



Advanced Integration

IB Mathematics AA HL | AHL 5.16

50 questions: substitution, integration by parts, repeated and cyclic integration

Learning goal	Select an effective integration method, execute it accurately and interpret definite integrals in context.
Syllabus link	AHL 5.16: substitution; integration by parts; repeated integration by parts. Less direct substitutions are supplied, as in the syllabus guidance.
Structure	15 short-response, 15 extended-response and 20 IB Paper 1-style questions.
Technology	Non-calculator throughout. All angles are measured in radians.

Instructions. Work on separate paper and give exact answers. Show your substitution or your choices of u and dv . Change the limits when evaluating a definite integral in a new variable. Include $+C$ in indefinite integrals. Distinguish signed integrals from total area or distance.

1. Short response | Questions 1–15

1. For each integral, select an efficient method: substitution, integration by parts once, repeated integration by parts, or cyclic integration by parts. Do not evaluate.

(a) $\int 4x \cos(x^2) dx$	(b) $\int x \ln x dx$
(c) $\int x^2 e^x dx$	(d) $\int e^x \cos x dx$

2. Find $\int 6x(x^2 + 1)^4 dx$.

3. Find $\int \frac{x}{\sqrt{x^2 + 9}} dx$.

4. Find $\int x e^{x^2} dx$.

5. Find $\int \frac{\cos(3x)}{2 + \sin(3x)} dx$.

6. Find $\int \frac{(\ln x)^2}{x} dx$, where $x > 0$.

7. Find $\int x \cos x dx$.

8. Find $\int x e^{2x} dx$.



9. Find $\int \ln x \, dx$, where $x > 0$.

10. Find $\int x \ln x \, dx$, where $x > 0$.

11. Find $\int \arcsin x \, dx$, where $-1 < x < 1$.

12. Find $\int \arctan x \, dx$.

13. Find $\int x^2 e^x \, dx$.

14. Find $\int x^2 \sin x \, dx$.

15. A student writes $\int x e^{-x} \, dx = -x e^{-x} + e^{-x} + C$. Identify the sign error and give the correct antiderivative.

2. Extended response | Questions 16–30

16. Use the substitution $u = x^2 + 4$ to find $\int x^3 \sqrt{x^2 + 4} \, dx$. State how you rewrite both x^2 and $x \, dx$.

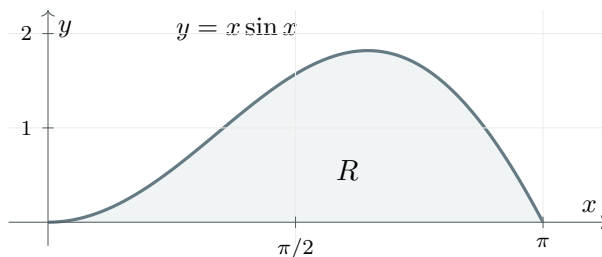
17. A function F satisfies $F'(x) = \frac{x^2}{1+x^3}$ on $x > -1$, and $F(0) = 1$.

- (a) Find $F(x)$ using substitution.
(b) Find $F(1)$.
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18. Use $u = \sqrt{x+1}$ to evaluate $\int_0^3 \frac{x}{\sqrt{x+1}} \, dx$.

- (a) Express x and dx in terms of u , and change the limits.
(b) Evaluate the transformed integral.

- 19.** The region R lies between $y = x \sin x$ and the x -axis for $0 \leq x \leq \pi$.



- (a) Use integration by parts to find an antiderivative of $x \sin x$.
(b) Find the exact area of R .

- 20.** Consider $I = \int (\ln x)^2 dx$, where $x > 0$.

- (a) Use integration by parts to express I in terms of $\int \ln x dx$.
(b) Hence evaluate $\int_1^e (\ln x)^2 dx$.

- 21.** A function F satisfies $F'(x) = x^2 \cos x$ and $F(0) = 2$.

- (a) Find $F(x)$ using integration by parts twice.
(b) Find $F(\pi)$.

- 22.** Consider $\int x^2 e^{2x} dx$.

- (a) Use repeated integration by parts to find this integral.
(b) Hence evaluate $\int_0^1 x^2 e^{2x} dx$.

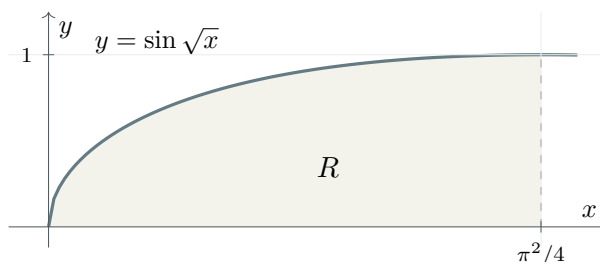
- 23.** Let $I = \int e^x \sin x dx$ and $J = \int e^x \cos x dx$.

- (a) Use integration by parts to obtain two equations linking I and J .
(b) Hence find I and evaluate $\int_0^\pi e^x \sin x dx$.

- 24.** Let $I = \int e^{2x} \cos(3x) dx$.

- (a) Apply integration by parts twice, displaying the equation in which I reappears.
(b) Hence find I .

- 25.** Use the substitution $u = \sqrt{x}$ to find an antiderivative of $\sin \sqrt{x}$ for $x > 0$. Hence find the area of the shaded region under the curve from $x = 0$ to $x = \frac{\pi^2}{4}$.

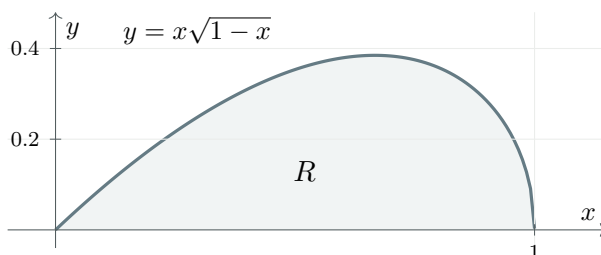


- 26.** Use the substitution $u = \sqrt{x}$ to evaluate $\int_1^{e^2} \frac{\ln x}{\sqrt{x}} dx$.

- (a) Show that the transformed integral is $4 \int_1^e \ln u du$.
(b) Hence evaluate it exactly.

- 27.** Use $u = x^2 + 1$ to find $\int \frac{x^3}{x^2 + 1} dx$. Hence evaluate $\int_0^1 \frac{x^3}{x^2 + 1} dx$.

- 28.** The diagram shows $y = x\sqrt{1-x}$ for $0 \leq x \leq 1$.



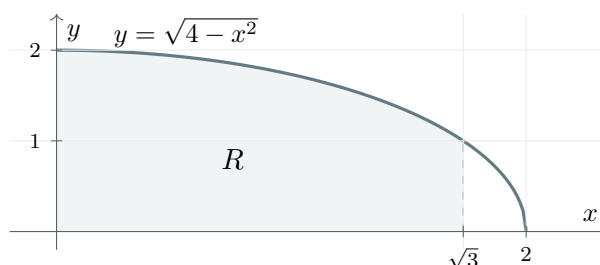
- (a) Using $u = 1 - x$, rewrite the area integral with correctly transformed limits.
(b) Find the area under the curve.

- 29.** Use the substitution $u = x^2$ to evaluate $\int_0^1 x^3 e^{x^2} dx$.

- (a) Express the integral entirely in terms of u .
(b) Use integration by parts to evaluate it.

- 30.** A function F satisfies $F'(x) = (ax + b)e^x$, where a and b are constants. It is given that $F(0) = 0$, $F'(0) = 1$ and $F(1) = e + 1$.

- (a) Find b .
(b) Use integration by parts to find a and the equation of F .

**3. IB exam-style | Paper 1 | Questions 31–50****31.** [Maximum mark: 6]Consider $I = \int_0^{\pi/3} \sin x \cos^3 x \, dx$.(a) Use a suitable substitution to rewrite I with new limits. [3 marks](b) Hence find the exact value of I . [3 marks]**32.** [Maximum mark: 7]Let $I = \int_1^{e^2} \frac{\ln x \sqrt{1 + (\ln x)^2}}{x} \, dx$.(a) Using $u = \ln x$, show that $I = \int_0^2 u \sqrt{1 + u^2} \, du$. [3 marks](b) Hence evaluate I exactly. [4 marks]**33.** [Maximum mark: 8]Consider $I = \int_0^{\sqrt{3}} \frac{x^3}{(1 + x^2)^2} \, dx$.(a) Using $u = 1 + x^2$, show that $I = \frac{1}{2} \int_1^4 \left(\frac{1}{u} - \frac{1}{u^2} \right) du$. [4 marks](b) Hence show that $I = \ln 2 - \frac{3}{8}$. [4 marks]**34.** [Maximum mark: 9]The region R lies under $y = \sqrt{4 - x^2}$ from $x = 0$ to $x = \sqrt{3}$.(a) Use the substitution $x = 2 \sin \theta$, with $0 \leq \theta \leq \pi/3$, to show that the area is $4 \int_0^{\pi/3} \cos^2 \theta \, d\theta$. [4 marks](b) Hence find the exact area of R . [5 marks]**35.** [Maximum mark: 10]Let $I = \int_{\sqrt{3}}^{2\sqrt{2}} \frac{1}{x\sqrt{x^2 + 1}} \, dx$. Use $u = \sqrt{x^2 + 1}$.(a) Show that $I = \int_2^3 \frac{1}{u^2 - 1} \, du$. [4 marks](b) Express $\frac{1}{u^2 - 1}$ in partial fractions. [2 marks](c) Hence find the exact value of I . [4 marks]



36. [Maximum mark: 9]

Consider $I = \int_0^{\ln 2} \frac{e^{2x}}{\sqrt{1+e^x}} dx$.

- (a) Using $u = \sqrt{1+e^x}$, show that $I = 2 \int_{\sqrt{2}}^{\sqrt{3}} (u^2 - 1) du$. [5 marks]
(b) Hence evaluate I exactly. [4 marks]

37. [Maximum mark: 7]

Let $I = \int_0^1 (x+1) \ln(x+1) dx$.

- (a) Use integration by parts to find an antiderivative of $(x+1) \ln(x+1)$. [4 marks]
(b) Hence evaluate I . [3 marks]

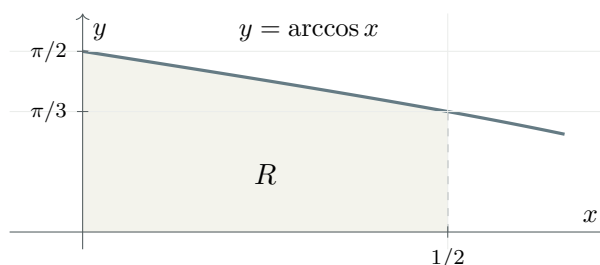
38. [Maximum mark: 9]

Consider $I = \int_0^1 x \arctan x dx$.

- (a) Use integration by parts to show that $I = \frac{\pi}{8} - \frac{1}{2} \int_0^1 \frac{x^2}{1+x^2} dx$. [4 marks]
(b) Use $\frac{x^2}{1+x^2} = 1 - \frac{1}{1+x^2}$ to evaluate I exactly. [5 marks]

39. [Maximum mark: 8]

The shaded region R lies under $y = \arccos x$ between $x = 0$ and $x = \frac{1}{2}$.



- (a) Use integration by parts to show that $\int \arccos x dx = x \arccos x - \sqrt{1-x^2} + C$. [5 marks]
(b) Hence find the exact area of R . [3 marks]

40. [Maximum mark: 8]

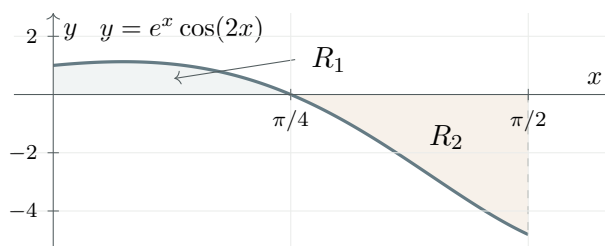
Consider $I = \int_0^{\pi/4} x \sec^2 x dx$.

- (a) Use integration by parts to express I in terms of $\int_0^{\pi/4} \tan x dx$. [3 marks]
(b) By differentiating $\ln(\cos x)$ on this interval, hence evaluate I . [5 marks]



41. [Maximum mark: 12]

The diagram shows $y = e^x \cos(2x)$ for $0 \leq x \leq \frac{\pi}{2}$.



- (a) Use integration by parts twice to show that $\int e^x \cos(2x) dx = \frac{e^x}{5} (\cos 2x + 2 \sin 2x) + C$. [6 marks]
- (b) Find $\int_0^{\pi/2} e^x \cos(2x) dx$. [2 marks]
- (c) Find the total area of R_1 and R_2 . [4 marks]

42. [Maximum mark: 9]

Consider $I = \int_0^1 x^3 e^{-x} dx$.

- (a) Using repeated integration by parts, show that an antiderivative of $x^3 e^{-x}$ is $-e^{-x}(x^3 + 3x^2 + 6x + 6)$. [6 marks]
- (b) Hence evaluate I . [3 marks]

43. [Maximum mark: 10]

For integers $n \geq 0$, define $I_n = \int_0^1 x^n e^x dx$.

- (a) Find I_0 . [1 marks]
- (b) For $n \geq 1$, use integration by parts to show that $I_n = e - nI_{n-1}$. [4 marks]
- (c) Hence find I_3 . [3 marks]
- (d) Explain why $I_3 < I_2$ without using decimal approximations. [2 marks]

44. [Maximum mark: 10]

For integers $n \geq 0$, let $J_n = \int_0^{\pi/2} \sin^n x dx$.

- (a) Write down J_0 and J_1 . [2 marks]
- (b) For $n \geq 2$, use integration by parts with $u = \sin^{n-1} x$ and $dv = \sin x dx$ to show that $J_n = \frac{n-1}{n} J_{n-2}$. [5 marks]
- (c) Hence find J_4 and J_5 . [3 marks]

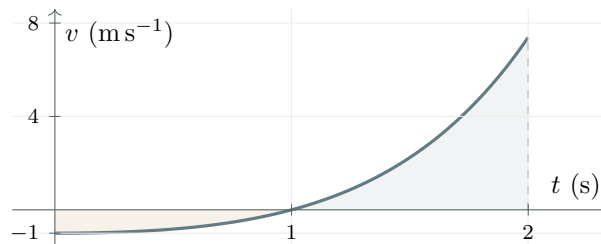
45. [Maximum mark: 9]

Consider $I = \int_1^e x(\ln x)^2 dx$.

- (a) Use integration by parts to express I in terms of $\int_1^e x \ln x dx$. [4 marks]
- (b) Use integration by parts again to find the exact value of I . [5 marks]

**46.** [Maximum mark: 11]

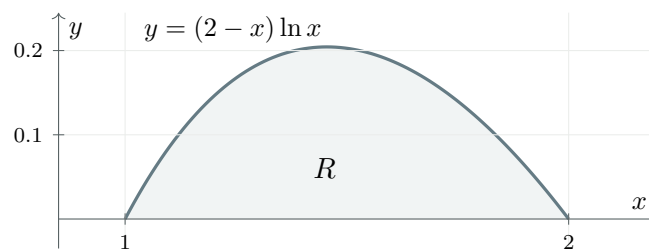
A particle moves in a straight line with velocity $v(t) = (t-1)e^t$ m s⁻¹ for $0 \leq t \leq 2$. Its initial displacement from a fixed origin is zero.



- (a) Find the time at which the particle changes direction. [2 marks]
(b) Use integration by parts to find the displacement $s(t)$. [4 marks]
(c) Find the displacement at $t = 2$ and the total distance travelled for $0 \leq t \leq 2$. [5 marks]

47. [Maximum mark: 9]

The curve $y = (2-x)\ln x$ encloses a region R with the x -axis between $x = 1$ and $x = 2$.



- (a) Explain why the curve is non-negative on this interval. [2 marks]
(b) Find an antiderivative of $(2-x)\ln x$. [4 marks]
(c) Hence find the exact area of R . [3 marks]



48. [Maximum mark: 9]

For a positive integer n , define $K_n = \int_0^\pi x \sin(nx) dx$.

- (a) Use integration by parts to show that $K_n = -\frac{\pi(-1)^n}{n}$. [5 marks]
- (b) Find K_2 and K_3 . [2 marks]
- (c) Explain why $K_2 < 0$ does not imply that $x \sin(2x)$ is negative throughout $[0, \pi]$. [2 marks]

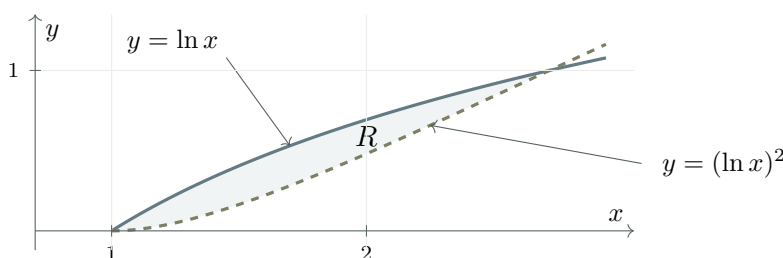
49. [Maximum mark: 10]

Consider $I = \int_1^{e^{\pi/2}} \sin(\ln x) dx$.

- (a) Using $u = \ln x$, show that $I = \int_0^{\pi/2} e^u \sin u du$. [3 marks]
- (b) Use integration by parts twice to find an antiderivative of $e^u \sin u$. [5 marks]
- (c) Hence evaluate I exactly. [2 marks]

50. [Maximum mark: 10]

The curves $y = \ln x$ and $y = (\ln x)^2$ enclose the region R shown.



- (a) Find the coordinates of their two intersections. [2 marks]
- (b) Use integration by parts to find $\int (\ln x)^2 dx$. [4 marks]
- (c) Determine which curve is above the other between the intersections, and hence find the exact area of R . [4 marks]



Answer key and solution outlines

Integration by parts: $\int u dv = uv - \int v du$. For definite integrals, evaluate the boundary term at both limits. Constants in intermediate indefinite steps are absorbed into the final C .

Section 1: Short response

1. (a) Substitution: $u = x^2$. (b) Parts once: differentiate $\ln x$. (c) Repeated parts: differentiate x^2 twice. (d) Cyclic parts: two applications return to the original integral.
2. $u = x^2 + 1, du = 2x dx$: $3 \int u^4 du = \frac{3}{5}(x^2 + 1)^5 + C$.
3. $u = x^2 + 9, du = 2x dx$: $\frac{1}{2} \int u^{-1/2} du = \sqrt{x^2 + 9} + C$.
4. $u = x^2, du = 2x dx$: $\frac{1}{2} \int e^u du = \frac{1}{2}e^{x^2} + C$.
5. $u = 2 + \sin(3x), du = 3 \cos(3x) dx$: $\frac{1}{3} \int \frac{1}{u} du = \frac{1}{3} \ln(2 + \sin 3x) + C$. The logarithm argument is always positive.
6. $u = \ln x, du = dx/x$: $\int u^2 du = \frac{1}{3}(\ln x)^3 + C$.
7. With $u = x, dv = \cos x dx$, parts gives $x \sin x - \int \sin x dx = x \sin x + \cos x + C$.
8. $u = x, dv = e^{2x} dx, v = \frac{1}{2}e^{2x}$: $\frac{1}{2}xe^{2x} - \frac{1}{2} \int e^{2x} dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C$.
9. Write $\ln x = 1 \cdot \ln x$. With $u = \ln x, dv = dx$, parts gives $x \ln x - \int 1 dx = x \ln x - x + C$.
10. $u = \ln x, dv = x dx$: $\frac{x^2}{2} \ln x - \frac{1}{2} \int x dx = \frac{x^2}{2} \ln x - \frac{x^2}{4} + C$.
11. $u = \arcsin x, dv = dx$: $x \arcsin x - \int \frac{x}{\sqrt{1-x^2}} dx = x \arcsin x + \sqrt{1-x^2} + C$. In the remaining integral use $w = 1 - x^2, dw = -2x dx$.
12. $u = \arctan x, dv = dx$: $x \arctan x - \int \frac{x}{1+x^2} dx = x \arctan x - \frac{1}{2} \ln(1+x^2) + C$.
13. Parts gives $I = x^2 e^x - 2 \int x e^x dx$. A second application gives $\int x e^x dx = x e^x - e^x$. Hence $I = e^x(x^2 - 2x + 2) + C$.
14. Parts gives $I = -x^2 \cos x + 2 \int x \cos x dx$. Thus $I = -x^2 \cos x + 2x \sin x + 2 \cos x + C$.
15. With $u = x, dv = e^{-x} dx, v = -e^{-x}$, parts gives $-x e^{-x} + \int e^{-x} dx = -(x+1)e^{-x} + C$. The last term must be $-e^{-x}$. Its derivative is $-e^{-x} + (x+1)e^{-x} = x e^{-x}$.

Section 2: Extended response

16. $x^2 = u - 4$ and $x dx = du/2$. Thus

$$I = \frac{1}{2} \int (u-4)u^{1/2} du = \frac{1}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C.$$

Hence $I = \frac{1}{5}(x^2 + 4)^{5/2} - \frac{4}{3}(x^2 + 4)^{3/2} + C$.

17. (a) $u = 1 + x^3, du = 3x^2 dx$, so $F = \frac{1}{3} \ln(1+x^3) + C$. Since $F(0) = 1, C = 1$. (b) $F(1) = 1 + \frac{1}{3} \ln 2$.



Section 2: Extended response (continued)

18. (a) $x = u^2 - 1$, $dx = 2u du$; $x = 0, 3$ correspond to $u = 1, 2$. (b)

$$I = 2 \int_1^2 (u^2 - 1) du = \left[\frac{2}{3}u^3 - 2u \right]_1^2 = \frac{8}{3}.$$

19. (a) $u = x$, $dv = \sin x dx$, $v = -\cos x$, so $\int x \sin x dx = -x \cos x + \sin x + C$. (b) The curve is non-negative on $[0, \pi]$: $A = [-x \cos x + \sin x]_0^\pi = \pi$ square units.

20. (a) $u = (\ln x)^2$, $dv = dx$, so $I = x(\ln x)^2 - 2 \int \ln x dx$. (b) Using $\int \ln x dx = x \ln x - x + C$,

$$I = x((\ln x)^2 - 2 \ln x + 2) + C.$$

The definite integral is $e - 2$.

21. (a) $\int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx = x^2 \sin x + 2x \cos x - 2 \sin x + C$. At $x = 0$, $C = 2$. (b) $F(\pi) = 2 - 2\pi$.

22. (a) With $u = x^2$, $dv = e^{2x} dx$,

$$I = \frac{1}{2}x^2 e^{2x} - \int x e^{2x} dx = e^{2x} \left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4} \right) + C.$$

- (b) Substitute 1 and 0: $I_{[0,1]} = (e^2 - 1)/4$.

23. (a) Up to constants, $I = e^x \sin x - J$ and $J = e^x \cos x + I$. (b) Therefore $2I = e^x(\sin x - \cos x)$, giving

$$I = \frac{e^x}{2}(\sin x - \cos x) + C.$$

The definite integral is $e^\pi/2 - (-1/2) = (e^\pi + 1)/2$.

24. (a) Put $J = \int e^{2x} \sin(3x) dx$. Up to constants,

$$I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{2}J, \quad J = \frac{1}{2}e^{2x} \sin 3x - \frac{3}{2}I.$$

Thus $I = \frac{1}{2}e^{2x} \cos 3x + \frac{3}{4}e^{2x} \sin 3x - \frac{9}{4}I$. (b) Rearranging gives $I = \frac{e^{2x}}{13}(2 \cos 3x + 3 \sin 3x) + C$.

25. $x = u^2$, $dx = 2u du$, so

$$\int \sin \sqrt{x} dx = 2 \int u \sin u du = 2(-u \cos u + \sin u) + C.$$

Thus an antiderivative is $2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x}$. The curve is non-negative on the interval, so $A = [2 \sin \sqrt{x} - 2\sqrt{x} \cos \sqrt{x}]_0^{\pi^2/4} = 2$ square units.

26. (a) $x = u^2$, $dx = 2u du$ and $\ln x = 2 \ln u$. The limits become 1 and e , giving $4 \int_1^e \ln u du$. (b) $4[u \ln u - u]_1^e = 4(0 - (-1)) = 4$.

27. $x^2 = u - 1$, $x dx = du/2$, so $I = \frac{1}{2} \int (1 - u^{-1}) du = \frac{1}{2}(x^2 + 1 - \ln(x^2 + 1)) + C$. For the definite integral, $u = 1$ to 2: $\frac{1}{2}[u - \ln u]_1^2 = (1 - \ln 2)/2$.

28. (a) $x = 1 - u$, $dx = -du$; the limits $x = 0, 1$ become $u = 1, 0$. Therefore $A = -\int_1^0 (1 - u)u^{1/2} du = \int_0^1 (u^{1/2} - u^{3/2}) du$. (b) $A = [\frac{2}{3}u^{3/2} - \frac{2}{5}u^{5/2}]_0^1 = \frac{4}{15}$ square units.

29. (a) $x^3 dx = x^2(x dx) = u du/2$, and the limits remain 0 and 1. Hence $I = \frac{1}{2} \int_0^1 u e^u du$. (b) $I = \frac{1}{2}[(u - 1)e^u]_0^1 = \frac{1}{2}$.

30. (a) $F'(0) = b = 1$. (b) Integrating gives $F = a(x - 1)e^x + e^x + C$. The condition $F(0) = 0$ gives $C = a - 1$. Then $F(1) = e + a - 1 = e + 1$, so $a = 2$, $C = 1$. Thus $F = (2x - 1)e^x + 1$.



Section 3: IB exam-style

31. (a) Take $u = \cos x$, $du = -\sin x dx$. The limits become 1 and $1/2$, so $I = -\int_1^{1/2} u^3 du = \int_{1/2}^1 u^3 du$. (b) $I = [u^4/4]_{1/2}^1 = 15/64$.

32. (a) $du = dx/x$ and $\ln 1 = 0$, $\ln(e^2) = 2$, so the transformed integral follows. (b) With $w = 1 + u^2$, $dw = 2u du$ and limits 1, 5,

$$I = \frac{1}{2} \int_1^5 w^{1/2} dw = \frac{1}{3} [w^{3/2}]_1^5 = \frac{5\sqrt{5} - 1}{3}.$$

33. (a) $du = 2x dx$, $x^2 = u - 1$ and $x^3 dx = (u - 1)du/2$. Thus the integrand becomes $(u - 1)/(2u^2)$. (b) $I = \frac{1}{2} [\ln u + u^{-1}]_1^4 = \frac{1}{2} (\ln 4 + 1/4 - 1) = \ln 2 - 3/8$.

34. (a) $dx = 2 \cos \theta d\theta$ and $\sqrt{4 - x^2} = 2\sqrt{1 - \sin^2 \theta} = 2 \cos \theta$ because $\cos \theta > 0$ on the interval. The limits are 0 and $\pi/3$. (b) Since $\cos^2 \theta = (1 + \cos 2\theta)/2$,

$$A = 2 \int_0^{\pi/3} (1 + \cos 2\theta) d\theta = [2\theta + \sin 2\theta]_0^{\pi/3} = \frac{2\pi}{3} + \frac{\sqrt{3}}{2}.$$

35. (a) $u^2 = x^2 + 1$ gives $dx = (u/x)du$, so $dx/(xu) = du/x^2 = du/(u^2 - 1)$. The endpoints give $u = 2, 3$. (b) $1/(u^2 - 1) = \frac{1}{2}[1/(u - 1) - 1/(u + 1)]$. (c)

$$I = \frac{1}{2} \left[\ln \frac{u - 1}{u + 1} \right]_2^3 = \frac{1}{2} \ln \frac{3}{2}.$$

36. (a) $e^x = u^2 - 1$ and $e^x dx = 2u du$. Hence $e^{2x} dx/u = 2(u^2 - 1)du$. At $x = 0, \ln 2$, $u = \sqrt{2}, \sqrt{3}$. (b)

$$I = \left[\frac{2}{3} u^3 - 2u \right]_{\sqrt{2}}^{\sqrt{3}} = 0 - \left(-\frac{2\sqrt{2}}{3} \right) = \frac{2\sqrt{2}}{3}.$$

37. (a) Take $u = \ln(x + 1)$, $dv = (x + 1)dx$, so $v = (x + 1)^2/2$. Then

$$\int (x + 1) \ln(x + 1) dx = \frac{(x + 1)^2}{2} \ln(x + 1) - \frac{(x + 1)^2}{4} + C.$$

(b) At 1 the expression is $2 \ln 2 - 1$; at 0 it is $-1/4$. Therefore $I = 2 \ln 2 - 3/4$.

38. (a) $u = \arctan x$, $dv = x dx$, giving boundary term $[x^2 \arctan x/2]_0^1 = \pi/8$. (b) The remaining integral is $[x - \arctan x]_0^1 = 1 - \pi/4$. Thus $I = \pi/8 - \frac{1}{2}(1 - \pi/4) = \pi/4 - 1/2$.

39. (a) $u = \arccos x$, $dv = dx$ gives $\int \arccos x dx = x \arccos x + \int x/\sqrt{1 - x^2} dx$. In the latter put $w = 1 - x^2$, $dw = -2x dx$, giving $-\sqrt{1 - x^2}$. (b) $A = [x \arccos x - \sqrt{1 - x^2}]_0^{1/2} = 1 + \pi/6 - \sqrt{3}/2$.

40. (a) With $u = x$, $dv = \sec^2 x dx$,

$$I = [x \tan x]_0^{\pi/4} - \int_0^{\pi/4} \tan x dx = \frac{\pi}{4} - \int_0^{\pi/4} \tan x dx.$$

(b) $(\ln \cos x)' = -\tan x$. Thus $\int_0^{\pi/4} \tan x dx = [-\ln \cos x]_0^{\pi/4} = \frac{1}{2} \ln 2$, and $I = \pi/4 - \frac{1}{2} \ln 2$.

41. (a) Put $I = \int e^x \cos 2x dx$ and $J = \int e^x \sin 2x dx$. Parts gives, up to constants, $I = e^x \cos 2x + 2J$ and $J = e^x \sin 2x - 2I$. Hence $5I = e^x(\cos 2x + 2 \sin 2x)$, giving the stated antiderivative H . (b) $H(0) = 1/5$, $H(\pi/2) = -e^{\pi/2}/5$, so the integral is $-(e^{\pi/2} + 1)/5$. (c) The sign changes at $x = \pi/4$, where $H = 2e^{\pi/4}/5$. Therefore

$$A = [H(\pi/4) - H(0)] - [H(\pi/2) - H(\pi/4)] = \frac{4e^{\pi/4} + e^{\pi/2} - 1}{5}.$$



Section 3: IB exam-style (continued)

42. (a) Successive parts with $dv = e^{-x}dx$ give

$$\begin{aligned}\int x^3 e^{-x} dx &= -x^3 e^{-x} + 3 \int x^2 e^{-x} dx, \\ \int x^2 e^{-x} dx &= -x^2 e^{-x} + 2 \int x e^{-x} dx, \\ \int x e^{-x} dx &= -(x+1)e^{-x} + C.\end{aligned}$$

Combining gives the stated antiderivative. (b) $I = -16/e - (-6) = 6 - 16/e$.

43. (a) $I_0 = e - 1$. (b) $u = x^n$, $dv = e^x dx$ gives $I_n = [x^n e^x]_0^1 - n \int_0^1 x^{n-1} e^x dx = e - nI_{n-1}$ for $n \geq 1$. (c) $I_1 = 1$, $I_2 = e - 2$, $I_3 = e - 3(e - 2) = 6 - 2e$. (d) For $0 < x < 1$, $x^3 e^x < x^2 e^x$, so integrating gives the strict inequality.

44. (a) $J_0 = \pi/2$, $J_1 = 1$. (b) The boundary term $[-\sin^{n-1} x \cos x]_0^{\pi/2}$ is zero for $n \geq 2$. Thus

$$J_n = (n-1) \int_0^{\pi/2} \sin^{n-2} x \cos^2 x dx = (n-1)(J_{n-2} - J_n).$$

Rearranging gives $nJ_n = (n-1)J_{n-2}$. (c) $J_4 = \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 3\pi/16$; $J_5 = \frac{4}{5} \cdot \frac{2}{3} \cdot 1 = 8/15$.

45. (a) $u = (\ln x)^2$, $dv = x dx$ gives $I = [\frac{1}{2}x^2(\ln x)^2]_1^e - \int_1^e x \ln x dx = e^2/2 - K$. (b) With $u = \ln x$, $dv = x dx$, $K = [\frac{1}{2}x^2 \ln x - \frac{1}{4}x^2]_1^e = e^2/4 + 1/4$. Hence $I = (e^2 - 1)/4$.

46. (a) $e^t > 0$, so $v = 0$ at $t = 1$; it is negative before and positive after. (b) Parts gives $\int (t-1)e^t dt = (t-2)e^t + C$. Since $s(0) = 0$, $C = 2$; $s(t) = (t-2)e^t + 2$ metres. (c) $s(1) = 2 - e$ and $s(2) = 2$. Total distance $= |s(1) - s(0)| + |s(2) - s(1)| = (e-2) + e = 2e - 2$ metres; final displacement is 2 metres.

47. (a) $2 - x \geq 0$ and $\ln x \geq 0$ on $[1, 2]$. (b) Split the integral as $2 \int \ln x dx - \int x \ln x dx$. Parts gives

$$H(x) = 2x \ln x - 2x - \frac{x^2}{2} \ln x + \frac{x^2}{4}.$$

(c) $H(2) = 2 \ln 2 - 3$, $H(1) = -7/4$; area $= 2 \ln 2 - 5/4$ square units.

48. (a) With $u = x$, $dv = \sin(nx)dx$, $v = -\cos(nx)/n$,

$$K_n = \left[-\frac{x \cos(nx)}{n} + \frac{\sin(nx)}{n^2} \right]_0^\pi = -\frac{\pi \cos(n\pi)}{n} = -\frac{\pi(-1)^n}{n}.$$

(b) $K_2 = -\pi/2$, $K_3 = \pi/3$. (c) The integrand is positive on $(0, \pi/2)$ and negative on $(\pi/2, \pi)$; the negative signed contribution has greater magnitude.

49. (a) $x = e^u$, $dx = e^u du$; the limits become $0, \pi/2$. (b) If $A = \int e^u \sin u du$ and $B = \int e^u \cos u du$, parts gives $A = e^u \sin u - B$ and $B = e^u \cos u + A$, up to constants. Thus $A = \frac{1}{2}e^u(\sin u - \cos u) + C$. (c)

$$I = \left[\frac{1}{2}e^u(\sin u - \cos u) \right]_0^{\pi/2} = \frac{e^{\pi/2} + 1}{2}.$$

50. (a) Let $z = \ln x$. Then $z = z^2$ gives $z = 0, 1$, so intersections $(1, 0)$ and $(e, 1)$. (b) Parts with $u = (\ln x)^2$, $dv = dx$ gives $x(\ln x)^2 - 2 \int \ln x dx = x[(\ln x)^2 - 2 \ln x + 2] + C$. (c) For $1 < x < e$, $0 < \ln x < 1$, so $\ln x > (\ln x)^2$. Therefore

$$A = \int_1^e [\ln x - (\ln x)^2] dx = [x \ln x - x]_1^e - (e - 2) = 1 - (e - 2) = 3 - e.$$