



Implicit Differentiation

Related Rates & Optimisation

IB Mathematics AA HL | AHL 5.14

50 questions: implicit curves, changing quantities and optimal designs

Learning goal	Differentiate implicit relations, connect rates and justify optimal values.
Syllabus link	AHL 5.14: implicit differentiation; related rates of change; optimisation using the chain rule or implicit constraints, including endpoint optima.
Structure	15 short-response, 15 extended-response and 20 IB-style questions.
Technology	Non-calculator throughout. All angles are measured in radians.

Instructions. Work on separate paper and give exact answers. Differentiate before substituting instantaneous values. Include units and interpret the sign of a rate. Use calculus in optimisation questions; state the feasible domain and justify each optimum, checking endpoints or limiting behaviour when relevant.

1. Short response | Questions 1–15

- Given $x^2 + y^2 = 25$, find $\frac{dy}{dx}$ in terms of x and y .
- Given $x^3 + y^3 = 9xy$, find $\frac{dy}{dx}$ in terms of x and y .
- Given $\ln y + xy = 2$, where $y > 0$, find $\frac{dy}{dx}$ in terms of x and y .
- Given $x^2 + xy + y^2 = 7$, find $\frac{dy}{dx}$.
- The relation $e^y + xy = 5$ defines y implicitly as a function of x . Find $\frac{dy}{dx}$.
- Given $x + \sin y = 2$, find $\frac{dy}{dx}$.
- Find the equation of the tangent to $x^2 + y^2 = 25$ at $(3, 4)$.
- Find the equation of the normal to $x^2 + 4y^2 = 20$ at $(2, 2)$.
- For the circle $x^2 + y^2 = 25$, find $\frac{d^2y}{dx^2}$ at $(3, 4)$.



10. Two positive lengths x and y , measured in metres, satisfy $x^2 + y^2 = 100$. When $x = 8$ and $y = 6$, x is increasing at 6 m s^{-1} . Find $\frac{dy}{dt}$.
11. The volume of a sphere is increasing at $12\pi \text{ cm}^3\text{s}^{-1}$. Find the rate at which its radius is increasing when $r = 2 \text{ cm}$.
12. The radius of a circle is increasing at 0.4 cm s^{-1} . Find the rate of change of its area when $r = 5 \text{ cm}$.
13. Positive numbers x and y satisfy $x^2 + y^2 = 50$. Find the maximum possible value of xy .
14. Find the absolute maximum and minimum of $f(x) = x^3 - 3x$ on $0 \leq x \leq 2$, stating where each occurs.
15. Find the absolute maximum and minimum of $g(x) = x + \frac{4}{x}$ on $1 \leq x \leq 4$, stating where each occurs.

2. Extended response | Questions 16–30

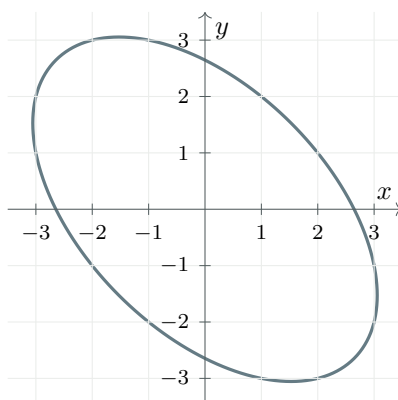
16. The curve

$$x^2y + y^3 = 10$$

passes through $(1, 2)$.

- (a) Find $\frac{dy}{dx}$ in terms of x and y .
- (b) Hence find the equation of the tangent at $(1, 2)$.

17. The curve $x^2 + xy + y^2 = 7$ is regarded locally as $y = y(x)$. Find the coordinates of all points on the curve where the tangent is horizontal. Mark these points on the diagram. The diagram is also used in Question 18.



18. For the same curve

$$x^2 + xy + y^2 = 7,$$

find the coordinates of all points where the tangent is vertical. Mark these points on the diagram in Question 17.



19. The curve

$$xy + x + y = 5$$

passes through $(1, 2)$. Find the equations of both the tangent and the normal at this point.

20. Consider the family of curves

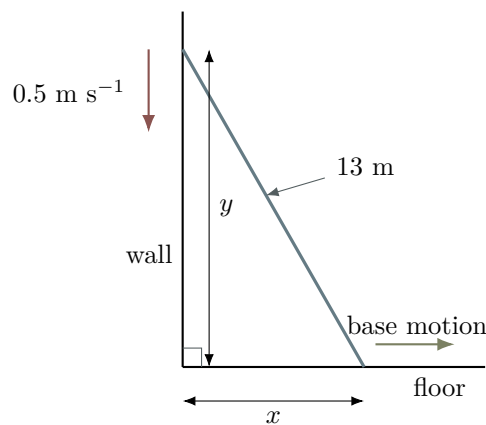
$$x^2 + y^2 + 2kxy = 37.$$

The point $(3, 2)$ lies on one member of the family.

(a) Find k .

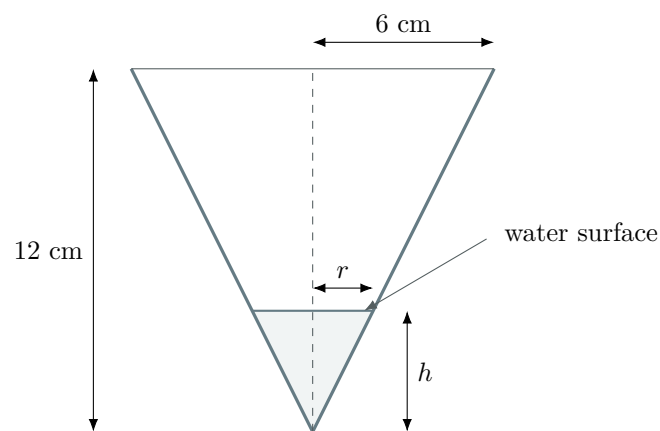
(b) Find the gradient of that curve at $(3, 2)$.

21. A 13-metre ladder rests against a vertical wall. The top slides down the wall at 0.5 m s^{-1} . Find the rate at which the base slides away from the wall when the top is 12 m above the ground.



Schematic; not drawn to scale.

22. Water is poured into an inverted cone of height 12 cm and top radius 6 cm at a rate of $9\pi \text{ cm}^3 \text{ s}^{-1}$. The water forms a similar cone. Find the rate at which the water depth is increasing when the depth is 4 cm.

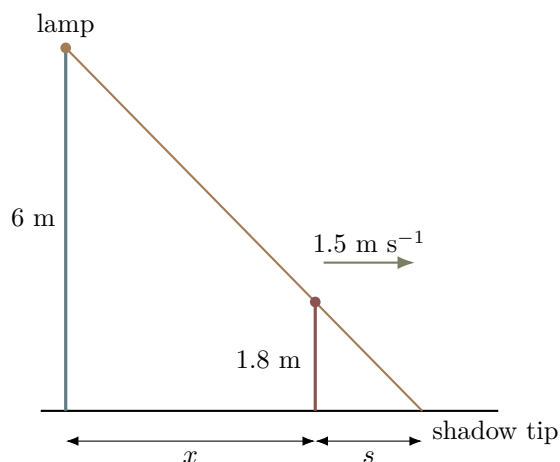


Vertical cross-section at the stated instant.

23. Point $A = (x, 0)$ moves along the positive x -axis at 2 m s^{-1} . Point $B = (0, y)$ moves down the positive y -axis at 1 m s^{-1} . Find the rate of change of the distance AB when $x = 3$ and $y = 4$.

24. A lamp is 6 m above the ground. A person of height 1.8 m walks directly away from the lamp at 1.5 m s^{-1} . If s is the length of the person's shadow, find

- (a) $\frac{ds}{dt}$;
(b) the speed of the tip of the shadow.



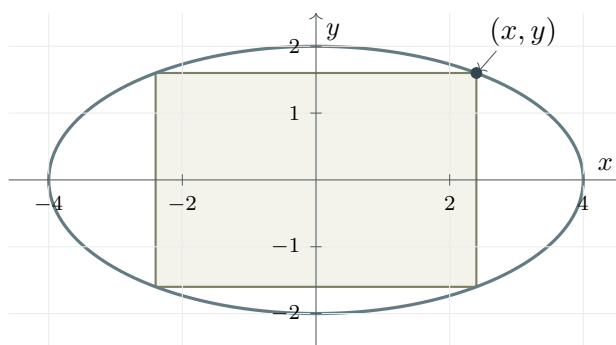
Schematic; x is the distance from the lamp to the person.

25. A 13-metre ladder makes an angle θ with the horizontal. Its base is sliding away from the wall at 0.7 m s^{-1} . Find $\frac{d\theta}{dt}$ when the base is 5 m from the wall.

26. A rectangle is centred at the origin with vertices $(\pm x, \pm y)$ on the ellipse

$$x^2 + 4y^2 = 16,$$

where $x, y > 0$. Use implicit differentiation to find the maximum area of the rectangle and the corresponding values of x and y .



A representative rectangle; its full side lengths are $2x$ and $2y$.



27. A point (x, y) lies on the branch $xy = 8$ in the first quadrant. Use implicit differentiation to find the point on the curve that is closest to the origin, and find this minimum distance.

28. An open-top box has a square base of side x cm and height h cm. Its surface area is fixed at 108 cm^2 .

(a) Show that $V = \frac{108x - x^3}{4}$.

(b) Hence find the dimensions of the box with maximum volume.

29. A design parameter x is restricted to $0 \leq x \leq 2$, and the cost is

$$C(x) = x^2 - 6x + 13.$$

Find the minimum cost and explain why checking the endpoint is essential in this problem.

30. A moving point $(x(t), y(t))$ remains on the curve

$$x^2 + xy + y^2 = 7.$$

Both coordinates are measured in metres and time is in seconds. At an instant when the point is at $(1, 2)$, $\frac{dx}{dt} = 2 \text{ m s}^{-1}$. Find $\frac{dy}{dt}$.

3. IB exam-style / Paper 1 | Questions 31–50

31. [Maximum mark: 7]

The curve C is defined by

$$x^2 + 3xy + 2y^2 = 12.$$

(a) Show that

$$\frac{dy}{dx} = -\frac{2x + 3y}{3x + 4y}.$$

[3 marks]

(b) Find the equation of the tangent to C at $(2, 1)$.

[2 marks]

(c) Find the equation of the normal to C at $(2, 1)$.

[2 marks]

32. [Maximum mark: 7]

The ellipse E has equation $x^2 + 4y^2 = 20$.

(a) Find $\frac{dy}{dx}$.

[2 marks]

(b) By differentiating the equation of E a second time, find $\frac{d^2y}{dx^2}$ at $(2, 2)$.

[4 marks]

(c) State whether the curve is concave up or concave down at $(2, 2)$.

[1 marks]



33. [Maximum mark: 10]

Consider the curve

$$x^2 + 2xy + 3y^2 = 12.$$

(a) Show that

$$\frac{dy}{dx} = -\frac{x+y}{x+3y}.$$

[3 marks]

(b) Find the coordinates of all points where the tangent is horizontal.

[3 marks]

(c) Hence, by considering $\frac{d^2y}{dx^2}$, classify each stationary point as a local maximum or local minimum of y .

[4 marks]

34. [Maximum mark: 9]

The curve C is given by

$$x^2 - xy + y^2 = 3.$$

(a) Find $\frac{dy}{dx}$.

[3 marks]

(b) Find all points on C where the tangent is horizontal.

[3 marks]

(c) Find all points on C where the tangent is vertical.

[3 marks]

35. [Maximum mark: 5]

A curve is defined implicitly by

$$e^y + xy = e.$$

(a) Show that

$$\frac{dy}{dx} = -\frac{y}{e^y + x}.$$

[3 marks]

(b) Find the equation of the tangent at $(0, 1)$.

[2 marks]

36. [Maximum mark: 5]

A right circular cylinder has constant volume $500\pi \text{ cm}^3$. Its radius r is increasing at 0.1 cm s^{-1} .

(a) Write an equation connecting r and the height h .

[1 marks]

(b) Find $\frac{dh}{dt}$ at the instant when $r = 5 \text{ cm}$.

[4 marks]

37. [Maximum mark: 6]

A spherical balloon has radius r , surface area A , and volume V . At an instant when $r = 3 \text{ m}$, the radius is increasing at 0.02 m s^{-1} .

(a) Find $\frac{dA}{dt}$.

[3 marks]

(b) Find $\frac{dV}{dt}$.

[3 marks]



38. [Maximum mark: 6]

Water is poured into an inverted conical tank of height 10 cm and top radius 4 cm at $8\pi \text{ cm}^3\text{s}^{-1}$. The water surface remains horizontal.

(a) Show that, when the water depth is h ,

$$V = \frac{4\pi}{75}h^3.$$

[3 marks]

(b) Hence find $\frac{dh}{dt}$ when $h = 5$ cm.

[3 marks]

39. [Maximum mark: 7]

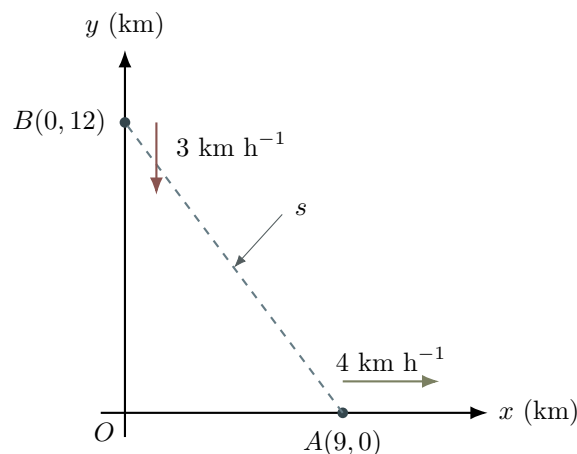
Two cyclists move on perpendicular roads through the origin. At a certain instant, cyclist A is at $(9, 0)$ km and is moving in the positive x -direction at 4 km h^{-1} . Vehicle B is at $(0, 12)$ km and is moving toward the origin at 3 km h^{-1} . Let s be the distance between them.

(a) Find $\frac{ds}{dt}$ at this instant.

[4 marks]

(b) Assuming both velocities remain constant, show that the cyclists are at a local minimum separation at this instant.

[3 marks]



40. [Maximum mark: 6]

A 10-metre ladder leans against a vertical wall. The base moves away from the wall at 0.8 m s^{-1} . Let x be the distance of the base from the wall, y the height of the top, and θ the angle between the ladder and the ground.

(a) Find $\frac{dy}{dt}$ when $x = 6$ m.

[3 marks]

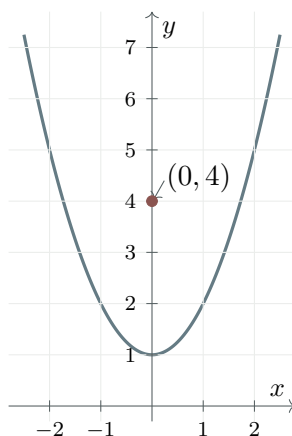
(b) Find $\frac{d\theta}{dt}$ at the same instant.

[3 marks]



41. [Maximum mark: 7]

A point P lies on the parabola $y = x^2 + 1$. Find the coordinates of the point or points on the parabola closest to $(0, 4)$, and find the minimum distance.



42. [Maximum mark: 5]

A rectangular enclosure has width x metres and length $(8 - x)$ metres, where construction restrictions require

$$0 < x \leq 3.$$

- (a) Find the maximum possible area. [3 marks]
(b) Explain why the unconstrained stationary point does not give the answer. [2 marks]

43. [Maximum mark: 6]

A rectangle has one vertex at the origin and the opposite vertex (x, y) on the curve

$$y = 12 - x^2,$$

where $0 < x < \sqrt{12}$.

- (a) Write the area A as a function of x . [1 marks]
(b) Find the dimensions of the rectangle with maximum area. [4 marks]
(c) Find the maximum area. [1 marks]

44. [Maximum mark: 6]

Positive numbers x and y satisfy

$$x^2 + 2y^2 = 18.$$

The quantity $P = xy$.

- (a) By implicit differentiation, find $\frac{dy}{dx}$. [2 marks]
(b) Hence find the maximum possible value of P . [4 marks]



45. [Maximum mark: 8]

A closed cylindrical container has total surface area $54\pi \text{ cm}^2$. Its radius is $r \text{ cm}$ and height is $h \text{ cm}$.

(a) Show that

$$h = \frac{27}{r} - r.$$

[2 marks]

(b) Find the dimensions that maximize the volume.

[5 marks]

(c) Find the maximum volume.

[1 marks]

46. [Maximum mark: 7]

A closed cylindrical can must have volume $250\pi \text{ cm}^3$. Its radius is $r \text{ cm}$ and its height is $h \text{ cm}$.

(a) Express the total surface area S in terms of r only.

[2 marks]

(b) Find the dimensions that minimize S .

[4 marks]

(c) Find the minimum surface area.

[1 marks]

47. [Maximum mark: 8]

A right circular cone has fixed slant height 10 cm . Its radius r and vertical height h therefore satisfy

$$r^2 + h^2 = 100.$$

(a) Show that the volume can be written as

$$V = \frac{\pi}{3}(100h - h^3).$$

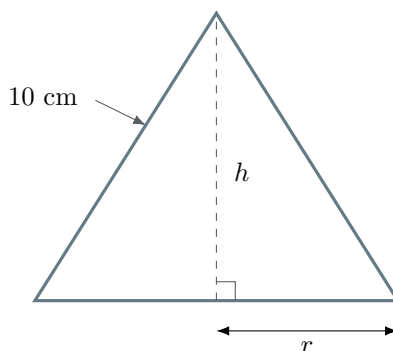
[2 marks]

(b) Hence find the values of r and h that maximize the volume.

[4 marks]

(c) Find the maximum volume in exact form.

[2 marks]



Axial cross-section; not drawn to scale.

48. [Maximum mark: 6]

A rectangle has side lengths $x(t)$ and $y(t)$, and its diagonal remains fixed at 10 cm . At an instant when $x = 6 \text{ cm}$, $y = 8 \text{ cm}$, and $\frac{dx}{dt} = 1 \text{ cm s}^{-1}$:

(a) find $\frac{dy}{dt}$;

[3 marks]

(b) hence find the rate of change of the area of the rectangle.

[3 marks]



49. [Maximum mark: 5]

The curve C_k is defined by

$$x^2 + kxy + y^2 = 8.$$

The point $(1, 3)$ lies on C_k .

- (a) Find k . [1 marks]
- (b) By implicit differentiation, show that the tangent to C_k at $(1, 3)$ is horizontal. [3 marks]
- (c) Hence state the equation of the normal at $(1, 3)$. [1 marks]

50. [Maximum mark: 11]

The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

has a point $P = (p, q)$ in the first quadrant. The tangent at P meets the positive coordinate axes at A and B .

- (a) By implicit differentiation, show that the tangent at P has equation

$$\frac{px}{16} + \frac{qy}{9} = 1.$$

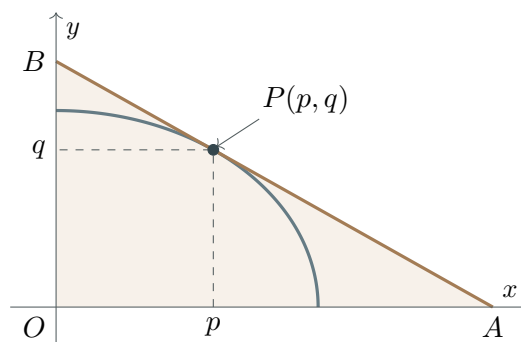
[4 marks]

- (b) Hence show that the area T of triangle OAB is

$$T = \frac{72}{pq}.$$

[2 marks]

- (c) Find the minimum possible value of T . [5 marks]



One possible position of P ; the shaded region is triangle OAB .

**Answer key and solution outlines**

Here $y' = dy/dx$, and a dot denotes differentiation with respect to time, for example $\dot{r} = dr/dt$. Derivative quotients apply where their denominators are nonzero.

Section 1: Short response

1. $2x + 2yy' = 0$, hence $y' = -x/y$ where $y \neq 0$.
2. $3x^2 + 3y^2y' = 9(y + xy')$. Collecting the y' terms gives $(3y^2 - 9x)y' = 9y - 3x^2$, so $y' = (3y - x^2)/(y^2 - 3x)$ where the denominator is nonzero.
3. $\frac{y'}{y} + y + xy' = 0$. Thus $(1 + xy)y' = -y^2$, giving $y' = -\frac{y^2}{1 + xy}$ where $1 + xy \neq 0$.
4. $2x + y + xy' + 2yy' = 0$, so $y' = -\frac{2x + y}{x + 2y}$ where $x + 2y \neq 0$.
5. $e^y y' + y + xy' = 0$, so $y' = -\frac{y}{e^y + x}$ where $e^y + x \neq 0$.
6. $1 + \cos y y' = 0$, so $y' = -1/\cos y = -\sec y$ where $\cos y \neq 0$.
7. At $(3, 4)$, $m = -\frac{3}{4}$. Tangent: $3x + 4y = 25$.
8. $\frac{dy}{dx} = -\frac{x}{4y}$, so $m_t = -\frac{1}{4}$ and $m_n = 4$. Normal: $y = 4x - 6$.
9. Differentiate $2x + 2yy' = 0$: $2 + 2(y')^2 + 2yy'' = 0$. At $(3, 4)$, $y' = -3/4$, so $y'' = -25/64$.
10. $2x\dot{x} + 2y\dot{y} = 0$. Thus $2(8)(6) + 2(6)\dot{y} = 0$, so $\dot{y} = -8 \text{ m s}^{-1}$.
11. $\dot{V} = 4\pi r^2 \dot{r}$. Hence $12\pi = 16\pi \dot{r}$, so $\dot{r} = 3/4 \text{ cm s}^{-1}$.
12. $\dot{A} = 2\pi r \dot{r} = 2\pi(5)(0.4) = 4\pi \text{ cm}^2 \text{ s}^{-1}$.
13. Let $P = xy$. Differentiating the constraint gives $y' = -x/y$, so $P' = y + xy' = (y^2 - x^2)/y$. Thus $P' = 0$ requires $x = y = 5$. For $0 < x < 5$, $P' > 0$; for $5 < x < \sqrt{50}$, $P' < 0$. Hence $P_{\max} = 25$.
14. $f'(x) = 3(x^2 - 1)$. Check $x = 0, 1, 2$: $f = 0, -2, 2$. Absolute minimum -2 at $x = 1$; absolute maximum 2 at $x = 2$.
15. $g'(x) = 1 - 4/x^2$, so the interior critical point is $x = 2$. Values: $g(1) = 5$, $g(2) = 4$, $g(4) = 5$. Minimum 4 at $x = 2$; maximum 5 at $x = 1, 4$.

Section 2: Extended response

16. $2xy + x^2y' + 3y^2y' = 0$, so $y' = -\frac{2xy}{x^2 + 3y^2}$. At $(1, 2)$, $m = -4/13$. Tangent: $y - 2 = -\frac{4}{13}(x - 1)$.
17. $y' = -(2x + y)/(x + 2y)$. Horizontal tangents require $2x + y = 0$, so $y = -2x$. Substitution gives $3x^2 = 7$. Points:

$$\left(\sqrt{\frac{7}{3}}, -2\sqrt{\frac{7}{3}}\right), \left(-\sqrt{\frac{7}{3}}, 2\sqrt{\frac{7}{3}}\right).$$

At both points, $x + 2y = -3x \neq 0$, so the derivative is defined and equals zero.



Section 2: Extended response (continued)

18. Vertical tangents require $x + 2y = 0$, so $x = -2y$. Then $3y^2 = 7$. Points:

$$\left(-2\sqrt{\frac{7}{3}}, \sqrt{\frac{7}{3}}\right), \left(2\sqrt{\frac{7}{3}}, -\sqrt{\frac{7}{3}}\right).$$

Here $2x + y = -3y \neq 0$. Equivalently, $dx/dy = 0$ with a nonzero denominator, confirming vertical tangents.

19. $y' = -(y+1)/(x+1)$, hence $m_t = -3/2$ at $(1, 2)$. Tangent $3x + 2y = 7$. Normal slope $2/3$, so $2x - 3y + 4 = 0$.

20. (a) $9 + 4 + 12k = 37 \Rightarrow k = 2$. (b) $y' = -(x+ky)/(y+kx)$, so $m = -7/8$.

21. With $x^2 + y^2 = 169$, at $y = 12$ we have $x = 5$. Differentiate: $x\dot{x} + y\dot{y} = 0$. Thus $5\dot{x} + 12(-0.5) = 0$, so $\dot{x} = 6/5 \text{ m s}^{-1}$.

22. Similarity gives $r = h/2$. Hence $V = \pi h^3/12$, so $\dot{V} = (\pi h^2/4)\dot{h}$. At $h = 4$: $9\pi = 4\pi\dot{h}$, giving $\dot{h} = 9/4 \text{ cm s}^{-1}$.

23. $s^2 = x^2 + y^2$, so $s\dot{s} = x\dot{x} + y\dot{y}$. At $(3, 4)$, $s = 5$, giving $5\dot{s} = 3(2) + 4(-1) = 2$. Thus $\dot{s} = 2/5 \text{ m s}^{-1}$.

24. Similar triangles: $6/(x+s) = 1.8/s$, hence $s = 3x/7$. Therefore $\dot{s} = 9/14 \text{ m s}^{-1}$. Tip speed $= \dot{x} + \dot{s} = 3/2 + 9/14 = 15/7 \text{ m s}^{-1}$.

25. $\cos \theta = x/13$. At $x = 5$, $\sin \theta = 12/13$. Thus $-\sin \theta \dot{\theta} = \dot{x}/13$, so $\dot{\theta} = -7/120 \text{ rad s}^{-1}$.

26. The full side lengths are $2x$ and $2y$, so $A = 4xy$. Implicit differentiation of $x^2 + 4y^2 = 16$ gives $y' = -x/(4y)$. Therefore

$$A' = 4y + 4xy' = \frac{4y^2 - x^2}{y} = \frac{16 - 2x^2}{y}.$$

Since $y > 0$, A' changes from positive to negative at $x = 2\sqrt{2}$. The constraint then gives $y = \sqrt{2}$. As $x \rightarrow 0^+$ or $x \rightarrow 4^-$, $A \rightarrow 0$, confirming the global maximum: $x = 2\sqrt{2}$, $y = \sqrt{2}$, $A_{\max} = 16$.

27. Minimizing distance D is equivalent to minimizing $D^2 = x^2 + y^2$. The constraint $xy = 8$ gives $y' = -y/x$. Thus

$$(D^2)' = 2x + 2yy' = 2x - \frac{2y^2}{x} = \frac{2(x^4 - 64)}{x^3}.$$

For $x > 0$, the derivative is negative before $x = 2\sqrt{2}$ and positive afterwards. Hence the unique closest point is $(2\sqrt{2}, 2\sqrt{2})$, and $D_{\min} = 4$.

28. The surface-area constraint is $x^2 + 4xh = 108$, so $h = (108 - x^2)/(4x)$ and

$$V = x^2h = \frac{108x - x^3}{4}, \quad 0 < x < 6\sqrt{3}.$$

Then $V' = (108 - 3x^2)/4$, which is zero at $x = 6$. The sign changes from positive to negative, so this gives a maximum; $V \rightarrow 0$ at both degenerate limiting ends. With $h = (108 - 36)/24 = 3$, the dimensions are $6 \text{ cm} \times 6 \text{ cm} \times 3 \text{ cm}$.

x	0	6	$6\sqrt{3}$	
$V'(x)$		+	0	-
$V(x)$			Max	

**Section 2: Extended response (continued)**

29. $C'(x) = 2x - 6 < 0$ on $[0, 2]$, so C decreases throughout the feasible interval. The minimum occurs at the endpoint $x = 2$: $C_{\min} = 5$. The unconstrained stationary point $x = 3$ is outside the domain.

30. Differentiate with respect to time:

$$(2x + y)\dot{x} + (x + 2y)\dot{y} = 0.$$

At $(1, 2)$ with $\dot{x} = 2$, this gives $4(2) + 5\dot{y} = 0$, so $\dot{y} = -\frac{8}{5} \text{ m s}^{-1}$. The y -coordinate is decreasing.

Section 3: IB exam-style / Paper 1

31. Differentiate:

$$2x + 3(y + xy') + 4yy' = 0$$

so $y' = -(2x + 3y)/(3x + 4y)$. At $(2, 1)$, $m_t = -7/10$. Tangent: $7x + 10y = 24$. Normal slope $10/7$, giving $10x - 7y - 13 = 0$.

32. $2x + 8yy' = 0 \Rightarrow y' = -x/(4y)$. Differentiate again:

$$2 + 8((y')^2 + yy'') = 0.$$

At $(2, 2)$, $y' = -1/4$, hence $y'' = -5/32$. Therefore the curve is **concave down** there.

33. $2x + 2(y + xy') + 6yy' = 0$, so $y' = -(x + y)/(x + 3y)$. Horizontal tangents satisfy $x + y = 0$, giving $(-\sqrt{6}, \sqrt{6})$ and $(\sqrt{6}, -\sqrt{6})$. From

$$1 + 2y' + 3(y')^2 + (x + 3y)y'' = 0,$$

at a stationary point $y'' = -1/(x + 3y)$. Thus $(-\sqrt{6}, \sqrt{6})$ is a **local maximum**, and $(\sqrt{6}, -\sqrt{6})$ is a **local minimum**. More explicitly, $y'' = -1/(2\sqrt{6}) < 0$ at $(-\sqrt{6}, \sqrt{6})$ and $y'' = 1/(2\sqrt{6}) > 0$ at $(\sqrt{6}, -\sqrt{6})$.

34. Differentiation gives

$$(-x + 2y)y' + 2x - y = 0, \quad y' = \frac{2x - y}{x - 2y}.$$

Horizontal: $y = 2x$, so $3x^2 = 3$: $(1, 2), (-1, -2)$. Vertical: $x = 2y$, so $3y^2 = 3$: $(2, 1), (-2, -1)$. In each case the other linear expression in the derivative is nonzero, so there are no indeterminate $0/0$ cases among these points.

35. $e^y y' + y + xy' = 0$, so $y' = -y/(e^y + x)$. At $(0, 1)$, $m = -1/e$. Tangent: $y - 1 = -\frac{1}{e}x$.

36. Constant volume gives $\pi r^2 h = 500\pi$, or $r^2 h = 500$. Differentiate:

$$2rh\dot{r} + r^2\dot{h} = 0.$$

When $r = 5$, $h = 20$. Thus $2(5)(20)(0.1) + 25\dot{h} = 0$, so $\dot{h} = -4/5 \text{ cm s}^{-1}$.

37. $A = 4\pi r^2$, so $\dot{A} = 8\pi r\dot{r} = 8\pi(3)(0.02) = 12\pi/25 \text{ m}^2\text{s}^{-1}$. Also $V = \frac{4}{3}\pi r^3$, so $\dot{V} = 4\pi r^2\dot{r} = 4\pi(9)(0.02) = 18\pi/25 \text{ m}^3\text{s}^{-1}$.

38. Similarity gives $r/h = 4/10 = 2/5$. Therefore

$$V = \frac{1}{3}\pi \left(\frac{2h}{5}\right)^2 h = \frac{4\pi}{75}h^3.$$

Then $\dot{V} = (4\pi/25)h^2\dot{h}$. At $h = 5$, $8\pi = 4\pi\dot{h}$, hence $\dot{h} = 2 \text{ cm s}^{-1}$.



Section 3: IB exam-style / Paper 1 (continued)

39. Let $t = 0$ at the stated instant. Since $s^2 = x^2 + y^2$, differentiation gives $s\dot{s} = x\dot{x} + y\dot{y}$. With $s = 15$, $\dot{x} = 4$ and $\dot{y} = -3$,

$$15\dot{s} = 9(4) + 12(-3) = 0, \quad \boxed{\dot{s} = 0 \text{ km h}^{-1}}.$$

Constant velocities give $x = 9 + 4t$ and $y = 12 - 3t$ locally. Hence $s^2 = (9 + 4t)^2 + (12 - 3t)^2 = 225 + 25t^2$, which is minimized at $t = 0$. Since square root is increasing, s also has a local minimum there (of 15 km).

40. $x^2 + y^2 = 100$. At $x = 6$, $y = 8$. Then $x\dot{x} + y\dot{y} = 0$, so $6(0.8) + 8\dot{y} = 0$, hence $\boxed{\dot{y} = -3/5 \text{ m s}^{-1}}$. Also $\cos \theta = x/10$, so $-\sin \theta \dot{\theta} = \dot{x}/10$. Since $\sin \theta = 4/5$, $\boxed{\dot{\theta} = -1/10 \text{ rad s}^{-1}}$.

41. Minimize

$$D^2 = x^2 + (x^2 - 3)^2.$$

Then

$$(D^2)' = 2x + 4x(x^2 - 3) = 2x(2x^2 - 5).$$

Critical values are $x = 0$, $\pm\sqrt{5/2}$. The minimum occurs at $x = \pm\sqrt{5/2}$, with $y = x^2 + 1 = 7/2$. Thus the closest points are

$$\boxed{\left(\pm\sqrt{\frac{5}{2}}, \frac{7}{2}\right)}$$

and $D^2 = 11/4$, so $\boxed{D_{\min} = \sqrt{11}/2}$. To justify the minimum, $(D^2)'' = 12x^2 - 10$: it is -10 at 0 and 20 at $x = \pm\sqrt{5/2}$. Also $D^2 \rightarrow \infty$ as $|x| \rightarrow \infty$, so the two minima are global.

42. $A = x(8 - x) = 8x - x^2$, with $A'(x) = 8 - 2x > 0$ throughout $(0, 3]$. Hence A is increasing on the entire feasible interval, so the maximum is at $x = 3$: $\boxed{A_{\max} = 15 \text{ m}^2}$. The unconstrained stationary point $x = 4$ is infeasible.

43. $A = x(12 - x^2) = 12x - x^3$. Then $A' = 12 - 3x^2$, so $x = 2$. Hence $y = 8$, and the maximum area is $\boxed{16}$, with dimensions $\boxed{2 \times 8}$. On $0 < x < \sqrt{12}$, A' is positive for $x < 2$ and negative for $x > 2$. The limiting areas at the two degenerate ends are zero, so this is the global maximum.

44. $2x + 4yy' = 0 \Rightarrow \boxed{y' = -x/(2y)}$. For $P = xy$,

$$P' = y + xy' = y - \frac{x^2}{2y}.$$

So $2y^2 = x^2$. Together with $x^2 + 2y^2 = 18$, this gives $x = 3$, $y = 3/\sqrt{2}$. Therefore

$$\boxed{P_{\max} = \frac{9}{\sqrt{2}} = \frac{9\sqrt{2}}{2}}.$$

Since $2y^2 = 18 - x^2$, $P' = (18 - 2x^2)/(2y)$ changes from positive to negative at $x = 3$. As either positive variable tends to zero, $P \rightarrow 0$, confirming the maximum.

45. Surface area: $2\pi r^2 + 2\pi rh = 54\pi$. Hence $\boxed{h = 27/r - r}$. Then

$$V = \pi r^2 h = 27\pi r - \pi r^3, \quad V' = 27\pi - 3\pi r^2.$$

Thus $r = 3$, $h = 6$, giving $\boxed{V_{\max} = 54\pi \text{ cm}^3}$. Feasibility requires $0 < r < 3\sqrt{3}$. Since V' changes from positive to negative at $r = 3$ and V tends to zero at both degenerate ends, this is the global maximum.



Section 3: IB exam-style / Paper 1 (continued)

46. Volume condition gives $h = 250/r^2$. Thus

$$S = 2\pi r^2 + 2\pi r h = \boxed{2\pi r^2 + \frac{500\pi}{r}}.$$

Then $S' = 4\pi r - 500\pi/r^2 = 0 \Rightarrow r^3 = 125 \Rightarrow r = 5$. Hence $h = 10$, and $\boxed{S_{\min} = 150\pi \text{ cm}^2}$. For $r > 0$, $S'' = 4\pi + 1000\pi/r^3 > 0$, and $S \rightarrow \infty$ as $r \rightarrow 0^+$ or $r \rightarrow \infty$. Thus the stationary value is the unique global minimum.

47. Since $r^2 = 100 - h^2$,

$$V = \frac{1}{3}\pi r^2 h = \boxed{\frac{\pi}{3}(100h - h^3)}.$$

Then $V' = (\pi/3)(100 - 3h^2) = 0$, so $h = 10/\sqrt{3} = 10\sqrt{3}/3$. Hence

$$r = \sqrt{100 - 100/3} = \boxed{\frac{10\sqrt{6}}{3}}.$$

The maximum volume is

$$\boxed{\frac{2000\sqrt{3}\pi}{27} \text{ cm}^3}.$$

The feasible range is $0 < h < 10$. The derivative is positive before $h = 10/\sqrt{3}$ and negative afterwards; the limiting volumes at $h = 0$ and $h = 10$ are zero. This confirms a global maximum.

48. $x^2 + y^2 = 100$, so $x\dot{x} + y\dot{y} = 0$. At $(6, 8)$: $6(1) + 8\dot{y} = 0$, hence $\boxed{\dot{y} = -3/4 \text{ cm s}^{-1}}$. Area $A = xy$, so

$$\dot{A} = y\dot{x} + x\dot{y} = 8(1) + 6(-3/4) = \boxed{7/2 \text{ cm}^2 \text{ s}^{-1}}.$$

49. From $(1, 3)$: $1 + 3k + 9 = 8$, so $\boxed{k = -2/3}$. Differentiating,

$$2x + k(y + xy') + 2yy' = 0, \quad y' = -\frac{2x + ky}{kx + 2y}.$$

At $(1, 3)$, the numerator is $2 + (-2/3)(3) = 0$, while the denominator is nonzero, so the tangent is horizontal. The normal is therefore vertical: $\boxed{x = 1}$.

50. (a) Differentiation gives $x/8 + (2y/9)y' = 0$, so at P , $m = -9p/(16q)$. The tangent is

$$y - q = -\frac{9p}{16q}(x - p) \implies 9px + 16qy = 9p^2 + 16q^2.$$

Since $p^2/16 + q^2/9 = 1$, the right side equals 144. Dividing by 144 gives $\boxed{px/16 + qy/9 = 1}$. (b) The intercepts are $A = (16/p, 0)$ and $B = (0, 9/q)$. Thus $T = \frac{1}{2} \frac{16}{p} \frac{9}{q} = \boxed{\frac{72}{pq}}$. (c) To minimize T , maximize $Q = pq$. Implicit differentiation of the constraint gives $dq/dp = -9p/(16q)$. Therefore

$$Q' = q + p \frac{dq}{dp} = \frac{16q^2 - 9p^2}{16q} = \frac{144 - 18p^2}{16q}.$$

With $0 < p < 4$ and $q > 0$, Q' changes from positive to negative at $p = 2\sqrt{2}$. Then $q = 3/\sqrt{2}$, $Q_{\max} = 6$, and $\boxed{T_{\min} = 12}$ square units. As either intercept point P approaches an axis, $pq \rightarrow 0^+$ and $T \rightarrow \infty$.