4$, so the function is no longer continuous at 0.



Section 2: Extended response (continued)

27. (a) A finite limit first requires $1 - a = 0$, so $a = 1$. Using $e^{2x} = 1 + 2x + 2x^2 + \dots$, the quotient is $(2 - b)/x + 2 + \dots$. Thus $b = 2$ is also required. **(b)** The resulting limit is $\boxed{2}$, with $\boxed{a = 1, b = 2}$.

28. (a) $\tan x - \sin x = (1/3 + 1/6)x^3 + \dots = \boxed{x^3/2} + \dots$. **(b)** Division by x^3 gives the limit $\boxed{1/2}$. **(c)** The linear terms cancel; the first surviving terms determine the limit.

29. (a) Initially $f(2) = g(2) = 0$. After one differentiation, $f'(2) = g'(2) = 0$ still gives $0/0$. **(b)** Using the stated hypotheses twice, $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)} \stackrel{H}{=} \lim_{x \rightarrow 2} \frac{f'(x)}{g'(x)} \stackrel{H}{=} \lim_{x \rightarrow 2} \frac{f''(x)}{g''(x)} = \frac{-12}{4} = \boxed{-3}$.

30. (a) The student replaces a limit of a function by a limit of its derivative, using the quotient rule. These limits need not agree. **(b)** The original form is $0/0$: $\lim_{x \rightarrow 0} \frac{\sin 2x}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2 \cos 2x}{1} = \boxed{2}$. **(c)** For the appropriate indeterminate form, l'Hôpital's rule uses f'/g' , whereas the derivative of f/g is $(f'g - fg')/g^2$.

Section 3: IB exam-style / Paper 1

31. (a) The form is $0/0$: $\lim_{x \rightarrow 0} \frac{e^{3x} - 1}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} 3e^{3x} = \boxed{3}$. **(b)** Since $x/\sin 2x \rightarrow 1/2$, the required limit is $3 \cdot 1/2 = \boxed{3/2}$. Equivalently, apply the rule once to obtain $3e^{3x}/(2 \cos 2x)$.

32. (a) The form is $0/0$ before each application: $L \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{4 \sin 4x}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{16 \cos 4x}{2} = \boxed{8}$. **(b)** $\cos 4x = 1 - 8x^2 + \dots$, so $(1 - \cos 4x)/x^2 = 8 + \dots \rightarrow \boxed{8}$.

33. (a) Since $\ln(1 + 2x) = 2x - 2x^2 + \dots$, the quotient is $(2 - m)/x - 2 + \dots$. A finite limit requires $\boxed{m = 2}$. **(b)** For $m = 2$, the form remains $0/0$ after the first application:

$$L \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{2/(1 + 2x) - 2}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-4/(1 + 2x)^2}{2} = \boxed{-2}.$$

34. (a) Continuity requires c to equal the limit. Two applications, each to $0/0$, give $c = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{e^x}{2} = \boxed{\frac{1}{2}}$. **(b)** From $e^x = 1 + x + x^2/2 + x^3/6 + \dots$, we get $f(x) = 1/2 + x/6 + \dots$. Thus $\frac{f(x) - f(0)}{x} = 1/6 + \dots \rightarrow \boxed{\frac{1}{6}}$. This is $f'(0)$, the tangent gradient at $(0, 1/2)$.

35. (a) Three applications to $0/0$ give

$$\lim_{x \rightarrow 0} \frac{\sin 3x - 3x}{x^3} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{3 \cos 3x - 3}{3x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-9 \sin 3x}{6x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-27 \cos 3x}{6} = \boxed{-\frac{9}{2}}.$$

(b) Rewrite as $\frac{\sin 3x - 3x}{x^3} \cdot \frac{x^2}{1 - \cos x}$. The limits are $-9/2$ and 2 , so the required limit is $\boxed{-9}$.

36. (a) The form is $0/0$: $k = \lim_{x \rightarrow 0} \frac{\sin 2x}{x} \stackrel{H}{=} \lim_{x \rightarrow 0} 2 \cos 2x = \boxed{2}$. **(b)** Replace the open circle at $(0, 2)$ by a filled circle; the other points are unchanged. **(c)** $\sin 2x = 2x - (4/3)x^3 + \dots$, so $h(x) = 2 - (4/3)x^2 + \dots$. Therefore $(h(x) - 2)/x^2 \rightarrow \boxed{-4/3}$.

37. (a) The form at 1 is $0/0$: $\lim_{x \rightarrow 1} f(x) \stackrel{H}{=} \lim_{x \rightarrow 1} \frac{4x - 3}{2x + 1} = \boxed{\frac{1}{3}}$. **(b)** $f(x) = \frac{(2x - 1)(x - 1)}{(x + 2)(x - 1)} = \frac{2x - 1}{x + 2}$ on the original domain. Letting $x \rightarrow 1$ again gives $1/3$. **(c)** The hole is at $\boxed{(1, \frac{1}{3})}$.



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38. (a) The form is $0/0$; one application gives $\lim_{x \rightarrow 0} \frac{1}{2\sqrt{9+x}} = \frac{1}{6}$. (b) The first two quotients have form $0/0$:

$$\lim_{x \rightarrow 0} \frac{\sqrt{9+x} - 3 - x/6}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/(2\sqrt{9+x}) - 1/6}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-1/(4(9+x)^{3/2})}{2} = \frac{-1}{216}.$$

39. (a) The form is ∞/∞ before each application: $\lim_{x \rightarrow \infty} f(x) \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2x+3}{e^x} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2}{e^x} = 0$. (b) The asymptote is $y = 0$. (c) Since $f(x) > 0$ for $x > 0$ and $f(x) \rightarrow 0$, its reciprocal tends to $+\infty$.

40. (a) The form is ∞/∞ : $\lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{1/x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{2}{\sqrt{x}} = 0$. (b) Again the form is ∞/∞ ; after one application, $\lim_{x \rightarrow \infty} \frac{(\ln x)^2}{\sqrt{x}} \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{2 \ln x / x}{1/(2\sqrt{x})} = \lim_{x \rightarrow \infty} \frac{4 \ln x}{\sqrt{x}} = 0$ by part (a).

41. (a) Expand $\sin(ax) = ax - a^3x^3/6 + \dots$. Then the quotient is $(a-b)/x^2 - a^3/6 + \dots$. A finite limit requires $b = a$. (b) The limit becomes $-a^3/6 = -4/3$, so $a^3 = 8$. Hence $a = b = 2$.

42. (a) Both successive quotients have form $0/0$:

$$L(k) \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{ke^{kx} - k}{2 \sin 2x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{k^2 e^{kx}}{4 \cos 2x} = \frac{k^2}{4}.$$

This also holds for $k = 0$, when the original numerator is identically zero. (b) $k^2/4 = 9$ gives $k = \pm 6$.

43. (a) Subtract the series term by term:

$$\sin x - x \cos x = \left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) - \left(x - \frac{x^3}{2} + \frac{x^5}{24}\right) + \dots = \frac{x^3}{3} - \frac{x^5}{30} + \dots$$

(b) After subtracting $x^3/3$ and dividing by x^5 , the limit is $-1/30$.

44. (a) As $t \rightarrow 0^+$, the form is $0/0$: $\lim_{t \rightarrow 0^+} \frac{80(1 - e^{-t/4})}{t} \stackrel{H}{=} \lim_{t \rightarrow 0^+} 20e^{-t/4} = 20 \text{ g s}^{-1}$. (b) $M'(t) = 20e^{-t/4}$, so $M'(0) = 20 \text{ g s}^{-1}$. The average rate over an interval beginning at 0 approaches the initial instantaneous rate. (c) Since $M(t) \rightarrow 80$ and $t \rightarrow \infty$, $A(t) \rightarrow 0 \text{ g s}^{-1}$. A finite numerator over an unbounded denominator is not an indeterminate form.

45. (a) At $+\infty$, the form is ∞/∞ : $\lim_{x \rightarrow \infty} R(x) \stackrel{H}{=} \lim_{x \rightarrow \infty} \frac{8e^{2x}}{4e^{2x}} = 2$. At $-\infty$, $e^{2x} \rightarrow 0$, so $R(x) \rightarrow 1/3$. (b) The asymptotes are $y = 2$ and $y = 1/3$. (c) At $-\infty$ the original quotient tends to $1/3$ directly; it is not of type $0/0$ or ∞/∞ .

46. (a) Subtracting the standard series gives $e^x - \cos x = x + x^2 + x^3/6 + \dots$. The quotient is $(1-a)/x^2 + (1-b)/x + 1/6 + \dots$. Both singular terms must vanish, so $a = 1$, $b = 1$. (b) The surviving constant gives $L = 1/6$. (c) Three applications. The numerator and denominator derivatives are

$$(e^x + \sin x - 1 - 2x, 3x^2), \quad (e^x + \cos x - 2, 6x), \quad (e^x - \sin x, 6).$$

The original quotient and the first two derived quotients all have form $0/0$.

47. (a) Since $\ln(1+ax) = ax - a^2x^2/2 + \dots$, the quotient is $(a-b)/x - a^2/2 + \dots$. Finiteness requires $b = a$. (b) $-a^2/2 = -8$ gives $a^2 = 16$, hence $(a, b) = (4, 4)$ or $(-4, -4)$. (c) The logarithm requires $1 + ax > 0$: $x > -1/4$ for $(4, 4)$ and $x < 1/4$ for $(-4, -4)$.

48. (a) Replace x by $2x$ in the supplied series: $\sqrt{1+2x} = 1 + x - \frac{x^2}{2} + \frac{x^3}{2} + \dots$. (b) The numerator is $x^3/2 + \dots$, and $\sin^3 x = x^3(1 + \dots)$. After cancelling x^3 , the limit is $1/2$.



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49. (a) The initial and first derived quotients have form $0/0$: $\lim_{x \rightarrow 0^+} \frac{e^x - 1 - x}{x^2} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{e^x - 1}{2x} \stackrel{H}{=} \lim_{x \rightarrow 0^+} \frac{e^x}{2} =$

$\boxed{1/2}$. (b) Write $F_p(x) = \frac{e^x - 1 - x}{x^2} x^{2-p}$. As $x \rightarrow 0^+$,

$$\lim F_p(x) = \begin{cases} 0, & p < 2, \\ 1/2, & p = 2, \\ +\infty, & p > 2. \end{cases}$$

(c) A finite right-hand limit gives continuity at the endpoint. Thus $\boxed{p \leq 2}$: set $F_p(0) = 0$ if $p < 2$, and $F_2(0) = 1/2$.

50. (a) The form is $0/0$ initially and after one application:

$$\lim_{x \rightarrow 0} f(x) \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{1/(1+x) - 1}{\sin x} \stackrel{H}{=} \lim_{x \rightarrow 0} \frac{-1/(1+x)^2}{\cos x} = \boxed{-1}.$$

(b) Set $f(0) = -1$. Then

$$f(x) = \frac{-x^2/2 + x^3/3 + \dots}{x^2/2 - x^4/24 + \dots} = \frac{-1 + (2/3)x + \dots}{1 - x^2/12 + \dots} = -1 + \frac{2}{3}x + \text{terms of degree at least 2}.$$

(c) Therefore $f'(0) = 2/3$. The tangent through $(0, -1)$ is $\boxed{y = \frac{2}{3}x - 1}$.