



Differential Equations

IB Mathematics AA HL | AHL 5.18

50 questions: numerical methods, exact solutions and mathematical models

Learning goal	Select and apply a method; solve initial-value problems and interpret their solutions.
Syllabus link	AHL 5.18: first-order differential equations; Euler's method; separation of variables, including logistic models; homogeneous equations using $y = vx$; linear equations using an integrating factor.
Structure	15 short-response, 15 extended-response and 20 original IB Paper 1-style questions, followed by worked solutions.
Technology	Non-calculator throughout. Give Euler estimates exactly or as terminating decimals. All angles are in radians.

Instructions. Work on separate paper. Show your method and use initial conditions to determine constants. Give exact answers unless an estimate is requested. State integrating factors and substitutions clearly. When separating variables, check for equilibrium solutions lost through division. Use physical models for $t \geq 0$.

1. Short response | Questions 1–15

1. For the first-order differential equation $y' = x + y$, state the gradient of the solution curve at $(0, 2)$ and write the equation of its tangent there.

2. State the most direct method for solving

$$\frac{dy}{dx} = x(y + 1).$$

3. Explain briefly why

$$\frac{dy}{dx} = \frac{x + y}{x}$$

is a homogeneous first-order differential equation.

4. Write

$$2y' + 4y = x$$

in the standard linear form $y' + P(x)y = Q(x)$, and state $P(x)$ and $Q(x)$.

5. Verify that $y = 2e^{3x}$ is a solution of $y' = 3y$.

6. Find the equilibrium solutions of

$$\frac{dy}{dx} = y(5 - y).$$

7. For $\frac{dy}{dx} = x + y$, $y(0) = 1$, use one Euler step with $h = 0.2$ to estimate $y(0.2)$.



8. For $\frac{dy}{dx} = y - x$, $y(0) = 2$, use Euler's method with $h = 0.5$ to estimate $y(1)$.

n	x_n	y_n	$f(x_n, y_n) = y_n - x_n$
0	0	2	
1	0.5		
2	1		

9. Separate the variables in

$$\frac{dy}{dx} = \frac{x}{y}.$$

Do not integrate.

10. Solve

$$\frac{dy}{dx} = 2x, \quad y(0) = 3.$$

11. Solve $\frac{dy}{dx} = -2y$, given $y(0) = 3$. State $\lim_{x \rightarrow \infty} y$.

12. State an integrating factor for

$$y' + 4y = x.$$

13. For $x > 0$, state an integrating factor for

$$y' - \frac{2}{x}y = x^2.$$

14. If $y = vx$, write $\frac{dy}{dx}$ in terms of v , x and $\frac{dv}{dx}$.

15. Determine whether

$$\frac{dy}{dx} = 1 + \left(\frac{y}{x}\right)^2$$

is a homogeneous first-order differential equation. If it is, state the substitution that should be used.

**2. Extended response | Questions 16–30**

16. Consider the initial-value problem

$$\frac{dy}{dx} = xe^{-y}, \quad y(0) = 0.$$

- (a) Find the particular solution explicitly.
(b) Determine its stationary point and explain why this point is a minimum.

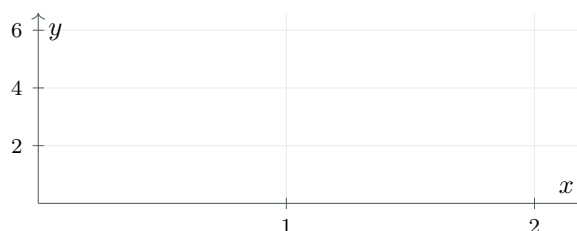
17. Obtain an implicit solution of

$$\frac{dy}{dx} = \frac{x}{1+y^2}, \quad y(0) = 0.$$

18. Consider

$$\frac{dy}{dx} = \frac{y}{x}, \quad y(1) = 3, \quad x > 0.$$

- (a) Find the particular solution.
(b) Sketch the solution on the supplied axes and mark the initial point.



19. A population P , measured in thousands, satisfies

$$\frac{dP}{dt} = 0.2P(10 - P), \quad P(0) = 2.$$

Time t is measured in years. Find $P(t)$ for $t \geq 0$.

20. Use the population model from Question 19.

- (a) Find the exact time at which $P = 5$.
(b) State the limiting population and sketch P against t , labelling the initial value and horizontal asymptote.

21. Find the complete set of solutions of

$$\frac{dy}{dx} = y(4 - y),$$

including any equilibrium solutions that could be lost during separation. State the sign of y' when $0 < y < 4$ and when $y > 4$.



22. Solve the homogeneous initial-value problem

$$\frac{dy}{dx} = \frac{y}{x} + \left(\frac{y}{x}\right)^2, \quad y(1) = 1, \quad x > 0.$$

State the largest interval containing $x = 1$ on which this particular solution is defined.

23. Solve

$$\frac{dy}{dx} = 1 - \frac{y}{x}, \quad y(1) = 0, \quad x > 0.$$

Solve once using $y = vx$ and once using an integrating factor.

24. Solve

$$\frac{dy}{dx} = \frac{x^2 + y^2}{xy}, \quad y(1) = 2, \quad x > 0,$$

choosing the branch that satisfies the initial condition. State the largest interval containing $x = 1$ on which the solution satisfies the differential equation.

25. Solve the linear initial-value problem

$$y' + 2y = e^x, \quad y(0) = 1.$$

26. Solve

$$y' - y = x, \quad y(0) = 2.$$

27. For $x > 0$, solve

$$xy' + 2y = x^3, \quad y(1) = 1.$$

28. A hot object cools in a room at 20°C . Its temperature T satisfies Newton's law of cooling

$$\frac{dT}{dt} = -k(T - 20),$$

where t is measured in minutes. The following measurements are recorded.

t (min)	T ($^\circ\text{C}$)
0	80
10	50

Find k exactly and obtain $T(t)$. Hence find $T(20)$.

29. A radioactive sample of initial mass 100 mg has half-life 5730 years. Model its mass M by $\frac{dM}{dt} = -kM$. Find $M(t)$ and the mass remaining after 11460 years.



30. For

$$\frac{dy}{dx} = x - y, \quad y(0) = 1,$$

use Euler's method with step size $h = 0.25$ to estimate $y(0.75)$. Show the successive (x, y) values.

n	x_n	y_n	$f(x_n, y_n) = x_n - y_n$
0	0	1	
1	0.25		
2	0.5		
3	0.75		

3. IB Paper 1-style questions | Questions 31–50

31. [Maximum mark: 8]

Consider $\frac{dy}{dx} = y^2 e^x$, with $y(0) = \frac{1}{2}$.

- (a) Find y explicitly in terms of x . [6 marks]
(b) State the largest interval containing 0 on which this solution is defined. [2 marks]

32. [Maximum mark: 7]

Consider the differential equation

$$\frac{dy}{dx} = -2y^2 \sin^2 x, \quad y \neq 0.$$

The solutions can be written in the form

$$y = \frac{a}{2x - \sin 2x + b},$$

where a and b are constants.

- (a) By solving the differential equation, find a . [5 marks]
(b) Given that $y = 1$ when $x = \frac{\pi}{2}$, find b . [2 marks]

33. [Maximum mark: 7]

You may use $\int x^2 \sin x \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C$. For $x > 0$, consider $x \frac{dy}{dx} + 2y = x \sin x$, with $y(\pi) = 1$.

- (a) Write the equation in standard linear form and state an integrating factor. [2 marks]
(b) Find the particular solution. [5 marks]

34. [Maximum mark: 7]

For $x > 0$, $y' + \frac{y}{x} = x^2$, with $y(1) = 0$.

- (a) Find the particular solution. [6 marks]
(b) Find the gradient of its tangent at $(1, 0)$. [1 mark]



35. [Maximum mark: 8]

Consider $y' = \frac{2x + y}{x - 2y}$, where $x > 0$ and $x \neq 2y$, with $y(1) = 0$.

(a) Use $y = vx$ to show that $x \frac{dv}{dx} = \frac{2(1 + v^2)}{1 - 2v}$. [3 marks]

(b) Hence show that $\arctan\left(\frac{y}{x}\right) - \ln\left(1 + \frac{y^2}{x^2}\right) = 2 \ln x$. [5 marks]

36. [Maximum mark: 8]

Consider $y' = 1 + \frac{y}{x}$, $x > 0$, with $y(e) = 2e$.

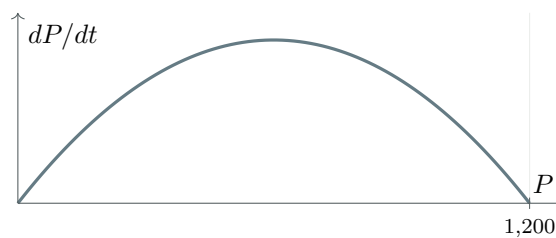
(a) Find the particular solution. [6 marks]

(b) Find where its curve crosses the x -axis. [2 marks]

37. [Maximum mark: 10]

A population P of insects is modelled, with t in weeks, by

$$\frac{dP}{dt} = 0.0005P(1200 - P), \quad P(0) = 200.$$



Growth rate as a function of population.

(a) Write down the two equilibrium population levels. [2 marks]

(b) Find P in terms of t . [4 marks]

(c) Find the exact time at which $P = 600$. [2 marks]

(d) Find the maximum possible value of $\frac{dP}{dt}$. [2 marks]

38. [Maximum mark: 10]

A culture is modelled by $\frac{dN}{dt} = 0.02N(50 - N)$, with $N(0) = 75$, where N is measured in thousands and t in days.

(a) Find $N(t)$ for $t \geq 0$. [5 marks]

(b) Find the exact time at which $N = 60$. [2 marks]

(c) State the limiting population. Sketch the solution for $t \geq 0$, indicating whether it increases or decreases. [3 marks]

**39.** [Maximum mark: 6]Euler's method is applied to $y' = x^2 - y$, $y(0) = 1$, using $h = 0.5$.(a) Write the recurrence relation for y_{n+1} .

[1 mark]

(b) Complete the table and obtain an estimate of $y(1.5)$.

[5 marks]

n	x_n	y_n	$f(x_n, y_n) = x_n^2 - y_n$
0	0	1	
1	0.5		
2	1		
3	1.5		

40. [Maximum mark: 7]Consider $y' = 2x$, $y(0) = 0$. The exact solution $y = x^2$ is shown.(a) Use Euler's method with $h = 0.5$ to estimate $y(1)$.

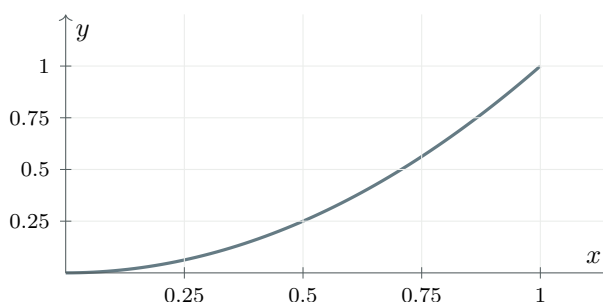
[2 marks]

(b) Repeat with $h = 0.25$.

[3 marks]

(c) Draw both Euler polygons on the supplied axes. Explain geometrically why both estimates are below the exact value.

[2 marks]

**41.** [Maximum mark: 8]A drink cools in a room at 18°C . Its temperature T satisfies

$$\frac{dT}{dt} = -k(T - 18), \quad k > 0.$$

Initially $T = 90$, and after 10 minutes $T = 54$.(a) Find k exactly.

[3 marks]

(b) Find T as a function of t .

[2 marks]

(c) Find the time when $T = 27$.

[3 marks]



42. [Maximum mark: 6]

A radioactive isotope has remaining fraction $R(t)$, where t is in years, satisfying

$$\frac{dR}{dt} = -kR, \quad R(0) = 1.$$

Given that $R(1600) = \frac{1}{2}$:

- (a) find k and hence $R(t)$; [4 marks]
(b) find the time when $R = \frac{1}{8}$. [2 marks]

43. [Maximum mark: 7]

A tank contains 50 litres of well-mixed solution. Liquid containing 1 gram of salt per litre enters at 5 litres per minute, and mixture leaves at the same volume rate. The mass of salt M grams at time t minutes satisfies

$$\frac{dM}{dt} + 0.1M = 5, \quad M(0) = 10.$$



- (a) Solve the differential equation. [5 marks]
(b) Find $\lim_{t \rightarrow \infty} M(t)$. [1 mark]
(c) Interpret this limit in terms of the model. [1 mark]

44. [Maximum mark: 7]

Consider the initial-value problem

$$y' - 2xy = e^{x^2}, \quad y(0) = 0.$$

- (a) Find an integrating factor. [2 marks]
(b) Hence solve the differential equation. [5 marks]

45. [Maximum mark: 7]

Euler's method is used for

$$\frac{dy}{dx} = x + y, \quad y(0) = 1,$$

with $h = 0.1$.

- (a) Write the recurrence relations for x_{n+1} and y_{n+1} . [2 marks]
(b) Find the Euler estimate of $y(0.3)$. [3 marks]
(c) The exact solution is $y = 2e^x - x - 1$. Use concavity to explain whether the Euler estimate is an overestimate or an underestimate. [2 marks]



46. [Maximum mark: 7]

Consider $y' = (1 + y^2) \cos x$, with $y(0) = 0$.

- (a) Find the particular solution. [5 marks]
(b) Explain why this solution is defined for every real x . [2 marks]
-

47. [Maximum mark: 7]

For $0 < x < \pi$, $y' + y \cot x = \sin x$, with $y(\pi/2) = 0$.

- (a) State an integrating factor. [2 marks]
(b) Hence find the particular solution. [5 marks]
-

48. [Maximum mark: 9]

Consider $y' = \left(\frac{y}{x}\right)^2$, $x > 0$, with $y(1) = 2$.

- (a) Use $y = vx$ to find the particular solution. [7 marks]
(b) State its largest interval of definition containing $x = 1$ and the equation of its vertical asymptote. [2 marks]
-

49. [Maximum mark: 9]

A student solves $y' = y(2 - y)$ by dividing by $y(2 - y)$.

- (a) Find the non-equilibrium solutions and identify all equilibrium solutions. [5 marks]
(b) Find the solution satisfying $y(0) = 3$. [2 marks]
(c) Explain why this solution decreases towards 2 for $x \geq 0$. [2 marks]
-

50. [Maximum mark: 8]

For $x > 0$, consider

$$x \frac{dy}{dx} + (1 - x)y = x^2 e^x.$$

Given that $y = \frac{4e}{3}$ when $x = 1$:

- (a) find an integrating factor after first writing the equation in standard linear form; [3 marks]
(b) find the particular solution; [4 marks]
(c) hence find $y(3)$. [1 mark]



Answer key and solution outlines

1. Short response | Questions 1–15

1. $y' = 0 + 2 = 2$. The tangent is $y - 2 = 2(x - 0)$, hence $y = 2x + 2$.
2. Separation of variables.
3. $\frac{x+y}{x} = 1 + \frac{y}{x}$ depends only on y/x , so it has the form $y' = F(y/x)$.
4. $y' + 2y = \frac{x}{2}$, so $P(x) = 2$ and $Q(x) = \frac{x}{2}$.
5. $y' = 6e^{3x} = 3(2e^{3x}) = 3y$.
6. $y = 0$ and $y = 5$.
7. $y_1 = 1 + 0.2(0 + 1) = 1.2$.
8. $y_1 = 2 + 0.5(2) = 3$ at $x = 0.5$; $y_2 = 3 + 0.5(3 - 0.5) = \frac{17}{4} = 4.25$. The table slopes are $2, 5/2, 13/4$ respectively.
9. $y dy = x dx$.
10. $y = x^2 + C$ and $C = 3$, so $y = x^2 + 3$.
11. $dy/y = -2 dx$ gives $y = Ce^{-2x}$. Thus $y = 3e^{-2x}$ and the limit is 0.
12. $\mu = e^{4x}$.
13. $\mu = e^{\int -2/x dx} = x^{-2}$ for $x > 0$.
14. $y' = v + x \frac{dv}{dx}$.
15. Yes. Use $y = vx$ (equivalently $v = y/x$).

2. Extended response | Questions 16–30

16. $e^y dy = x dx$, so $e^y = \frac{x^2}{2} + C$. From $y(0) = 0$, $C = 1$. Hence

$$y = \ln\left(1 + \frac{x^2}{2}\right).$$

The derivative has the sign of x , so the stationary point $(0, 0)$ is a minimum.

17. $(1 + y^2) dy = x dx$. Thus

$$y + \frac{y^3}{3} = \frac{x^2}{2} + C.$$

The initial condition gives $C = 0$, so

$$y + \frac{y^3}{3} = \frac{x^2}{2}.$$

18. $\frac{dy}{y} = \frac{dx}{x}$ gives $y = Cx$ for $x > 0$. Since $y(1) = 3$, $y = 3x$. The sketch is the ray $y = 3x$, $x > 0$, through $(1, 3)$; the origin is excluded.

19. Separate using partial fractions:

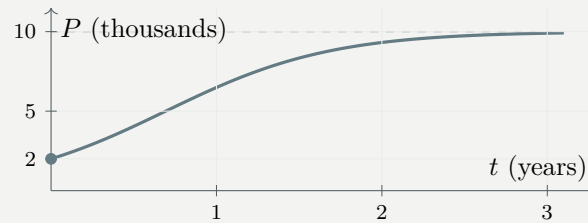
$$\frac{dP}{P(10-P)} = 0.2 dt \implies \ln\left(\frac{P}{10-P}\right) = 2t + C.$$

From $P(0) = 2$, $\frac{P}{10-P} = \frac{1}{4}e^{2t}$. Therefore

$$P(t) = \frac{10}{1 + 4e^{-2t}}.$$

2. Extended response | Questions 16–30 (continued)

20. $5 = 10/(1 + 4e^{-2t})$ gives $4e^{-2t} = 1$, so $t = \ln 2$. The limiting population is 10 thousand. The curve rises from $(0, 2)$, changes concavity at $P = 5$, and approaches $P = 10$ from below.



21. Equilibria: $y = 0$ and $y = 4$. For non-equilibrium solutions,

$$\frac{dy}{y(4-y)} = dx \implies \frac{1}{4} \ln \left| \frac{y}{4-y} \right| = x + C,$$

so

$$y = \frac{4}{1 + Ce^{-4x}}.$$

Together with $y = 0$, this gives the complete set; $C = 0$ includes $y = 4$. Each nonconstant solution is restricted to an interval on which its denominator is nonzero. For $0 < y < 4$, $y' > 0$; for $y > 4$, $y' < 0$.

22. Let $y = vx$. Then $y' = v + xv'$ and

$$v + xv' = v + v^2 \implies xv' = v^2.$$

Hence $-1/v = \ln x + C$. Since $v(1) = 1$, $C = -1$, so $v = 1/(1 - \ln x)$ and

$$y = \frac{x}{1 - \ln x}.$$

Since $x > 0$ and $1 - \ln x \neq 0$, the required interval is $0 < x < e$.

23. With $y = vx$,

$$v + xv' = 1 - v \implies xv' = 1 - 2v.$$

Integration gives $y = \frac{x}{2} + \frac{C}{x}$. Using $y(1) = 0$ gives $C = -\frac{1}{2}$, hence

$$y = \frac{x}{2} - \frac{1}{2x}.$$

Alternatively, $y' + y/x = 1$ has integrating factor x . Then $(xy)' = x$, giving $xy = x^2/2 + C$ and the same result.

24. Set $y = vx$. Then

$$v + xv' = \frac{1}{v} + v \implies xv' = \frac{1}{v}.$$

Thus $v dv = dx/x$, so $v^2 = 2 \ln x + C$. Since $v(1) = 2$, $C = 4$. The positive branch is required:

$$y = x\sqrt{2 \ln x + 4}.$$

The equation also requires $y \neq 0$, so $2 \ln x + 4 > 0$. The interval is $x > e^{-2}$; the endpoint is excluded.

25. $\mu = e^{2x}$. Then $(e^{2x}y)' = e^{3x}$, so

$$y = \frac{1}{3}e^x + Ce^{-2x}.$$

Using $y(0) = 1$ gives $C = \frac{2}{3}$:

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-2x}.$$

**2. Extended response | Questions 16–30 (continued)**

26. $\mu = e^{-x}$. Thus $(e^{-x}y)' = xe^{-x}$ and

$$e^{-x}y = -(x+1)e^{-x} + C.$$

So $y = -(x+1) + Ce^x$. The condition gives $C = 3$:

$$y = 3e^x - x - 1.$$

27. Divide by x : $y' + \frac{2}{x}y = x^2$. With $\mu = x^2$,

$$(x^2y)' = x^4.$$

Hence $y = \frac{x^3}{5} + C/x^2$. The condition gives $C = 4/5$:

$$y = \frac{x^3}{5} + \frac{4}{5x^2}.$$

28. $T - 20 = Ce^{-kt}$. Since $T(0) = 80$, $C = 60$. Since $T(10) - 20 = 30$, $e^{-10k} = 1/2$, so $k = \ln 2/10$. Therefore

$$T(t) = 20 + 60 \cdot 2^{-t/10}, \quad T(20) = 35^\circ\text{C}.$$

29. $M = 100e^{-kt}$. Half-life gives $e^{-5730k} = 1/2$, so $k = \ln 2/5730$. Hence

$$M(t) = 100 \cdot 2^{-t/5730}, \quad M(11460) = 25 \text{ mg}.$$

30. Using $y_{n+1} = y_n + 0.25(x_n - y_n)$:

$$(0, 1) \rightarrow \left(\frac{1}{4}, \frac{3}{4}\right) \rightarrow \left(\frac{1}{2}, \frac{5}{8}\right) \rightarrow \left(\frac{3}{4}, \frac{19}{32}\right).$$

Thus $y(0.75) \approx \frac{19}{32} = 0.59375$. The table slopes are $-1, -1/2, -1/8, 5/32$ respectively.

3. IB Paper 1-style questions | Questions 31–50

31. Separate:

$$\int y^{-2} dy = \int e^x dx \implies -\frac{1}{y} = e^x + C.$$

Using $y(0) = 1/2$ gives $C = -3$. Therefore

$$y = \frac{1}{3 - e^x}.$$

The denominator vanishes at $x = \ln 3$, so the required interval is $-\infty < x < \ln 3$.

32.

(a) $y^{-2}dy = -2\sin^2 x dx = -(1 - \cos 2x) dx$. Hence

$$-\frac{1}{y} = -x + \frac{1}{2}\sin 2x + C,$$

so $\frac{1}{y} = x - \frac{1}{2}\sin 2x + C_1$ and

$$y = \frac{2}{2x - \sin 2x + b}.$$

Thus $a = 2$.

(b) At $x = \pi/2$, $1 = 2/(\pi + b)$, so $b = 2 - \pi$.

**3. IB Paper 1-style questions | Questions 31–50 (continued)**

33. Divide by x :

$$y' + \frac{2}{x}y = \sin x,$$

so $\mu = x^2$. Then

$$(x^2y)' = x^2 \sin x.$$

Using the supplied integral,

$$x^2y = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

At $x = \pi$, $\pi^2 = \pi^2 - 2 + C$, so $C = 2$. Hence

$$y = \frac{-x^2 \cos x + 2x \sin x + 2 \cos x + 2}{x^2}.$$

34. $\mu = x$. Then $(xy)' = x^3$, so $xy = x^4/4 + C$. Using $y(1) = 0$ gives $C = -1/4$:

$$y = \frac{x^4 - 1}{4x}.$$

At $(1, 0)$, $y' = 1^2 - 0 = \boxed{1}$.

35. Set $y = vx$, so $y' = v + xv'$. Then

$$v + xv' = \frac{2+v}{1-2v} \implies xv' = \frac{2(1+v^2)}{1-2v}.$$

Thus

$$\frac{1-2v}{1+v^2} dv = 2 \frac{dx}{x}.$$

Integrating,

$$\arctan v - \ln(1+v^2) = 2 \ln x + C.$$

At $(1, 0)$, $v = 0$, so $C = 0$. Substitute $v = y/x$ to obtain the required result.

36. With $y = vx$,

$$v + xv' = 1 + v \implies v' = \frac{1}{x}.$$

Hence $v = \ln x + C$ and $y = x(\ln x + C)$. Since $y(e) = 2e$, $2 = 1 + C$, so

$$y = x(\ln x + 1).$$

Since $x > 0$, $y = 0$ requires $\ln x + 1 = 0$, so the intercept is $\boxed{(e^{-1}, 0)}$.

37.

(a) $\boxed{P = 0, 1200}$.

(b) Separation gives $\ln \frac{P}{1200-P} = 0.6t + C$. Since $P(0) = 200$, the ratio is $1/5$. Thus

$$P = \frac{1200}{1 + 5e^{-0.6t}}.$$

(c) $600 = 1200/(1 + 5e^{-0.6t})$ gives $e^{-0.6t} = 1/5$, so

$$t = \frac{\ln 5}{0.6} = \frac{5}{3} \ln 5 \text{ weeks}.$$

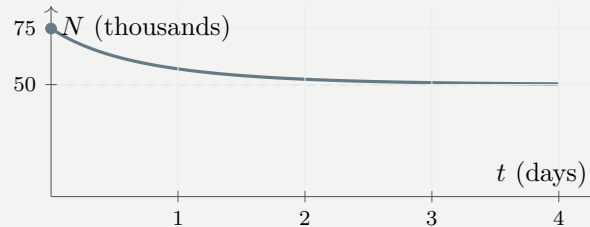
(d) The quadratic $0.0005P(1200 - P)$ is maximal at $P = 600$, giving

$$\boxed{180 \text{ insects per week}}.$$

3. IB Paper 1-style questions | Questions 31–50 (continued)

38.

- (a) Separation gives $\ln |N/(50 - N)| = t + C$. Using $N(0) = 75$ gives $N/(50 - N) = -3e^t$. Rearranging, $N = 50/(1 - \frac{1}{3}e^{-t})$.
- (b) $60 = 50/(1 - e^{-t}/3)$ gives $e^{-t} = 1/2$, so $t = \ln 2$ days.
- (c) $N \rightarrow 50$: the limiting population is 50 000. For $N > 50$, $N' < 0$, so the curve decreases from 75 towards the asymptote $N = 50$.



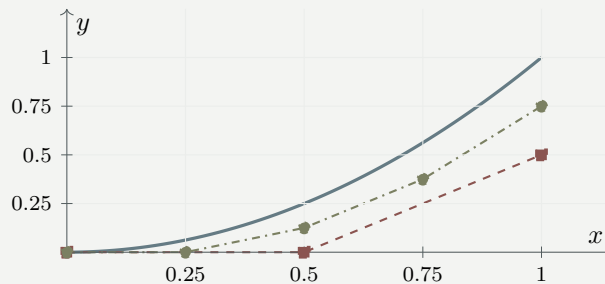
39. $y_{n+1} = y_n + \frac{1}{2}(x_n^2 - y_n)$.

$$(0, 1) \rightarrow (\frac{1}{2}, \frac{1}{2}) \rightarrow (1, \frac{3}{8}) \rightarrow (\frac{3}{2}, \frac{11}{16}).$$

Therefore $y(1.5) \approx \frac{11}{16}$. The slopes at the first three points are $-1, -1/4, 5/8$; at the final point the slope is $25/16$.

40.

- (a) $h = 0.5$: $(0, 0) \rightarrow (0.5, 0) \rightarrow (1, 0.5)$, so 0.5 .
- (b) $h = 0.25$: $(0, 0) \rightarrow (0.25, 0) \rightarrow (0.5, 0.125) \rightarrow (0.75, 0.375) \rightarrow (1, 0.75)$, so 0.75 .
- (c) The solution is concave up ($y'' = 2 > 0$). Each forward Euler segment uses the slope at its left endpoint, so its polygon lies below the curve. The $h = 0.25$ estimate is closer to the exact value 1.



Solid: exact; dashed squares: $h = 0.5$; dash-dot circles: $h = 0.25$.

41.

- (a) $T - 18 = 72e^{-kt}$. At $t = 10$, $36 = 72e^{-10k}$, so $k = \frac{\ln 2}{10}$.
- (b) $T = 18 + 72 \cdot 2^{-t/10}$.
- (c) $27 - 18 = 9 = 72 \cdot 2^{-t/10}$, so $2^{-t/10} = 1/8 = 2^{-3}$ and $t = 30$ min.

42.

- (a) $R = e^{-kt}$ and $e^{-1600k} = 1/2$, so $k = \frac{\ln 2}{1600}$ and

$$R(t) = 2^{-t/1600}.$$

- (b) $1/8 = 2^{-3}$, so $t = 4800$ years.



3. IB Paper 1-style questions | Questions 31–50 (continued)

43.

(a) $\mu = e^{0.1t}$. Then $(e^{0.1t}M)' = 5e^{0.1t}$, so $M = 50 + Ce^{-0.1t}$. From $M(0) = 10$, $C = -40$:

$$M = 50 - 40e^{-t/10}.$$

(b) 50 g .

(c) The mass approaches 50 g, so the concentration approaches the incoming concentration 1 g/L. At equilibrium the salt inflow and outflow rates are both 5 g/min.

44.

(a) $P(x) = -2x$, so $\mu = e^{-x^2}$.(b) $(e^{-x^2}y)' = 1$, hence $e^{-x^2}y = x + C$ and $y = e^{x^2}(x + C)$. Since $y(0) = 0$, $C = 0$:

$$y = xe^{x^2}.$$

45.

(a) $x_{n+1} = x_n + 0.1$, $y_{n+1} = y_n + 0.1(x_n + y_n)$.(b) $(0, 1) \rightarrow (0.1, 1.1) \rightarrow (0.2, 1.22) \rightarrow (0.3, 1.362)$, so $y(0.3) \approx 1.362$.(c) $y'' = 2e^x > 0$, so the curve is concave up. Euler's forward tangent steps lie below it; the estimate is an underestimate.46. $\frac{dy}{1+y^2} = \cos x \, dx$, so

$$\arctan y = \sin x + C.$$

The condition gives $C = 0$, hence

$$y = \tan(\sin x).$$

Since $-1 \leq \sin x \leq 1$ and $[-1, 1] \subset (-\pi/2, \pi/2)$, $\tan(\sin x)$ has no poles for real x .47. $\mu = e^{\int \cot x \, dx} = \sin x$ on $0 < x < \pi$. Thus

$$(y \sin x)' = \sin^2 x,$$

so

$$y \sin x = \frac{x}{2} - \frac{\sin 2x}{4} + C.$$

At $x = \pi/2$, $y = 0$, giving $C = -\pi/4$. Therefore

$$y = \frac{2x - \sin 2x - \pi}{4 \sin x}.$$

48. Let $y = vx$. Then

$$v + xv' = v^2 \implies \frac{dv}{v(v-1)} = \frac{dx}{x}.$$

So

$$\ln \left| \frac{v-1}{v} \right| = \ln x + C,$$

and hence $(v-1)/v = Cx$. Thus $v = 1/(1-Cx)$. Since $v(1) = 2$, $C = 1/2$. Therefore

$$y = \frac{2x}{2-x}.$$

The required interval is $0 < x < 2$, and the vertical asymptote is $x = 2$. The substitution divides by $v(v-1)$; the extra solutions $y = 0$ and $y = x$ do not satisfy the given initial condition.

**3. IB Paper 1-style questions | Questions 31–50 (continued)****49.**

- (a) $\frac{1}{2} \ln |y/(2-y)| = x + C$ gives $y = 2/(1 + Ae^{-2x})$, with $A \neq 0$ for a non-equilibrium solution, on intervals where the denominator is nonzero. Equilibria are $y = 0$ and $y = 2$; the latter corresponds to $A = 0$ in the same formula.
- (b) $3 = 2/(1 + A)$ gives $A = -1/3$, so $y = 2/(1 - e^{-2x}/3)$.
- (c) For $x \geq 0$ the denominator increases from $2/3$ towards 1, so $y > 2$ and $y' = y(2-y) < 0$. The expression tends to 2.

50.

- (a) Divide by x : $y' + (1/x - 1)y = xe^x$. Therefore

$$\mu = e^{\int (1/x - 1) dx} = xe^{-x}.$$

- (b) $(xe^{-x}y)' = x^2$, so

$$xe^{-x}y = \frac{x^3}{3} + C.$$

Hence $y = e^x \left(\frac{x^2}{3} + \frac{C}{x} \right)$. The condition $y(1) = 4e/3$ gives $C = 1$:

$$y = e^x \left(\frac{x^2}{3} + \frac{1}{x} \right).$$

- (c) $y(3) = \frac{10e^3}{3}$.