



Further Derivatives & Antiderivatives

IB Mathematics AA HL | AHL 5.15

50 questions: trigonometric, exponential, logarithmic and inverse functions

Learning goal	Differentiate further functions and recognize their antiderivatives, including rational integrands rearranged using partial fractions.
Syllabus link	AHL 5.15: further derivatives; corresponding indefinite integrals; linear composites; families of curves; partial fractions (AHL 1.11).
Structure	15 short-response, 15 extended-response and 20 IB Paper 1-style questions.
Technology	Non-calculator throughout. All angles are measured in radians.

Instructions. Work on separate paper and show your reasoning. Give exact answers and include $+C$ in indefinite integrals. Use absolute values in logarithmic antiderivatives where needed. Inverse trigonometric functions use their principal values; $\arcsin x$ means inverse sine, not $1/\sin x$.

1. Short response | Questions 1–15

1. Differentiate $y = \tan(3x - 1)$.
2. Differentiate $y = \sec(2x + \pi)$.
3. Differentiate $y = \operatorname{cosec}(5x)$.
4. Differentiate $y = \cot(1 - 4x)$.
5. Differentiate $y = 7^{2x-3}$.
6. Differentiate $y = \log_3(5x + 2)$ and state its real domain.
7. Differentiate $y = \arcsin(3x)$ and state the interval on which the derivative is finite.
8. Differentiate $y = \arccos(2x - 1)$.
9. Differentiate $y = \arctan(4x + 1)$.
10. Find $\int 6 \sec^2(2x) dx$.



11. Find $\int 3 \operatorname{cosec}(3x) \cot(3x) dx$.

12. Find $\int 5^{2x+1} dx$.

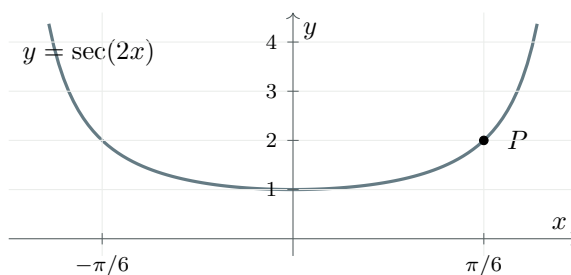
13. Find $\int \frac{1}{9+x^2} dx$.

14. Find $\int \frac{1}{\sqrt{16-x^2}} dx$.

15. Express $\frac{3}{(x+1)(x+4)}$ in partial fractions, and hence integrate it.

2. Extended response | Questions 16–30

16. The graph shows $y = \sec(2x)$ on $-\frac{\pi}{4} < x < \frac{\pi}{4}$. The point P has x -coordinate $\frac{\pi}{6}$.



- (a) Find the coordinates of P .
(b) Find the equation of the tangent at P .

17. Let $f(x) = \operatorname{cosec} x + \cot x$, where $0 < x < \pi$.

- (a) Show that $f'(x) = -\frac{1}{1 - \cos x}$.
(b) Find the equation of the tangent at $x = \frac{\pi}{2}$.

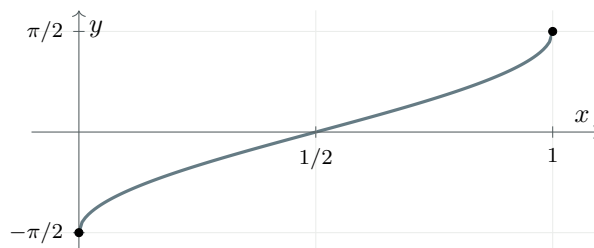
18. Let $f(x) = 2^{3x-1}$.

- (a) Find $f'(x)$ and $f'(0)$.
(b) Evaluate $\int_0^1 f(x) dx$.

19. Let $g(x) = \log_4(2x+1)$.

- (a) State the domain of g .
(b) Find $g'(x)$ and the tangent at $x = 0$.

20. The graph shows $f(x) = \arcsin(2x - 1)$.

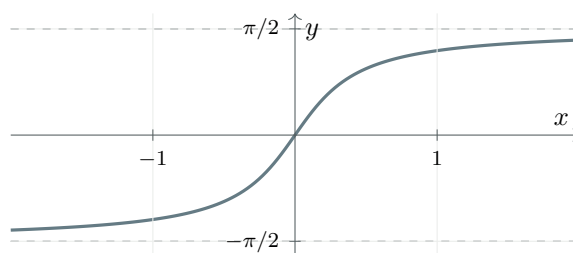


- State the domain and range of f .
- Find $f'(x)$ and the tangent at $x = \frac{1}{2}$.
- Explain why the finite-derivative domain differs from the domain of f .

21. Let $y = \arccos\left(\frac{x}{3}\right)$, where $-3 \leq x \leq 3$.

- Find $\frac{dy}{dx}$.
- Find the equation of the normal at $x = 0$.

22. The graph shows $f(x) = \arctan(3x)$.



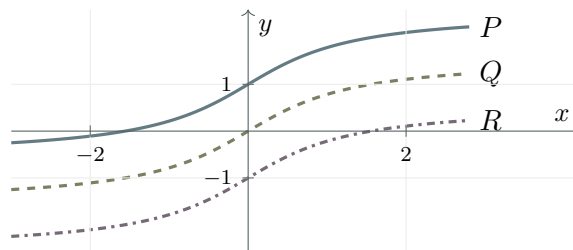
- Find $f'(x)$ and explain why f is increasing on $\mathbb{R}$.
- Find the greatest possible tangent gradient and state where it occurs.
- Find $f''(x)$ and classify the concavity on either side of $x = 0$.

23. Find each indefinite integral.

- $\int 4 \sec(2x + 1) \tan(2x + 1) dx.$
- $\int 5 \operatorname{cosec}^2(5x - 2) dx.$
- $\int \frac{1}{(x + 2) \ln 5} dx$, where $x > -2$.
- $\int \frac{1}{1 + 16x^2} dx.$
- $\int \frac{-3}{\sqrt{1 - 9x^2}} dx.$



- 24.** Every curve in a family satisfies $\frac{dy}{dx} = \frac{1}{1+x^2}$. Curves P , Q and R pass through $(0, 1)$, $(0, 0)$ and $(0, -1)$ respectively.



- (a) Find the general equation of the family and the equation of each labelled curve.
(b) Explain why any two curves in the family have the same tangent gradient at a given x .

- 25.** A function F satisfies $F'(x) = \frac{6}{\sqrt{9-4x^2}}$ and $F(0) = \pi$.

- (a) Find $F(x)$ on $-\frac{3}{2} < x < \frac{3}{2}$.
(b) Find $F\left(\frac{3}{4}\right)$.

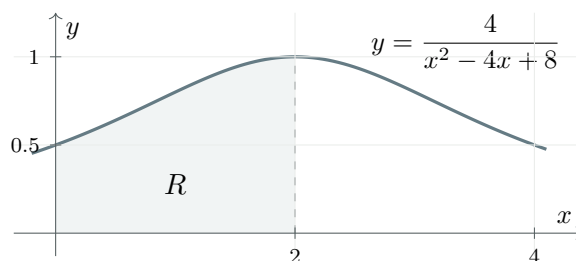
- 26.** Consider $\frac{5x+7}{(x+1)(x+2)}$.

- (a) Express it in partial fractions.
(b) Hence evaluate $\int_0^1 \frac{5x+7}{(x+1)(x+2)} dx$.

- 27.** Consider $\frac{2x+1}{x(x+1)^2}$.

- (a) Write it in the form $\frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$.
(b) Hence find its indefinite integral.

- 28.** The region R is bounded by $y = \frac{4}{x^2 - 4x + 8}$, the x -axis, $x = 0$ and $x = 2$.



- (a) Complete the square in the denominator.
(b) Find the exact area of R .



29. A student writes the following three statements. Identify the error in each and give the correct result.

(a) $\frac{d}{dx} \arccos(2x) = \frac{2}{\sqrt{1-4x^2}}.$

(b) $\int \frac{1}{4+x^2} dx = \arctan\left(\frac{x}{2}\right) + C.$

(c) $\int 2^{3x} dx = \frac{2^{3x}}{3} + C.$

30. A curve satisfies $y' = \sec^2 x - \operatorname{cosec}^2 x$ for $0 < x < \frac{\pi}{2}$ and passes through $\left(\frac{\pi}{4}, 5\right).$

(a) Find the equation of the curve.

(b) Find and classify its stationary point, justifying your answer using the sign of y' .

3. IB exam-style | Paper 1 | Questions 31–50

31. [Maximum mark: 7]

Let $f(x) = \tan(2x) - x$, where $-\frac{\pi}{4} < x < \frac{\pi}{4}.$

(a) Find $f'(x).$ [2 marks]

(b) Show that f is strictly increasing throughout its domain. [2 marks]

(c) Find the equations of the tangent and normal at the origin. [3 marks]

32. [Maximum mark: 9]

Let $f(x) = \sec x - \tan x$ on $-\frac{\pi}{3} \leq x \leq \frac{\pi}{3}.$

(a) Find $f'(x)$ and show that f is decreasing on this interval. [4 marks]

(b) Hence find the range of $f.$ [2 marks]

(c) Evaluate $\int_0^{\pi/3} (\sec x \tan x - \sec^2 x) dx.$ [3 marks]

33. [Maximum mark: 7]

Let $g(x) = \operatorname{cosec}(2x),$ where $0 < x < \frac{\pi}{2}.$

(a) Find $g'(x).$ [2 marks]

(b) Find the stationary point and show that it is a minimum. [4 marks]

(c) Write down the equation of the tangent at the stationary point. [1 marks]

34. [Maximum mark: 6]

The function $f(x) = a^{2x},$ where $a > 0$ and $a \neq 1,$ has $f'(0) = 4 \ln 3.$

(a) Find $a.$ [3 marks]

(b) Hence evaluate $\int_0^{1/2} f(x) dx.$ [3 marks]



35. [Maximum mark: 8]

Let $f(x) = \log_b(3x + 1)$, where $b > 1$. It is given that $f'(0) = \frac{3}{\ln 5}$.

- (a) Find b . [3 marks]
- (b) Find the point on the curve where the tangent gradient is $\frac{1}{\ln 5}$. [3 marks]
- (c) Find the equation of this tangent. [2 marks]

36. [Maximum mark: 8]

Let $f(x) = \arcsin\left(\frac{x}{2}\right)$, where $-2 \leq x \leq 2$.

- (a) Find $f'(x)$ for $-2 < x < 2$. [2 marks]
- (b) Find the equation of the tangent at $x = 1$. [3 marks]
- (c) Find the area bounded by $y = f'(x)$, the x -axis, $x = 0$ and $x = 1$. [3 marks]

37. [Maximum mark: 8]

Let $f(x) = \arccos(1 - 2x)$, where $0 \leq x \leq 1$.

- (a) Show that $f'(x) = \frac{1}{\sqrt{x - x^2}}$ for $0 < x < 1$. [3 marks]
- (b) Find the equation of the tangent at $x = \frac{1}{2}$. [2 marks]
- (c) Evaluate $\int_{1/4}^{3/4} \frac{1}{\sqrt{x - x^2}} dx$. [3 marks]

38. [Maximum mark: 8]

Consider the curve $y = \arctan\left(\frac{x}{2}\right)$.

- (a) Find $\frac{dy}{dx}$. [2 marks]
- (b) Find both points where the tangent gradient is $\frac{1}{4}$. [3 marks]
- (c) Find the equations of the two tangents and their y -intercepts. [3 marks]

39. [Maximum mark: 6]

A function F satisfies $F'(x) = 2 \sec(2x) \tan(2x) - 3 \operatorname{cosec}^2(3x)$ on $0 < x < \frac{\pi}{4}$, and $F\left(\frac{\pi}{6}\right) = 5$.

- (a) Find $F(x)$. [4 marks]
- (b) Find $F\left(\frac{\pi}{12}\right)$. [2 marks]

40. [Maximum mark: 7]

For $-1 < x < 1$, let $h(x) = \arcsin x + \arccos x$.

- (a) Show that $h'(x) = 0$. [2 marks]
- (b) Hence show that $\arcsin x + \arccos x = \frac{\pi}{2}$ throughout this interval. [2 marks]
- (c) Evaluate $\int_{-1/2}^{1/2} \frac{1}{\sqrt{1 - x^2}} dx$. [3 marks]



41. [Maximum mark: 7]

Consider the function $f(x) = \frac{3x+2}{x(x+2)}$, where $x > 0$.

(a) Express $f(x)$ in partial fractions. [3 marks]

(b) Hence show that $\int_1^2 f(x) dx = \ln\left(\frac{32}{9}\right)$. [4 marks]

42. [Maximum mark: 9]

A function F satisfies $F'(x) = \frac{2}{x^2-9}$ on $-3 < x < 3$, with $F(0) = 0$.

(a) Express $\frac{2}{x^2-9}$ in partial fractions. [3 marks]

(b) Find $F(x)$. [3 marks]

(c) State whether F is increasing or decreasing, and evaluate $\int_0^1 \frac{2}{x^2-9} dx$. [3 marks]

43. [Maximum mark: 9]

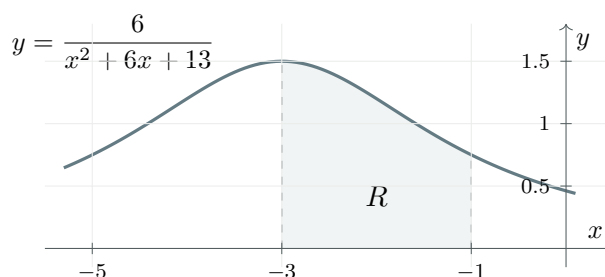
Consider $f(x) = \frac{x^2+1}{x(x+1)^2}$, where $x > 0$.

(a) Find constants A , B and C such that $f(x) = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$. [4 marks]

(b) Hence evaluate $\int_1^2 f(x) dx$, giving your answer as a rational number plus one logarithm. [5 marks]

44. [Maximum mark: 12]

The region R is bounded by $y = \frac{6}{x^2+6x+13}$, the x -axis, $x = -3$ and $x = -1$.



(a) Show that the area of R is $\frac{3\pi}{4}$. [4 marks]

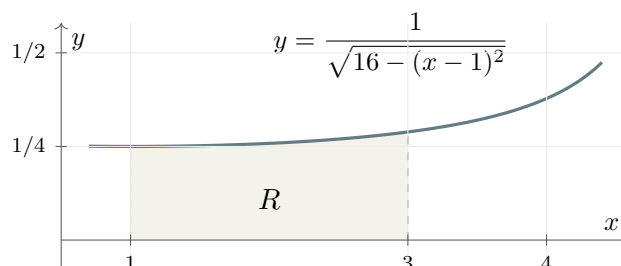
(b) Using the double-angle formula for tangent, show that $\tan\left(\frac{\pi}{8}\right) = \sqrt{2} - 1$. [3 marks]

(c) The line $x = k$ divides R into two equal areas. Find k in exact surd form. [5 marks]



45. [Maximum mark: 8]

The diagram shows the region R under $y = \frac{1}{\sqrt{16 - (x - 1)^2}}$ between $x = 1$ and $x = 3$.



- (a) Find the area of R . [3 marks]
- (b) Find a , where $1 < a < 5$, such that the area under the same curve from $x = 1$ to $x = a$ is $\frac{\pi}{4}$. [5 marks]

46. [Maximum mark: 9]

A particle moves along a straight line. Its velocity at time t seconds is $v(t) = \frac{4}{1 + 4t^2}$ m s⁻¹ for $0 \leq t \leq \frac{\sqrt{3}}{2}$. Its initial displacement from a fixed origin is $-\frac{\pi}{3}$ m.

- (a) Find its displacement $s(t)$. [3 marks]
- (b) Find the time at which it passes through the origin. [3 marks]
- (c) Find the total distance travelled on the stated time interval. [3 marks]

47. [Maximum mark: 10]

Let $f(x) = 3^x + 3^{-x}$, where $x \in \mathbb{R}$.

- (a) Find $f'(x)$ and $f''(x)$. [4 marks]
- (b) Find and classify the stationary point. [3 marks]
- (c) Evaluate $\int_{-1}^1 f(x) dx$. [3 marks]

48. [Maximum mark: 8]

Let $F(x) = 2 \ln |x - 1| - \ln |x + 2|$, where $x \neq 1, -2$.

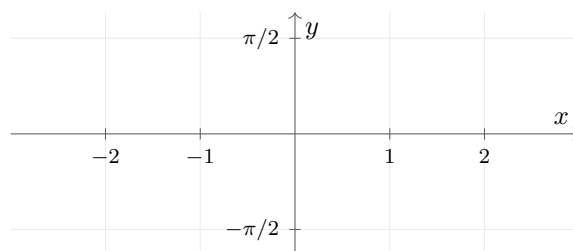
- (a) Show that $F'(x) = \frac{x + 5}{x^2 + x - 2}$. [3 marks]
- (b) Hence evaluate $\int_0^{1/2} \frac{x + 5}{x^2 + x - 2} dx$. [3 marks]
- (c) Explain why your answer is negative although the numerator is positive on the interval. [2 marks]



49. [Maximum mark: 8]

For $x \neq 0$, define $h(x) = \arctan x + \arctan\left(\frac{1}{x}\right)$.

- (a) Show that $h'(x) = 0$ for every $x \neq 0$. [3 marks]
- (b) Find $h(1)$ and $h(-1)$, and hence determine $h(x)$ for $x > 0$ and for $x < 0$. [3 marks]
- (c) Sketch $y = h(x)$ on the axes below. Explain why $h' = 0$ does not imply one constant value across the whole domain. [2 marks]



50. [Maximum mark: 12]

Let $f(x) = \frac{x^2 + 4x + 7}{x^2 + 4x + 8}$.

- (a) Show that $f(x) = 1 - \frac{1}{(x+2)^2 + 4}$. [2 marks]
- (b) Find the function F satisfying $F' = f$ and $F(-2) = 0$. [4 marks]
- (c) Find the area bounded by $y = f(x)$, the x -axis, $x = -2$ and $x = 0$. [3 marks]
- (d) Find $k > -2$ such that the area between $y = 1$ and $y = f(x)$ from $x = -2$ to $x = k$ is $\frac{\pi}{12}$. [3 marks]



Answer key and solution outlines

Section 1: Short response

1. $\frac{d}{dx} \tan u = \sec^2 u u'$, so $y' = 3 \sec^2(3x - 1)$.
2. $\frac{d}{dx} \sec u = \sec u \tan u u'$, so $y' = 2 \sec(2x + \pi) \tan(2x + \pi)$.
3. $\frac{d}{dx} \operatorname{cosec} u = -\operatorname{cosec} u \cot u u'$, so $y' = -5 \operatorname{cosec}(5x) \cot(5x)$.
4. $\frac{d}{dx} \cot u = -\operatorname{cosec}^2 u u'$, so $y' = 4 \operatorname{cosec}^2(1 - 4x)$.
5. $\frac{d}{dx} a^u = a^u \ln a u'$, so $y' = 2 \ln 7 7^{2x-3}$.
6. $y = \frac{\ln(5x+2)}{\ln 3}$, so $y' = \frac{5}{(5x+2) \ln 3}$. Domain: $x > -\frac{2}{5}$.
7. $y' = \frac{3}{\sqrt{1-9x^2}}$, finite for $-\frac{1}{3} < x < \frac{1}{3}$.
8. $y' = -\frac{2}{\sqrt{1-(2x-1)^2}}$, for $0 < x < 1$.
9. $y' = \frac{4}{1+(4x+1)^2}$, for all real x .
10. Since $(\tan 2x)' = 2 \sec^2(2x)$, the integral is $3 \tan(2x) + C$.
11. Since $(\operatorname{cosec} 3x)' = -3 \operatorname{cosec}(3x) \cot(3x)$, the integral is $-\operatorname{cosec}(3x) + C$.
12. $\int a^{mx+b} dx = \frac{a^{mx+b}}{m \ln a} + C$, so the answer is $\frac{5^{2x+1}}{2 \ln 5} + C$.
13. Write $9 + x^2 = 9[1 + (x/3)^2]$. The integral is $\frac{1}{3} \arctan\left(\frac{x}{3}\right) + C$.
14. Write $\sqrt{16 - x^2} = 4\sqrt{1 - (x/4)^2}$. The integral is $\arcsin\left(\frac{x}{4}\right) + C$, for $-4 < x < 4$.
15. $3 = A(x+4) + B(x+1)$ gives $A = 1$, $B = -1$. Thus the integral is $\ln|x+1| - \ln|x+4| + C = \ln\left|\frac{x+1}{x+4}\right| + C$.

Section 2: Extended response

16. (a) $P = (\pi/6, 2)$. (b) $y' = 2 \sec(2x) \tan(2x)$, so $m = 2(2)(\sqrt{3}) = 4\sqrt{3}$. Tangent: $y - 2 = 4\sqrt{3}(x - \pi/6)$.
17. (a) $f' = -\operatorname{cosec} x \cot x - \operatorname{cosec}^2 x = -\frac{1 + \cos x}{\sin^2 x} = -\frac{1}{1 - \cos x}$. (b) $f(\pi/2) = 1$, $f'(\pi/2) = -1$; $y - 1 = -(x - \pi/2)$.
18. (a) $f' = 3 \ln 2 2^{3x-1}$, so $f'(0) = \frac{3}{2} \ln 2$. (b) $\left[\frac{2^{3x-1}}{3 \ln 2} \right]_0^1 = \frac{4 - 1/2}{3 \ln 2} = \frac{7}{6 \ln 2}$.
19. (a) $x > -1/2$. (b) $g' = \frac{2}{(2x+1) \ln 4}$. Since $g(0) = 0$ and $g'(0) = 1/\ln 2$, the tangent is $y = x/\ln 2$.



Section 2: Extended response (continued)

20. (a) Domain $[0, 1]$; range $[-\pi/2, \pi/2]$. (b) $f' = \frac{2}{\sqrt{1-(2x-1)^2}}$; $f(1/2) = 0$, $f'(1/2) = 2$, so $y = 2x - 1$. (c) f is defined at both endpoints, but the denominator of f' is zero there; the derivative is finite only on $(0, 1)$.

21. (a) $y' = -\frac{1/3}{\sqrt{1-x^2/9}} = -\frac{1}{\sqrt{9-x^2}}$ for $-3 < x < 3$. (b) The point is $(0, \pi/2)$, the tangent gradient is $-1/3$, and the normal gradient is 3. Thus $y = 3x + \pi/2$.

22. (a) $f' = \frac{3}{1+9x^2} > 0$. (b) The denominator is at least 1, with equality at $x = 0$, so the greatest gradient is 3. (c) $f'' = -\frac{54x}{(1+9x^2)^2}$: concave up for $x < 0$, concave down for $x > 0$; $(0, 0)$ is a point of inflexion.

23. (a) $2\sec(2x+1) + C$. (b) $-\cot(5x-2) + C$. (c) $\frac{\ln(x+2)}{\ln 5} + C = \log_5(x+2) + C$. (d) $\frac{1}{4}\arctan(4x) + C$. (e) $\arccos(3x) + C$ (equivalently $-\arcsin(3x) + C$). Each chain-rule factor is reversed when integrating.

24. (a) $y = \arctan x + C$. Hence $P : y = \arctan x + 1$, $Q : y = \arctan x$, $R : y = \arctan x - 1$. (b) The constant changes the vertical position but its derivative is zero. The curves are vertical translations and share derivative $1/(1+x^2)$.

25. (a) $\frac{d}{dx} \arcsin(2x/3) = \frac{2}{\sqrt{9-4x^2}}$, so $F = 3\arcsin(2x/3) + C$. The condition gives $C = \pi$. (b) $F(3/4) = 3\arcsin(1/2) + \pi = 3\pi/2$.

26. (a) $5x + 7 = A(x+2) + B(x+1)$, so $A + B = 5$, $2A + B = 7$: $A = 2$, $B = 3$. (b) $[2\ln(x+1) + 3\ln(x+2)]_0^1 = 2\ln 2 + 3\ln(3/2) = \ln(27/2)$.

27. (a) $2x + 1 = A(x+1)^2 + Bx(x+1) + Cx$. At $x = 0$, $A = 1$; comparison gives $B = -1$, $C = 1$. (b) $\ln|x| - \ln|x+1| - \frac{1}{x+1} + C = \ln\left|\frac{x}{x+1}\right| - \frac{1}{x+1} + C$.

28. (a) $x^2 - 4x + 8 = (x-2)^2 + 4$. (b) The curve is positive, so $A = \int_0^2 \frac{4}{(x-2)^2 + 4} dx = [2\arctan((x-2)/2)]_0^2 = \pi/2$ square units.

29. (a) Missing negative sign: $-2/\sqrt{1-4x^2}$. (b) Missing factor $1/2$: $\frac{1}{2}\arctan(x/2) + C$. (c) Missing $\ln 2$ in the denominator: $2^{3x}/(3\ln 2) + C$. Differentiating each corrected antiderivative recovers its integrand.

30. (a) $y = \tan x + \cot x + C$; at $x = \pi/4$, $5 = 1 + 1 + C$, so $C = 3$. (b) $y' = 0$ gives $\sin^2 x = \cos^2 x$, hence $x = \pi/4$. The derivative is negative before and positive after this value: a minimum at $(\pi/4, 5)$.

x	0	$\pi/4$	$\pi/2$
y'		0	
		-	+
y		Min	

Section 3: IB exam-style

31. (a) $f' = 2\sec^2(2x) - 1$. (b) $\sec^2(2x) \geq 1$, so $f' \geq 1 > 0$. (c) $f(0) = 0$ and $f'(0) = 1$; tangent $y = x$, normal $y = -x$.

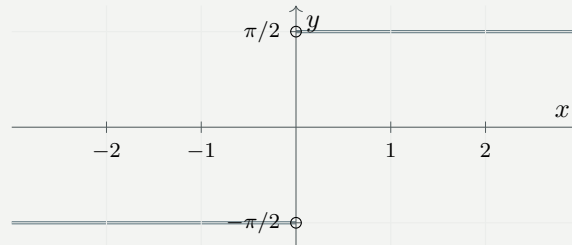


Section 3: IB exam-style (continued)

- 32.** (a) $f' = \sec x(\tan x - \sec x)$. Here $\sec x > 0$ and $\sec x - \tan x = (1 - \sin x)/\cos x > 0$, so $f' < 0$.
(b) Endpoint values give $[2 - \sqrt{3}, 2 + \sqrt{3}]$. (c) $[\sec x - \tan x]_0^{\pi/3} = 1 - \sqrt{3}$.
- 33.** (a) $g' = -2 \operatorname{cosec}(2x) \cot(2x)$. (b) $\operatorname{cosec}(2x) > 0$, so stationarity requires $\cot(2x) = 0$: $x = \pi/4$, $g = 1$. On the stated domain $\sin(2x) \leq 1$ and is positive, so $g \geq 1$; this is a global minimum. (c) $y = 1$.
- 34.** (a) $f' = 2 \ln a a^{2x}$ gives $2 \ln a = 4 \ln 3$, so $a = 9$. (b) $[9^{2x}/(2 \ln 9)]_0^{1/2} = 8/(2 \ln 9) = 2/\ln 3$.
- 35.** (a) $f' = 3/[(3x+1) \ln b]$. Thus $\ln b = \ln 5$ and $b = 5$. (b) $3/(3x+1) = 1$ gives $x = 2/3$ and $y = \log_5 3$. (c) $y - \log_5 3 = \frac{x - 2/3}{\ln 5}$.
- 36.** (a) $f' = 1/\sqrt{4-x^2}$. (b) $f(1) = \pi/6$, $f'(1) = 1/\sqrt{3}$; $y - \pi/6 = (x-1)/\sqrt{3}$. (c) $f' > 0$, so $A = \int_0^1 f'(x) dx = f(1) - f(0) = \pi/6$ square units.
- 37.** (a) $f' = 2/\sqrt{1-(1-2x)^2} = 2/\sqrt{4x-4x^2} = 1/\sqrt{x-x^2}$. (b) $y - \pi/2 = 2(x-1/2)$. (c) $[\arccos(1-2x)]_{1/4}^{3/4} = 2\pi/3 - \pi/3 = \pi/3$.
- 38.** (a) $y' = 2/(4+x^2)$. (b) $2/(4+x^2) = 1/4$ gives $x = \pm 2$; points $(2, \pi/4)$ and $(-2, -\pi/4)$. (c) $y = x/4 + \pi/4 - 1/2$ and $y = x/4 - \pi/4 + 1/2$; intercepts $\pi/4 - 1/2$ and $1/2 - \pi/4$.
- 39.** (a) $F = \sec(2x) + \cot(3x) + C$. At $x = \pi/6$, $5 = 2 + 0 + C$, so $F = \sec(2x) + \cot(3x) + 3$. (b) $F(\pi/12) = \sec(\pi/6) + \cot(\pi/4) + 3 = \frac{2}{\sqrt{3}} + 4$.
- 40.** (a) $h' = 1/\sqrt{1-x^2} - 1/\sqrt{1-x^2} = 0$. (b) h is constant on the interval; $h(0) = 0 + \pi/2 = \pi/2$.
(c) $[\arcsin x]_{-1/2}^{1/2} = \pi/6 - (-\pi/6) = \pi/3$.
- 41.** (a) $3x+2 = A(x+2) + Bx$ gives $A = 1$, $B = 2$, so $f = 1/x + 2/(x+2)$. (b) $[\ln x + 2 \ln(x+2)]_1^2 = \ln 2 + 2 \ln(4/3) = \ln(32/9)$.
- 42.** (a) $2/(x^2-9) = \frac{1}{3(x-3)} - \frac{1}{3(x+3)}$. (b) $F = \frac{1}{3} \ln \left| \frac{x-3}{x+3} \right| + C$. At 0, the logarithm is $\ln 1 = 0$, so $C = 0$. Equivalently, $F = \frac{1}{3} \ln \frac{3-x}{3+x}$ on this domain. (c) $F' < 0$, so decreasing; the integral is $F(1) - F(0) = \frac{1}{3} \ln(1/2) = -\frac{1}{3} \ln 2$.
- 43.** (a) $x^2+1 = A(x+1)^2 + Bx(x+1) + Cx$. Comparing coefficients: $A = 1$, $A+B = 1$, $2A+B+C = 0$, so $B = 0$, $C = -2$. (b) $\int f dx = \ln x + 2/(x+1) + K$. Thus $\int_1^2 f dx = \ln 2 + 2/3 - 1 = \ln 2 - 1/3$.
- 44.** (a) $x^2 + 6x + 13 = (x+3)^2 + 4$; $A = [3 \arctan((x+3)/2)]_{-3}^{-1} = 3\pi/4$. (b) Put $t = \tan(\pi/8) > 0$. Then $1 = 2t/(1-t^2)$, so $t^2 + 2t - 1 = 0$ and $t = \sqrt{2} - 1$. (c) $3 \arctan((k+3)/2) = 3\pi/8$. Thus $(k+3)/2 = \sqrt{2} - 1$ and $k = 2\sqrt{2} - 5$, which lies between -3 and -1 .
- 45.** (a) An antiderivative is $\arcsin((x-1)/4)$. Thus $A = \arcsin(1/2) - \arcsin 0 = \pi/6$. (b) $\arcsin((a-1)/4) = \pi/4$, so $(a-1)/4 = \sqrt{2}/2$ and $a = 1 + 2\sqrt{2}$.
- 46.** (a) $s = 2 \arctan(2t) + C$. Since $s(0) = -\pi/3$, $s(t) = 2 \arctan(2t) - \pi/3$ metres. (b) $\arctan(2t) = \pi/6$, so $t = 1/(2\sqrt{3}) = \sqrt{3}/6$ seconds. (c) $v > 0$, so distance $= \int_0^{\sqrt{3}/2} v dt = 2 \arctan \sqrt{3} = 2\pi/3$ metres.
- 47.** (a) $f' = \ln 3(3^x - 3^{-x})$ and $f'' = (\ln 3)^2(3^x + 3^{-x})$. (b) $3^{2x} = 1$ gives $x = 0$, $f(0) = 2$; $f'' > 0$, so a minimum at $(0, 2)$. (c) $\left[\frac{3^x - 3^{-x}}{\ln 3} \right]_{-1}^1 = \frac{16}{3 \ln 3}$.
- 48.** (a) $F' = 2/(x-1) - 1/(x+2) = [2(x+2) - (x-1)]/[(x-1)(x+2)] = (x+5)/(x^2+x-2)$. (b) $F(1/2) - F(0) = 2 \ln(1/2) - \ln(5/2) + \ln 2 = -\ln 5$. (c) The denominator $(x-1)(x+2)$ is negative on $[0, 1/2]$, so the integrand is negative.

Section 3: IB exam-style (continued)

49. (a) $h' = \frac{1}{1+x^2} + \frac{-1/x^2}{1+1/x^2} = 0$. (b) $h(1) = \pi/2$ and $h(-1) = -\pi/2$. Hence $h = \pi/2$ for $x > 0$ and $h = -\pi/2$ for $x < 0$. (c) Draw horizontal half-lines at these heights with open circles at $(0, \pm\pi/2)$. The domain consists of two separate intervals; a zero derivative gives a constant on each interval, not necessarily the same constant on both.



50. (a) The numerator is one less than the denominator, and $x^2 + 4x + 8 = (x + 2)^2 + 4$. (b) $F = x - \frac{1}{2} \arctan((x + 2)/2) + C$. At -2 , $0 = -2 + C$, so $F = x + 2 - \frac{1}{2} \arctan((x + 2)/2)$. (c) $f \geq 3/4 > 0$, so $A = F(0) - F(-2) = 2 - \pi/8$. (d) $1 - f = 1/[(x + 2)^2 + 4]$, so $\frac{1}{2} \arctan((k + 2)/2) = \pi/12$. Hence $k = -2 + 2 \tan(\pi/6) = -2 + \frac{2\sqrt{3}}{3}$.