



Areas and Volumes of Revolution

IB Mathematics AA HL | AHL 5.17

50 questions: area relative to the y -axis; rotation about either coordinate axis

Learning goal	Select the variable, limits and radii; calculate exact areas and volumes and interpret the result.
Syllabus link	AHL 5.17: area enclosed by a curve and the y -axis; volumes of revolution about the x -axis or y -axis.
Structure	15 short-response, 15 extended-response and 20 IB Paper 1-style questions.
Technology	Non-calculator throughout. All angles are measured in radians.

Instructions. Work on separate paper. Give exact answers and show the integral used. Every rotation is one complete turn (2π radians). For each volume, identify the axis and radius or radii. Give areas in square units and volumes in cubic units unless other units are specified.

1. Short response | Questions 1–15

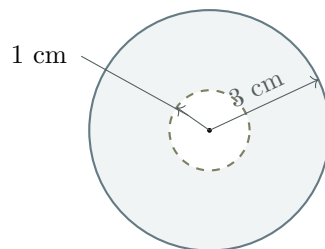
1. Complete the table with an integral. Assume $f(x) \geq 0$ on $[a, b]$ and $g(y) \geq 0$ on $[c, d]$. Do not evaluate.

Quantity	Integral
Area between $x = g(y)$ and the y -axis	
Volume about the x -axis under $y = f(x)$	
Volume about the y -axis beside $x = g(y)$	

2. Find the area between $x = 3y^2$ and the y -axis for $0 \leq y \leq 2$.
3. Find the area bounded by $x = 5 - y$, the y -axis, $y = 1$ and $y = 3$.
4. The curve $y = \ln x$ is written as $x = g(y)$. Find the area between this curve and the y -axis for $0 \leq y \leq \ln 3$.
5. Find the geometrical area between $x = y^2 - 1$ and the y -axis for $-1 \leq y \leq 1$.
6. The region under $y = \sqrt{3x}$, above the x -axis, for $0 \leq x \leq 2$ is rotated about the x -axis. Find its volume.
7. Find the volume formed by rotating the region bounded by $y = 2x$, $y = 0$ and $x = 3$ about the x -axis.
8. Find the volume formed by rotating the region between $x = y + 1$ and the y -axis, for $0 \leq y \leq 2$, about the y -axis.



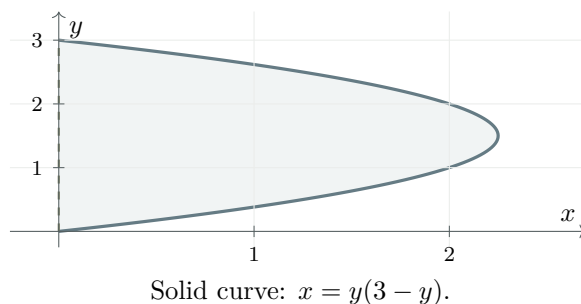
9. Rotate the region under $y = \cos x$, above the x -axis, for $0 \leq x \leq \pi/2$ about the x -axis. Find the volume.
10. Rotate the region between $x = \sqrt{9 - y^2}$ and the y -axis, for $0 \leq y \leq 3$, about the y -axis. Find the volume.
11. A solid has annular cross-sections perpendicular to the y -axis. At height y , the outer radius is $y + 2$ and the inner radius is y , for $0 \leq y \leq 1$. Find its volume.
12. The region between $y = x^2$ ($x \geq 0$) and the y -axis, for $1 \leq y \leq 4$, is rotated about the y -axis. State the radius and write a volume integral. Do not evaluate.
13. A student proposes $V = \pi \left(\int_a^b f(x) dx \right)^2$ for rotation about the x -axis. Correct the formula and explain the error.
14. A region is stretched by factor 3 horizontally and factor 2 vertically. Its rotation about the x -axis originally has volume V . Find the new volume in terms of V .
15. A washer has outer radius 3 cm and inner radius 1 cm. Find its area and explain why $\pi(3 - 1)^2$ is incorrect.





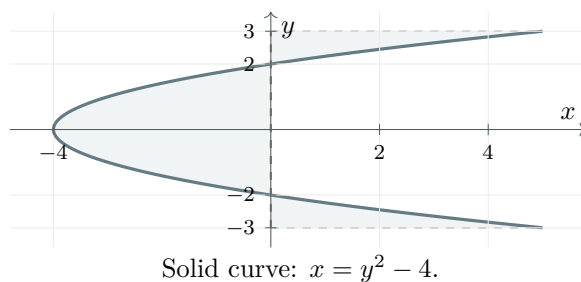
2. Extended response | Questions 16–30

16. The region R lies between $x = y(3 - y)$ and the y -axis.



- (a) Find the y -coordinates where the curve meets the axis.
(b) Find the area of R .

17. Consider $x = y^2 - 4$ for $-3 \leq y \leq 3$.



- (a) Evaluate $\int_{-3}^3 (y^2 - 4) dy$.
(b) Find the total area between the curve and the y -axis on this interval.

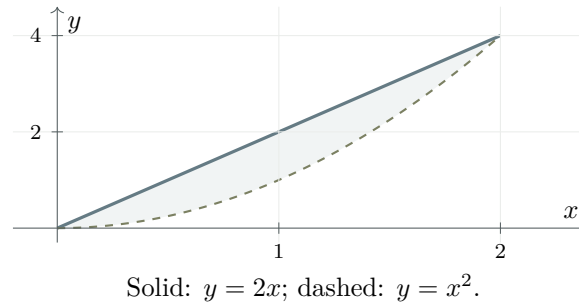
18. The region between $x = \ln y$ and the y -axis has $1 \leq y \leq e$.

- (a) Use integration by parts to find its area.
(b) Find the volume when this region is rotated about the y -axis.

19. The curve is $y = 2\sqrt{x+1}$. A region lies between the curve and the y -axis for $2 \leq y \leq 4$.

- (a) Express x in terms of y and find the area.
(b) Find the volume generated by rotation about the y -axis.

- 20.** The curves $y = 2x$ and $y = x^2$ enclose a region.



- Find the intersections and the enclosed area.
- Find the volume when the region is rotated about the x -axis.
- Express both curves as functions of y and find the volume about the y -axis.

- 21.** The region between $x = e^{y/2}$ and the y -axis has $0 \leq y \leq \ln 4$.

- Find its area.
- Find its volume of revolution about the y -axis.

- 22.** The region under $y = \sqrt{2x+1}$, above the x -axis, has $0 \leq x \leq a$, where $a > 0$. Its volume about the x -axis is 12π .

- Form an equation in a .
- Find a .

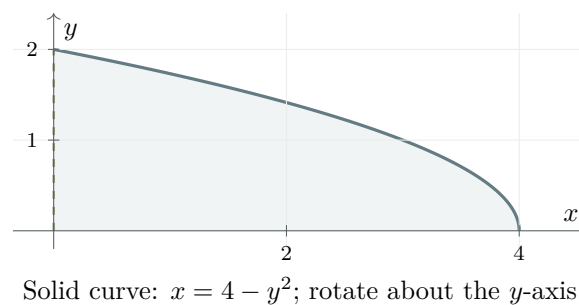
- 23.** The region under $y = \sin x$, above the x -axis, has $0 \leq x \leq \pi$.

- Find its area.
- Find the volume formed by rotating it about the x -axis, using a double-angle identity.

- 24.** A region is bounded by $y = 3 - x$, $y = 1$ and the y -axis.

- State the x -limits and the inner and outer radii for rotation about the x -axis.
- Find the volume.

- 25.** The shaded region is bounded by $x = 4 - y^2$ and the coordinate axes in the first quadrant.



- Find its area by integrating with respect to y .
- Find its volume of revolution about the y -axis.



26. For $b > 1$, the region under $y = 1/x$, above the x -axis, between $x = 1$ and $x = b$ is rotated about the x -axis.

- (a) Show that $V = \pi(1 - 1/b)$.
(b) Find b if $V = 3\pi/4$.

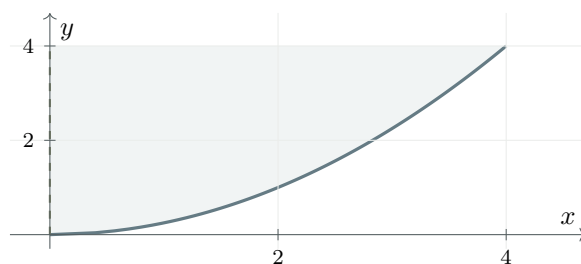
27. The region $y \leq x \leq 2y + 1$, $0 \leq y \leq 1$, is rotated about the y -axis. A student uses $\pi \int_0^1 (y + 1)^2 dy$.

- (a) Explain the error using the radii of a typical cross-section.
(b) Write the correct integral and evaluate it.

28. For $a, b > 0$, the right half of the ellipse $x^2/a^2 + y^2/b^2 = 1$ encloses a region with the y -axis.

- (a) Express x^2 as a function of y and state the limits.
(b) Show that rotation about the y -axis gives $V = 4\pi a^2 b/3$.

29. A bowl is formed by rotating the region $0 \leq x \leq 2\sqrt{y}$, $0 \leq y \leq 4$, about the y -axis.



Right-hand interior profile: $x^2 = 4y$; lengths in cm.

- (a) Show that water filled to depth h cm has volume $2\pi h^2$ cm³.
(b) Find the depth when the bowl is half full by volume.

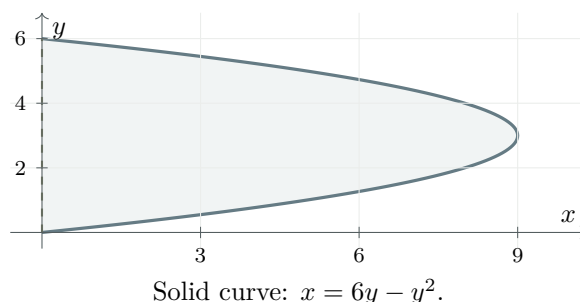
30. A non-negative function f on $[0, a]$ generates volume V when its region above the x -axis is rotated about that axis. Define $g(x) = 3f(2x)$ for $0 \leq x \leq a/2$.

- (a) Write an integral for the new volume W .
(b) Use $u = 2x$ to express W in terms of V .

3. IB Paper 1-style questions | Questions 31–50

31. [Maximum mark: 9]

The curve $x = 6y - y^2$ encloses a region R with the y -axis.



- Find the y -coordinates of the intersections with the axis. [2 marks]
- Find the area of R . [3 marks]
- Find the volume generated when R is rotated about the y -axis. [4 marks]

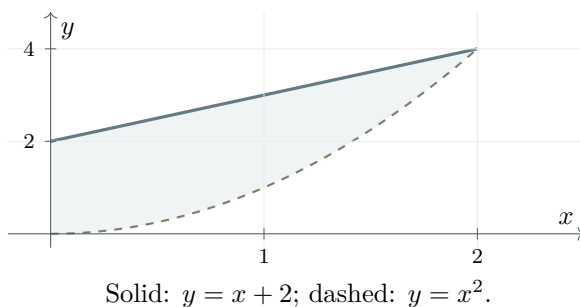
32. [Maximum mark: 10]

Let $x = y^3 - y$, with $0 \leq y \leq 2$. The region between this curve and the y -axis is rotated about that axis.

- Find where the curve crosses the y -axis. [2 marks]
- Find the total planar area between the curve and the axis. [4 marks]
- Find the volume of revolution. Explain why no split at $y = 1$ is needed. [4 marks]

33. [Maximum mark: 9]

The first-quadrant region R is bounded by $y = x + 2$, $y = x^2$ and the y -axis.



- Find the intersection of the curves in the first quadrant. [2 marks]
- Find the area of R . [3 marks]
- Find the volume when R is rotated about the x -axis. [4 marks]

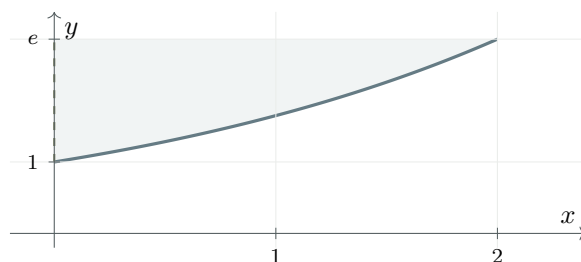
34. [Maximum mark: 12]

The curves $y = 3\sqrt{x}$ and $y = x$ enclose a region R . Let V_x and V_y be its volumes of revolution about the x -axis and y -axis respectively.

- Find the two intersection points. [2 marks]
- Find V_x . [4 marks]
- Find V_y using horizontal slices. [4 marks]
- Find $V_x : V_y$ in its simplest form. [2 marks]

35. [Maximum mark: 10]

The region R is bounded by $x = 2 \ln y$, the y -axis and $y = e$.



Solid curve: $x = 2 \ln y$.

- State the lower y -limit for R . [1 mark]
- Find the area of R . [3 marks]
- Use integration by parts to find $\int (\ln y)^2 dy$. [3 marks]
- Find the volume generated by rotating R about the y -axis. [3 marks]

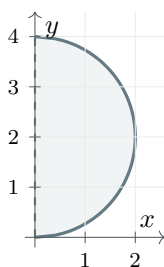
36. [Maximum mark: 9]

The region under $y = \sqrt{xe^x}$, above the x -axis, for $0 \leq x \leq \ln 2$ is rotated about the x -axis.

- Write an integral for the volume. [2 marks]
- Use integration by parts to find $\int xe^x dx$. [4 marks]
- Hence find the exact volume. [3 marks]

37. [Maximum mark: 12]

The region between $x^2 = 4y - y^2$ ($x \geq 0$) and the y -axis is rotated about the y -axis to form a sphere.



Solid curve: $x^2 = 4y - y^2$, $x \geq 0$.

- Find the centre and radius of the circular profile.
Calculate the sphere's volume by integration. [5 marks]
- A cylindrical hole of radius $\sqrt{3}$ is drilled along the y -axis. Find the y -limits of the remaining solid. [3 marks]
- Find the volume of the remaining solid. [4 marks]

38. [Maximum mark: 7]

Let $f(x) = \sec(2x)$, for $0 \leq x \leq \pi/8$.

- Find the range of f . [3 marks]
- Find the volume formed by rotating the region under f , above the x -axis, over this interval about the x -axis. [4 marks]



39. [Maximum mark: 8]

For $k > 0$, the first-quadrant region under $y = \sqrt{k^2 - x^2}$ and above the x -axis is rotated about the x -axis.

- (a) State the x -limits and show that $V = 2\pi k^3/3$. [5 marks]
(b) Find k if $V = 18\pi$. [3 marks]

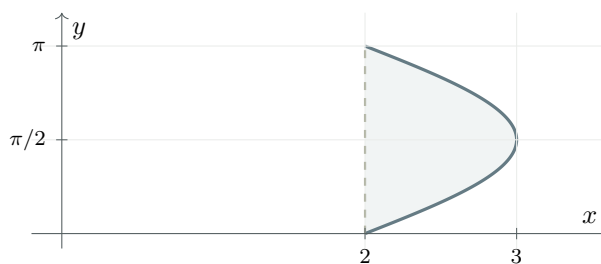
40. [Maximum mark: 10]

A region lies between $x = y^{1/3}$ and the y -axis for $0 \leq y \leq a$, where $a > 0$. Its volume about the y -axis is $96\pi/5$.

- (a) Show that its volume is $3\pi a^{5/3}/5$. [4 marks]
(b) Find a . [3 marks]
(c) Find the area of the original region. [3 marks]

41. [Maximum mark: 10]

The region R lies between $x = 2 + \sin y$ and $x = 2$ for $0 \leq y \leq \pi$.



Solid: $x = 2 + \sin y$; dashed: $x = 2$.

- (a) Find the area of R . [3 marks]
(b) Write a volume integral for rotating R about the y -axis. [3 marks]
(c) Evaluate this integral exactly. [4 marks]

42. [Maximum mark: 10]

Let $f(x) = (x+1)/\sqrt{x}$, for $1 \leq x \leq 4$. The region under f , above the x -axis, is denoted by R .

- (a) Express $f(x)$ as a sum of powers and find the area of R . [4 marks]
(b) Show that $f(x)^2 = x + 2 + 1/x$. [2 marks]
(c) Find the volume when R is rotated about the x -axis. [4 marks]

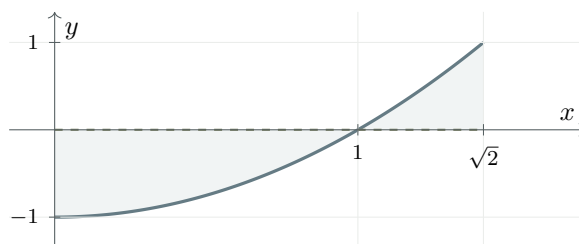
43. [Maximum mark: 12]

The tangent to $y = \ln x$ at $x = 1$ is a line L . The region R is bounded by the curve, L and $x = e$. You may use $\int (\ln x)^2 dx = x((\ln x)^2 - 2 \ln x + 2) + C$.

- (a) Show that L has equation $y = x - 1$. [2 marks]
(b) Explain why $x - 1 \geq \ln x$ for $1 \leq x \leq e$. [2 marks]
(c) Find the area of R . [3 marks]
(d) Find the volume when R is rotated about the x -axis. [5 marks]

44. [Maximum mark: 10]

The region between $y = x^2 - 1$ and the x -axis, for $0 \leq x \leq \sqrt{2}$, is shown.



Solid curve: $y = x^2 - 1$.

- Find its total geometrical area. [4 marks]
- Find the volume formed by rotating the whole region about the x -axis. [4 marks]
- Explain why a negative y -coordinate does not give a negative volume. [2 marks]

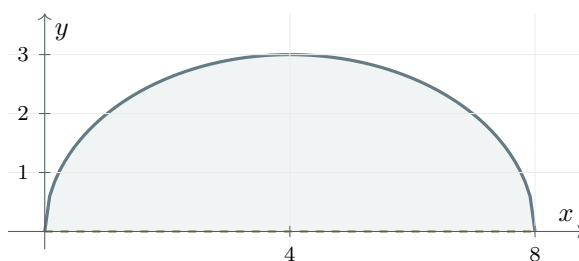
45. [Maximum mark: 9]

For $k > 0$, the region $0 \leq x \leq k/(y+1)$, $0 \leq y \leq 3$, has area $4 \ln 2$.

- Find its area in terms of k , and hence find k . [4 marks]
- Find the volume when the region is rotated about the y -axis. [5 marks]

46. [Maximum mark: 12]

A solid is formed by rotating the region under $y = a\sqrt{x(8-x)}$, above the x -axis, for $0 \leq x \leq 8$, about the x -axis. Its maximum radius is 3 cm, and $a > 0$.



Upper profile of the solid; x and y are in cm.

- Find where the maximum radius occurs and determine a . [4 marks]
- Find the volume of the solid. [5 marks]
- A cylinder has radius 3 cm and length 8 cm.
Express the solid's volume as a fraction of the cylinder's volume. [3 marks]

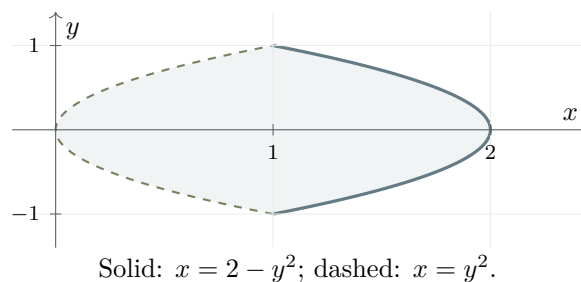
47. [Maximum mark: 10]

Let $m, H > 0$. Region R_1 lies under $y = mx$, above the x -axis, for $0 \leq x \leq H$. Region R_2 lies under $y = \sqrt{cx}$ over the same interval, where $c > 0$. Each is rotated about the x -axis.

- Find the volume generated by R_1 in terms of m and H . [3 marks]
- Find the volume generated by R_2 in terms of c and H . [3 marks]
- Find c in terms of m, H so that the volumes are equal. [4 marks]

48. [Maximum mark: 10]

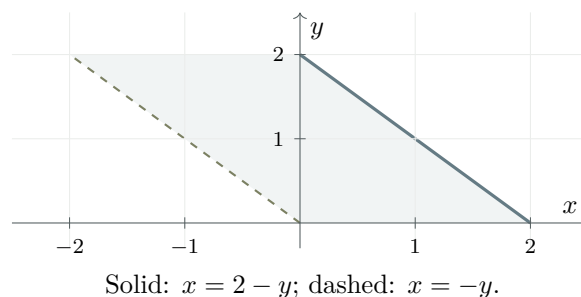
The curves $x = y^2$ and $x = 2 - y^2$ enclose a region R .



- (a) Find their intersection points. [2 marks]
- (b) Find the area of R by integrating with respect to y . [3 marks]
- (c) Find the volume formed by rotating R about the y -axis. [5 marks]

49. [Maximum mark: 12]

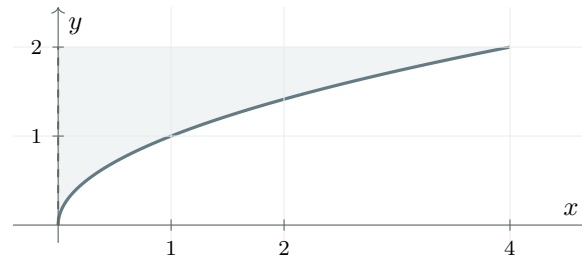
The region R is defined by $-y \leq x \leq 2 - y$, $0 \leq y \leq 2$. It straddles the y -axis.



- (a) Find the area of R . [2 marks]
- (b) When R is rotated about the y -axis, explain why each horizontal cross-section is a disk. State its radius for $0 \leq y \leq 1$ and for $1 \leq y \leq 2$. [4 marks]
- (c) Find the volume of the resulting solid. [4 marks]
- (d) Explain why adding the two separate volumes from the left and right of the axis would be incorrect. [2 marks]

50. [Maximum mark: 11]

A container is formed by rotating the region between $x = y^2$ and the y -axis, for $0 \leq y \leq 2$, about the y -axis. Water is filled from the bottom; wall thickness is ignored.



Right-hand interior profile: $x = y^2$; lengths in cm.

- | | |
|--|-----------|
| (a) Show that water at depth h cm has volume $V(h) = \pi h^5/5 \text{ cm}^3$. | [4 marks] |
| (b) Find the capacity of the container. | [2 marks] |
| (c) Find the fraction of its capacity occupied at half its full height. | [2 marks] |
| (d) Find the exact depth when the container is half full by volume. | [3 marks] |

**Answer key and solution outlines**

All unlabelled areas are in square units; all unlabelled volumes are in cubic units.

1. Short response | Questions 1–15

1. $A = \int_c^d g(y) dy$; $V_x = \pi \int_a^b [f(x)]^2 dx$; $V_y = \pi \int_c^d [g(y)]^2 dy$. A slice is perpendicular to the axis of rotation.
2. $A = \int_0^2 3y^2 dy = [y^3]_0^2 = 8$.
3. $A = \int_1^3 (5 - y) dy = [5y - y^2/2]_1^3 = 6$.
4. $x = e^y$, so $A = \int_0^{\ln 3} e^y dy = 3 - 1 = 2$.
5. The curve is to the left of the axis. $A = \int_{-1}^1 (1 - y^2) dy = 4/3$.
6. $V = \pi \int_0^2 (\sqrt{3x})^2 dx = \pi [3x^2/2]_0^2 = 6\pi$.
7. $V = \pi \int_0^3 (2x)^2 dx = 4\pi [x^3/3]_0^3 = 36\pi$.
8. $V = \pi \int_0^2 (y + 1)^2 dy = \pi [(y + 1)^3/3]_0^2 = 26\pi/3$.
9. $\cos^2 x = (1 + \cos 2x)/2$, so $V = \pi [x/2 + \sin 2x/4]_0^{\pi/2} = \pi^2/4$.
10. $V = \pi \int_0^3 (9 - y^2) dy = \pi [9y - y^3/3]_0^3 = 18\pi$.
11. $V = \pi \int_0^1 ((y + 2)^2 - y^2) dy = \pi \int_0^1 (4y + 4) dy = 6\pi$.
12. $x = \sqrt{y}$ is the radius. $V = \pi \int_1^4 (\sqrt{y})^2 dy = \pi \int_1^4 y dy$.
13. The radius must be squared before integrating: $V = \pi \int_a^b f(x)^2 dx$. The integral of the area of each circular slice gives volume; squaring the total planar area does not.
14. Slice thickness is multiplied by 3 and cross-sectional area by 2^2 . The new volume is $12V$.
15. Area = $\pi(3^2 - 1^2) = 8\pi \text{ cm}^2$. Subtract the areas of two disks, not their radii before squaring.

2. Extended response | Questions 16–30

16. (a) $y(3 - y) = 0$ gives $y = 0, 3$. (b) $x \geq 0$ between these values, so $A = \int_0^3 (3y - y^2) dy = [3y^2/2 - y^3/3]_0^3 = 9/2$.
17. (a) $[y^3/3 - 4y]_{-3}^3 = -6$. (b) The sign changes at $y = \pm 2$. By symmetry, $A = 2\{\int_0^2 (4 - y^2) dy + \int_2^3 (y^2 - 4) dy\} = 2(16/3 + 7/3) = 46/3$.
18. (a) With $u = \ln y$, $dv = dy$, $A = [y \ln y - y]_1^e = 1$. (b) Parts again gives $\int (\ln y)^2 dy = y((\ln y)^2 - 2 \ln y + 2) + C$. Thus $V = \pi [y((\ln y)^2 - 2 \ln y + 2)]_1^e = \pi(e - 2)$.
19. (a) $x = y^2/4 - 1$. $A = [y^3/12 - y]_2^4 = 8/3$. (b) $V = \pi \int_2^4 (y^2/4 - 1)^2 dy = \pi [y^5/80 - y^3/6 + y]_2^4 = \frac{76\pi}{15}$.
20. (a) $x^2 = 2x$ gives $(0, 0), (2, 4)$; $A = \int_0^2 (2x - x^2) dx = 4/3$. (b) $V_x = \pi \int_0^2 (4x^2 - x^4) dx = 64\pi/15$. (c) $x = y/2$ and $x = \sqrt{y}$; outer radius $\sqrt{y}$. $V_y = \pi \int_0^4 (y - y^2/4) dy = 8\pi/3$.
21. (a) $A = [2e^{y/2}]_0^{\ln 4} = 2$. (b) $V = \pi \int_0^{\ln 4} e^y dy = 3\pi$. Squaring $e^{y/2}$ gives e^y .
22. $V = \pi \int_0^a (2x + 1) dx = \pi(a^2 + a)$. Hence $a^2 + a - 12 = (a + 4)(a - 3) = 0$; since $a > 0$, $a = 3$.
23. (a) $A = [-\cos x]_0^\pi = 2$. (b) $\sin^2 x = (1 - \cos 2x)/2$, so $V = \pi [x/2 - \sin 2x/4]_0^\pi = \pi^2/2$.
24. (a) $3 - x = 1$ at $x = 2$. For $0 \leq x \leq 2$, $R = 3 - x$, $r = 1$. (b) $V = \pi \int_0^2 ((3 - x)^2 - 1) dx = \pi [-(3 - x)^3/3 + x]_0^2 = 20\pi/3$.
25. (a) $A = \int_0^2 (4 - y^2) dy = 16/3$. (b) $V = \pi \int_0^2 (4 - y^2)^2 dy = \pi [16y - 8y^3/3 + y^5/5]_0^2 = 256\pi/15$.
26. (a) $V = \pi \int_1^b x^{-2} dx = \pi [-x^{-1}]_1^b = \pi(1 - 1/b)$. (b) $1 - 1/b = 3/4$, so $b = 4$.

**2. Extended response | Questions 16–30 (continued)**

- 27.** (a) The region does not reach the axis (except at its lower edge). A cross-section is a washer with $R = 2y + 1$, $r = y$; $R - r$ is not the disk radius. (b) $V = \pi \int_0^1 ((2y + 1)^2 - y^2) dy = \pi \int_0^1 (3y^2 + 4y + 1) dy = 4\pi$.
- 28.** (a) $x^2 = a^2(1 - y^2/b^2)$, $-b \leq y \leq b$. (b) $V = \pi a^2 \int_{-b}^b (1 - y^2/b^2) dy = \pi a^2 [y - y^3/(3b^2)]_{-b}^b = 4\pi a^2 b/3$.
- 29.** (a) At height y , radius $x = 2\sqrt{y}$. $V(h) = \pi \int_0^h 4y dy = 2\pi h^2$. (b) Capacity is $32\pi \text{ cm}^3$; $2\pi h^2 = 16\pi$ gives $h = 2\sqrt{2} \text{ cm}$.
- 30.** (a) $W = 9\pi \int_0^{a/2} f(2x)^2 dx$. (b) $dx = du/2$, with $u = 0, a$. Thus $W = \frac{9}{2}\pi \int_0^a f(u)^2 du = \frac{9}{2}V$.

3. IB Paper 1-style questions | Questions 31–50

- 31.** (a) $y(6 - y) = 0$: $y = 0, 6$. (b) $A = [3y^2 - y^3/3]_0^6 = 36$. (c) $V = \pi \int_0^6 (36y^2 - 12y^3 + y^4) dy = \pi [12y^3 - 3y^4 + y^5/5]_0^6 = 1296\pi/5$.
- 32.** (a) $y(y-1)(y+1) = 0$ gives $y = 0, 1$ in the interval. (b) $A = -\int_0^1 (y^3 - y) dy + \int_1^2 (y^3 - y) dy = 1/4 + 9/4 = 5/2$. (c) Radius is $|y^3 - y|$, whose square is $(y^3 - y)^2$. $V = \pi [y^7/7 - 2y^5/5 + y^3/3]_0^2 = 856\pi/105$.
- 33.** (a) $x^2 = x + 2$ gives $x = 2, -1$; the required point is $(2, 4)$. (b) $A = \int_0^2 (x + 2 - x^2) dx = 10/3$. (c) $R = x + 2$, $r = x^2$. $V = \pi \int_0^2 ((x + 2)^2 - x^4) dx = \pi [x^3/3 + 2x^2 + 4x - x^5/5]_0^2 = 184\pi/15$.
- 34.** (a) $3\sqrt{x} = x$ gives $(0, 0), (9, 9)$. (b) $V_x = \pi \int_0^9 (9x - x^2) dx = 243\pi/2$. (c) $x = y^2/9$ and $x = y$; $V_y = \pi \int_0^9 (y^2 - y^4/81) dy = 486\pi/5$. (d) $V_x : V_y = 5 : 4$.
- 35.** (a) $y = 1$. (b) $A = 2[y \ln y - y]_1^e = 2$. (c) Set $u = (\ln y)^2$, $dv = dy$: $I = y(\ln y)^2 - 2 \int \ln y dy = y(\ln y)^2 - 2(y \ln y - y) + C$. (d) $V = 4\pi \int_1^e (\ln y)^2 dy = 4\pi(e - 2)$.
- 36.** (a) $V = \pi \int_0^{\ln 2} x e^x dx$. (b) $u = x$, $dv = e^x dx$ gives $\int x e^x dx = x e^x - e^x + C$. (c) $V = \pi[(x - 1)e^x]_0^{\ln 2} = \pi(2 \ln 2 - 1)$.
- 37.** (a) $x^2 + (y - 2)^2 = 4$: centre $(0, 2)$ in the profile, radius 2. $V = \pi \int_0^4 (4y - y^2) dy = 32\pi/3$. (b) The outer radius equals $\sqrt{3}$ when $4y - y^2 = 3$, so $y = 1, 3$. (c) Washers: $V = \pi \int_1^3 (4y - y^2 - 3) dy = 4\pi/3$.
- 38.** (a) $0 \leq 2x \leq \pi/4$; $\cos 2x$ decreases from 1 to $1/\sqrt{2}$. Hence $1 \leq f(x) \leq \sqrt{2}$. (b) $V = \pi \int_0^{\pi/8} \sec^2(2x) dx = \frac{\pi}{2} [\tan 2x]_0^{\pi/8} = \pi/2$.
- 39.** (a) $0 \leq x \leq k$ and $V = \pi \int_0^k (k^2 - x^2) dx = \pi[k^2x - x^3/3]_0^k = 2\pi k^3/3$. (b) $k^3 = 27$, so $k = 3$.
- 40.** (a) $V = \pi \int_0^a y^{2/3} dy = 3\pi a^{5/3}/5$. (b) $a^{5/3} = 32$, hence $a = 8$. (c) $A = \int_0^8 y^{1/3} dy = [3y^{4/3}/4]_0^8 = 12$.
- 41.** (a) $A = \int_0^\pi \sin y dy = 2$. (b) $V = \pi \int_0^\pi ((2 + \sin y)^2 - 4) dy$. (c) Expand to $4 \sin y + \sin^2 y$. Their integrals are 8 and $\pi/2$, so $V = 8\pi + \pi^2/2$.
- 42.** (a) $f = x^{1/2} + x^{-1/2}$; $A = [2x^{3/2}/3 + 2x^{1/2}]_1^4 = 20/3$. (b) $(x + 1)^2/x = x + 2 + 1/x$. (c) $V = \pi [x^2/2 + 2x + \ln x]_1^4 = \pi(27/2 + \ln 4)$.
- 43.** (a) Gradient $1/x = 1$ at $(1, 0)$, hence $y = x - 1$. (b) $h = x - 1 - \ln x$ has $h(1) = 0$ and $h' = 1 - 1/x \geq 0$. (c) $A = [(x - 1)^2/2 - x \ln x + x]_1^e = (e^2 - 2e - 1)/2$. (d) $V = \pi \int_1^e ((x - 1)^2 - (\ln x)^2) dx = \pi\{(e - 1)^3/3 - (e - 2)\} = \pi(e^3/3 - e^2 + 5/3)$.
- 44.** (a) Split at $x = 1$: $A = \int_0^1 (1 - x^2) dx + \int_1^{\sqrt{2}} (x^2 - 1) dx = (4 - \sqrt{2})/3$. (b) $V = \pi \int_0^{\sqrt{2}} (x^2 - 1)^2 dx = \pi [x^5/5 - 2x^3/3 + x]_0^{\sqrt{2}} = 7\pi\sqrt{2}/15$. (c) Radius is $|y|$; circular cross-sectional area is $\pi|y|^2 = \pi y^2 \geq 0$.
- 45.** (a) $A = k \int_0^3 (y + 1)^{-1} dy = k \ln 4 = 2k \ln 2$; $k = 2$. (b) $V = 4\pi \int_0^3 (y + 1)^{-2} dy = 4\pi[-1/(y + 1)]_0^3 = 3\pi$.
- 46.** (a) $x(8 - x) = 16 - (x - 4)^2$, maximal at $x = 4$. Thus $4a = 3$, so $a = 3/4$. (b) $V = \frac{9\pi}{16} \int_0^8 (8x - x^2) dx = \frac{9\pi}{16} [4x^2 - x^3/3]_0^8 = 48\pi \text{ cm}^3$. (c) The cylinder has volume $72\pi \text{ cm}^3$; the fraction is $2/3$.
- 47.** (a) $V_1 = \pi m^2 \int_0^H x^2 dx = \pi m^2 H^3/3$. (b) $V_2 = \pi c \int_0^H x dx = \pi c H^2/2$. (c) Equating and dividing by πH^2 gives $m^2 H/3 = c/2$, so $c = 2m^2 H/3$.

**3. IB Paper 1-style questions | Questions 31–50 (continued)**

48. (a) $y^2 = 2 - y^2$ gives $(1, -1), (1, 1)$. (b) $A = \int_{-1}^1 (2 - 2y^2) dy = 8/3$. (c) Both curves have $x \geq 0$ in the region; $R = 2 - y^2$, $r = y^2$. $V = \pi \int_{-1}^1 ((2 - y^2)^2 - y^4) dy = \pi \int_{-1}^1 (4 - 4y^2) dy = 16\pi/3$.

49. (a) Width is $(2 - y) - (-y) = 2$, so $A = 4$. (b) Each slice includes the axis; its rotation fills a disk. The radius is the larger of y and $2 - y$: $2 - y$ on $[0, 1]$, y on $[1, 2]$. (c) $V = \pi \int_0^1 (2 - y)^2 dy + \pi \int_1^2 y^2 dy = 14\pi/3$. (d) The rotated left and right portions overlap; adding their volumes counts the overlap twice.

50. (a) Radius at height y is y^2 , so $V(h) = \pi \int_0^h (y^2)^2 dy = \pi h^5/5$. (b) $V(2) = 32\pi/5 \text{ cm}^3$. (c) Half height is $h = 1$; $V(1)/V(2) = 1/32$. (d) $\pi h^5/5 = 16\pi/5$ gives $h = 16^{1/5} = 2^{4/5} \text{ cm}$.