



Maclaurin Series

IB Mathematics AA HL | AHL 5.19

50 questions: standard expansions, series manipulation and local approximations

Learning goal	Build and manipulate Maclaurin series; use them to approximate, evaluate limits and develop solutions of differential equations.
Syllabus link	AHL 5.19: expansions of e^x , $\sin x$, $\cos x$, $\ln(1+x)$ and $(1+x)^p$ ($p \in \mathbb{Q}$); substitution, products, integration, differentiation and series from differential equations.
Structure	15 short-response, 15 extended-response and 20 original IB Paper 1-style questions, followed by worked solutions.
Technology	Non-calculator throughout. All angles are in radians. Decimal data needed for comparisons are supplied.

Instructions. Work on separate paper. Expand about $x = 0$ in ascending powers and show the method used. P_n denotes the Maclaurin polynomial through degree n . Write $+\dots$ for omitted terms, and use $\approx$ for a numerical approximation. Retain enough terms before multiplying, dividing or differentiating. Open intervals requested exclude endpoint tests.

1. Short response | Questions 1–15

1. Write the Maclaurin polynomial $P_4(x)$ for a function f in terms of its derivatives at 0.

2. The following derivative values are given. Find $P_3(x)$.

$f(0)$	$f'(0)$	$f''(0)$	$f^{(3)}(0)$
2	-1	6	-12

3. The coefficient of x^5 in the Maclaurin series of f is $-3/20$. Find $f^{(5)}(0)$.

4. Expand e^{-2x} up to and including the term in x^3 .

5. Find the first three non-zero terms in the Maclaurin series of $\sin(3x)$.

6. Expand $\cos(2x)$ up to and including the term in x^4 .

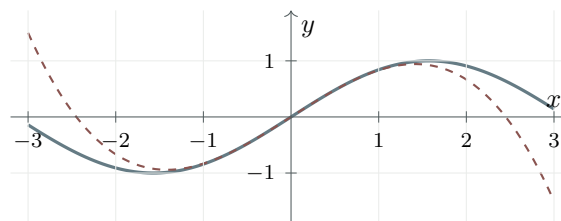
7. Expand $\ln(1+3x)$ through the term in x^3 . State the open interval obtained from $|3x| < 1$.

8. Write the expansion of $(1+x)^p$, where $p \in \mathbb{Q}$, up to and including the term in x^3 .

9. Expand $\sqrt{1-2x}$ through the term in x^3 .



10. Expand $(1 + 2x)^{-1}$ through the term in x^3 . State the open interval of convergence.
11. Find the first four non-zero terms in the Maclaurin series of e^{-x^2} .
12. Expand $\sin(x^2)$ up to and including the term in x^6 .
13. An analytic function has expansion $f(x) = 1 - 2x + 3x^2 - 4x^3 + \dots$. Find $f'(x)$ through the term in x^2 .
14. By integrating the series for $\cos t$, find the first three non-zero terms of $\int_0^x \cos t \, dt$.
15. The graphs of $y = \sin x$ and $y = x - x^3/6$ are shown. Identify each curve, then explain why $\sin x \approx x - x^3/6$ is a local approximation.



Curve A: solid. Curve B: dashed.

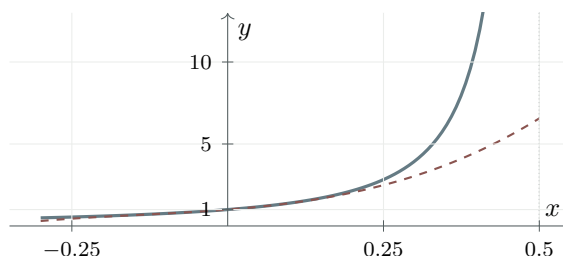
2. Extended response | Questions 16–30

16. Let $f(x) = \ln(2 + x)$.
- (a) Find $f(0)$ and its first three derivatives at 0.
- (b) Hence find its Maclaurin series through the term in x^3 .
17. Consider $f(x) = \sqrt{4 + x}$.
- (a) Write f as a multiple of $(1 + u)^{1/2}$ and expand through x^3 .
- (b) Use the cubic polynomial to approximate $\sqrt{4.2}$, giving a fraction.
- (c) State the open interval obtained from $|u| < 1$.



18. Let $f(x) = (1 - 2x)^{-3/2}$.

- (a) Find the expansion through x^3 .
- (b) Use it to approximate $(0.9)^{-3/2}$ as a fraction.
- (c) Use the graph to explain why the cubic becomes unreliable as $x \rightarrow \frac{1}{2}^-$.



Solid: f . Dashed: its cubic Maclaurin polynomial. Dotted: $x = \frac{1}{2}$.

19. Let $f(x) = e^{2x} \sin x$.

- (a) Find the expansion through x^4 by multiplying standard series.
- (b) Find $f^{(3)}(0)$.

20. Let $g(x) = e^{-x} \cos x$.

- (a) Find the expansion through x^4 .
- (b) Hence find $g''(0)$ and $g^{(4)}(0)$.

21. Find the Maclaurin series of $\frac{\ln(1+x)}{1+x}$ through x^4 , showing the product used.

22. Use the geometric series for $(1 - 2x)^{-1}$.

- (a) By differentiating, obtain the series for $(1 - 2x)^{-2}$ through x^3 .
- (b) Find the coefficient of x^n in $(1 - 2x)^{-2}$ for $n \geq 0$.

23. You are given $\frac{d}{dx} \arctan(2x) = \frac{2}{1 + 4x^2}$.

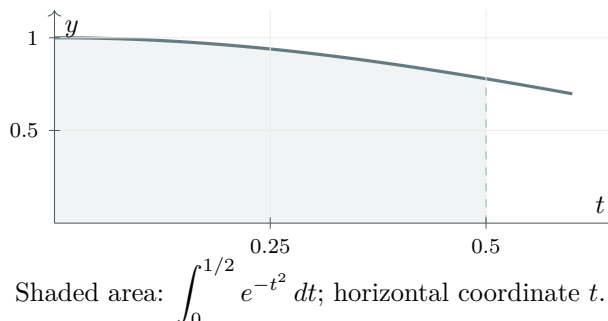
- (a) Use a geometric series and integration to find the first four non-zero terms of $\arctan(2x)$.
- (b) State the open interval obtained from $|4x^2| < 1$.

24. Given $\frac{d}{dx} \arcsin x = (1 - x^2)^{-1/2}$ and $\arcsin 0 = 0$, find the Maclaurin series of $\arcsin x$ through x^7 .



25. Define $F(x) = \int_0^x e^{-t^2} dt$.

- (a) Find $F(x)$ through x^7 .
(b) Use these four terms to approximate the shaded area as an exact fraction.



26. Define $H(x) = \int_0^x \frac{\ln(1+t)}{t} dt$, with integrand value 1 at $t = 0$. Find $H(x)$ through x^4 . Explain how the integration constant is determined.

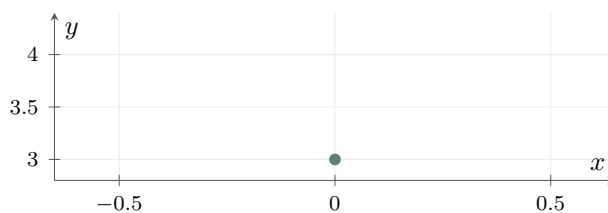
27. Find the value of a for which $\lim_{x \rightarrow 0} \frac{\sin(2x) - ax}{x^3}$ is finite. Then evaluate the limit.

28. The expansion of $(1 + kx)^p$ begins $1 + 6x + 12x^2 + \dots$.

- (a) Find p and k .
(b) Find the coefficient of x^3 and explain whether the expansion terminates.

29. An analytic solution satisfies $y' = 2xy$, $y(0) = 3$.

- (a) Use coefficient comparison to find its series through x^6 .
(b) Sketch the resulting polynomial on the supplied axes for $-0.5 \leq x \leq 0.5$.



30. An analytic solution satisfies $y' = x + y$, $y(0) = 2$. By substituting a power series and comparing coefficients, find y through x^4 .

**3. IB Paper 1-style questions | Questions 31–50****31.** [Maximum mark: 8]Let $f(x) = e^{-3x}$.

- (a) Use successive derivatives to obtain the Maclaurin series through x^4 . [5 marks]
- (b) Use the quartic polynomial to approximate $e^{-3/10}$ as a fraction. [2 marks]
- (c) State $f^{(5)}(0)$. [1 mark]
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32. [Maximum mark: 6]Consider $f(x) = \ln(1 - 2x)$.

- (a) Find the Maclaurin series through x^4 . [4 marks]
- (b) Hence evaluate $\lim_{x \rightarrow 0} \frac{\ln(1 - 2x) + 2x + 2x^2}{x^3}$. [2 marks]
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33. [Maximum mark: 9]Let $f(x) = e^{2x} \cos x$.

- (a) Find its Maclaurin series through x^4 . [5 marks]
- (b) Hence find $f^{(3)}(0)$ and $f^{(4)}(0)$. [2 marks]
- (c) Evaluate $\lim_{x \rightarrow 0} \frac{f(x) - 1 - 2x - \frac{3}{2}x^2}{x^3}$. [2 marks]
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34. [Maximum mark: 7]Let $g(x) = \ln(1 + x) \sin(2x)$.

- (a) Find its Maclaurin series through x^4 . [5 marks]
- (b) Hence evaluate $\lim_{x \rightarrow 0} \frac{g(x)}{x^2}$. [2 marks]
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35. [Maximum mark: 8]Let $f(x) = e^{\sin(2x)}$.

- (a) By substituting the series for $\sin(2x)$ into that for e^u , find $f(x)$ through x^4 . [6 marks]
- (b) Hence find $f^{(3)}(0)$ and $f^{(4)}(0)$. [2 marks]
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36. [Maximum mark: 8]Let $f(x) = \ln(1 + \sin x)$.

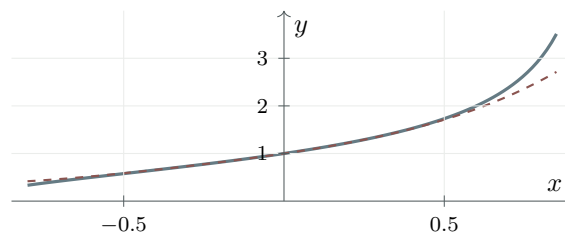
- (a) Find its Maclaurin series through x^4 . [6 marks]
- (b) Hence evaluate $\lim_{x \rightarrow 0} \frac{f(x) - x + \frac{1}{2}x^2}{x^3}$. [2 marks]



37. [Maximum mark: 9]

For $-1 < x < 1$, let $f(x) = \sqrt{\frac{1+x}{1-x}}$.

- (a) Use a product of two binomial series to find $f(x)$ through x^4 . [6 marks]
(b) Hence approximate $\sqrt{11/9}$, giving a terminating decimal. [2 marks]
(c) Use the graph to compare the reliability of the same polynomial at $x = 0.1$ and $x = 0.8$. [1 mark]



Solid: $f(x)$. Dashed: its quartic Maclaurin polynomial.

38. [Maximum mark: 7]

In the expansion of $(1 + kx)^{-2}$, where $k \neq 0$, the coefficient of x^3 is four times the coefficient of x^2 .

- (a) Find k . [4 marks]
(b) Write the first four terms and state the open interval of convergence. [3 marks]

39. [Maximum mark: 8]

Let $F(x) = (1 + x)^p e^{-px}$, where $p > 0$ is rational.

- (a) Show that $F(x) = 1 - \frac{p}{2}x^2 + \frac{p}{3}x^3 + \dots$. [5 marks]
(b) Given that the coefficient of x^2 is -3 , find p and $F^{(3)}(0)$. [3 marks]

40. [Maximum mark: 8]

For $x \neq 0$ near 0, let $g(x) = \frac{e^{2x} - 1}{\sin x}$.

- (a) Find the expansion of g through x^3 . [6 marks]
(b) State the value of $g(0)$ that makes g continuous at 0, and find $g'(0)$ for this extension. [2 marks]

41. [Maximum mark: 9]

An analytic solution satisfies $y' = 2(1 + y^2)$ and $y(0) = 0$.

- (a) By comparing coefficients, find its Maclaurin series through x^5 . [7 marks]
(b) Given that the solution is $y = \tan(2x)$ near zero, evaluate $\lim_{x \rightarrow 0} \frac{\tan(2x) - 2x}{x^3}$. [2 marks]

42. [Maximum mark: 9]

An analytic solution satisfies $y' = x - y^2$ and $y(0) = 1$.

- (a) Use successive differentiation to find $y'(0)$, $y''(0)$, $y^{(3)}(0)$ and $y^{(4)}(0)$. [6 marks]
(b) Hence find the Maclaurin series through x^4 . [3 marks]



43. [Maximum mark: 9]

An analytic solution satisfies $y'' = xy$, $y(0) = 1$ and $y'(0) = 2$. Write $y = \sum_{n=0}^{\infty} a_n x^n$.

- (a) Show that $a_2 = 0$ and $(n+2)(n+1)a_{n+2} = a_{n-1}$ for $n \geq 1$. [4 marks]
- (b) Hence find the Maclaurin series through x^6 . [5 marks]

44. [Maximum mark: 8]

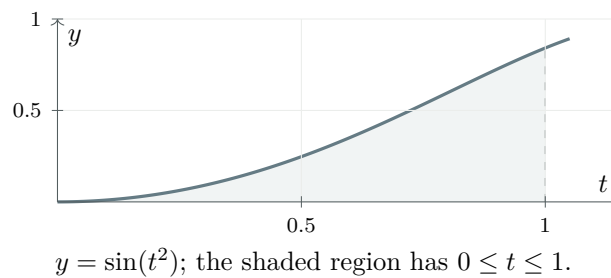
An analytic solution satisfies $(1-x)y' = 2y$ and $y(0) = 1$.

- (a) Show that its coefficients satisfy $a_{n+1} = \frac{n+2}{n+1}a_n$ for $n \geq 0$. [4 marks]
- (b) Find the first five terms and identify the corresponding binomial function. [3 marks]
- (c) State the open interval of convergence of the series. [1 mark]

45. [Maximum mark: 7]

The shaded area is $A = \int_0^1 \sin(t^2) dt$.

- (a) Find the first three non-zero terms in the expansion of $\sin(t^2)$. [3 marks]
- (b) Integrate those terms to obtain an approximation to A as an exact fraction. [4 marks]



46. [Maximum mark: 7]

Define $H(x) = \int_0^x \ln(1+t^2) dt$.

- (a) Find the first four non-zero terms in the Maclaurin series of H . [5 marks]
- (b) Evaluate $\lim_{x \rightarrow 0} \frac{H(x) - x^3/3}{x^5}$. [2 marks]

47. [Maximum mark: 8]

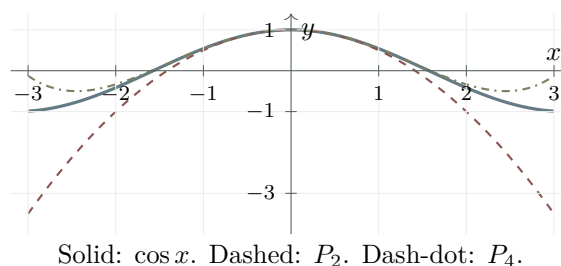
Let $f(x) = \sin^2 x$.

- (a) Use a double-angle identity to find its Maclaurin series through x^6 . [4 marks]
- (b) Differentiate your result to obtain the series for $2 \sin x \cos x$ through x^5 . [3 marks]
- (c) Check these coefficients using a standard trigonometric identity. [1 mark]

48. [Maximum mark: 7]

Let P_2 and P_4 be the Maclaurin polynomials of degrees 2 and 4 for $\cos x$.

- Write down $P_2(x)$ and $P_4(x)$. [2 marks]
- Given $\cos 1 \approx 0.5403$, estimate each absolute error at $x = 1$ to four decimal places. State which approximation is closer. [3 marks]
- Explain why neither finite polynomial represents the bounded, periodic behaviour of $\cos x$ for all real x . [2 marks]



49. [Maximum mark: 8]

Let $F(x) = (1 + ax)e^x - 1 - bx$, where a, b are constants.

- Find a, b such that both the coefficients of x and x^2 in F are zero. [5 marks]
- For these values, evaluate $\lim_{x \rightarrow 0} \frac{F(x)}{x^3}$. [3 marks]

50. [Maximum mark: 12]

An analytic solution satisfies $y' = e^x - y$ and $y(0) = 0$.

- Show that $y'' = y$, and find $y'(0)$. [3 marks]
- Hence obtain the Maclaurin series through x^5 . [4 marks]
- Evaluate $\lim_{x \rightarrow 0} \frac{y(x) - \sin x}{x^3}$. [2 marks]
- Use your three non-zero terms to approximate $\int_0^a y(x) dx$ for small positive a . [3 marks]



Answer key and solution outlines

1. Short response | Questions 1–15

1. $P_4(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4.$

2. $P_3 = 2 - x + (6/2!)x^2 - (12/3!)x^3 = \boxed{2 - x + 3x^2 - 2x^3}.$

3. $f^{(5)}(0)/5! = -3/20$, so $\boxed{f^{(5)}(0) = -18}.$

4. Put $u = -2x$ in $e^u = 1 + u + u^2/2! + u^3/3! + \dots$: $\boxed{1 - 2x + 2x^2 - \frac{4}{3}x^3 + \dots}.$

5. $\sin(3x) = 3x - (3x)^3/3! + (3x)^5/5! + \dots = \boxed{3x - \frac{9}{2}x^3 + \frac{81}{40}x^5 + \dots}.$

6. $\cos(2x) = 1 - (2x)^2/2! + (2x)^4/4! + \dots = \boxed{1 - 2x^2 + \frac{2}{3}x^4 + \dots}.$

7. $\ln(1+u) = u - u^2/2 + u^3/3 + \dots$ gives $\boxed{3x - \frac{9}{2}x^2 + 9x^3 + \dots}.$ The interval is $\boxed{-\frac{1}{3} < x < \frac{1}{3}}.$

8. $\boxed{1 + px + \frac{p(p-1)}{2}x^2 + \frac{p(p-1)(p-2)}{6}x^3 + \dots}.$

9. Use $p = \frac{1}{2}$, $u = -2x$ in the binomial series: $\boxed{1 - x - \frac{1}{2}x^2 - \frac{1}{2}x^3 + \dots}.$

10. The geometric series with ratio $-2x$ gives $\boxed{1 - 2x + 4x^2 - 8x^3 + \dots}$, for $\boxed{-\frac{1}{2} < x < \frac{1}{2}}.$

11. Substitute $u = -x^2$ into e^u : $\boxed{1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \dots}.$

12. $\sin(x^2) = x^2 - (x^2)^3/3! + \dots = \boxed{x^2 - \frac{1}{6}x^6 + \dots}.$ There is no x^4 term.

13. Differentiate each displayed term: $\boxed{f'(x) = -2 + 6x - 12x^2 + \dots}.$

14. $\int_0^x (1 - t^2/2 + t^4/24 + \dots) dt = \boxed{x - x^3/6 + x^5/120 + \dots}.$ The lower limit makes the constant zero.

15. A is $\sin x$ and B is $x - x^3/6$. They agree closely near $x = 0$, but the cubic departs from the sine curve farther away. The omitted terms are not identically zero, so the finite polynomial is not equal to $\sin x$ for every x .

2. Extended response | Questions 16–30

16. $f' = 1/(2+x)$, $f'' = -1/(2+x)^2$, $f^{(3)} = 2/(2+x)^3$. Thus $f(0) = \ln 2$, $f'(0) = 1/2$, $f''(0) = -1/4$, $f^{(3)}(0) = 1/4$. Dividing the derivative values by the relevant factorials gives $\boxed{\ln 2 + x/2 - x^2/8 + x^3/24 + \dots}.$

17. (a) $\sqrt{4+x} = 2(1+x/4)^{1/2}$, so $\boxed{f(x) = 2 + x/4 - x^2/64 + x^3/512 + \dots}.$ (b) Put $x = 1/5$: $\sqrt{4.2} \approx 2 + 1/20 - 1/1600 + 1/64000 = \boxed{131161/64000}.$ (c) $|x/4| < 1$, so $\boxed{-4 < x < 4}.$

18. (a) The binomial coefficients give $\boxed{f(x) = 1 + 3x + \frac{15}{2}x^2 + \frac{35}{2}x^3 + \dots}.$ (b) $1 - 2x = 0.9$ gives $x = 1/20$, so the approximation is $1 + 3/20 + 15/800 + 35/16000 = \boxed{3747/3200}.$ (c) $f(x) \rightarrow +\infty$, whereas the cubic remains finite at $1/2$. Closeness near zero does not persist near the vertical asymptote.

19. Multiply $(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \dots)(x - x^3/6 + \dots)$. The x^3 coefficient is $2 - 1/6 = 11/6$ and the x^4 coefficient is $4/3 - 2/6 = 1$. Hence $\boxed{f = x + 2x^2 + \frac{11}{6}x^3 + x^4 + \dots}$ and $\boxed{f^{(3)}(0) = 3!(11/6) = 11}.$

20. Multiply $(1 - x + x^2/2 - x^3/6 + x^4/24)(1 - x^2/2 + x^4/24)$ and retain powers through 4. The x^2 terms cancel; the x^3 coefficient is $1/2 - 1/6 = 1/3$, and the x^4 coefficient is $1/24 - 1/4 + 1/24 = -1/6$. Thus $\boxed{g = 1 - x + x^3/3 - x^4/6 + \dots}$; $\boxed{g''(0) = 0}$ and $\boxed{g^{(4)}(0) = -4}.$



2. Extended response | Questions 16–30 (continued)

21. Use $(x - x^2/2 + x^3/3 - x^4/4 + \dots)(1 - x + x^2 - x^3 + \dots)$. The coefficients of x^2, x^3, x^4 are $-1 - 1/2$, $1 + 1/2 + 1/3$, and $-1 - 1/2 - 1/3 - 1/4$. Hence $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \dots$.

22. (a) Start one power higher: $(1 - 2x)^{-1} = 1 + 2x + 4x^2 + 8x^3 + 16x^4 + \dots$. Its derivative is $2(1 - 2x)^{-2} = 2 + 8x + 24x^2 + 64x^3 + \dots$. Divide by 2: $1 + 4x + 12x^2 + 32x^3 + \dots$. (b) Differentiating $2^{n+1}x^{n+1}$ and dividing by 2 gives $(n+1)2^n$.

23. $2/(1 + 4x^2) = 2 - 8x^2 + 32x^4 - 128x^6 + \dots$. Integrate term by term; $\arctan 0 = 0$ makes the constant zero. Thus $2x - \frac{8}{3}x^3 + \frac{32}{5}x^5 - \frac{128}{7}x^7 + \dots$. The requested interval is $-\frac{1}{2} < x < \frac{1}{2}$.

24. First expand $(1 - x^2)^{-1/2} = 1 + x^2/2 + 3x^4/8 + 5x^6/16 + \dots$. Integrate and use the initial value: $\arcsin x = x + x^3/6 + 3x^5/40 + 5x^7/112 + \dots$.

25. (a) Integrate $1 - t^2 + t^4/2 - t^6/6 + \dots$: $F(x) = x - x^3/3 + x^5/10 - x^7/42 + \dots$. (b) $F(1/2) \approx 1/2 - 1/24 + 1/320 - 1/5376 = 4133/8960$.

26. Divide the logarithmic series by t to obtain $1 - t/2 + t^2/3 - t^3/4 + \dots$. Integration gives $H(x) = x - x^2/4 + x^3/9 - x^4/16 + \dots$. The definition gives $H(0) = 0$, so the constant is zero.

27. $\sin(2x) = 2x - \frac{4}{3}x^3 + \dots$. After division the expression is $(2 - a)/x^2 - 4/3 + \dots$. Finiteness requires $a = 2$; the limit is then $-4/3$.

28. (a) $pk = 6$ and $p(p-1)k^2/2 = 12$. Since $p \neq 0$, substitution of $k = 6/p$ gives $18(p-1)/p = 12$, so $p = 3, k = 2$. (b) $(1 + 2x)^3 = 1 + 6x + 12x^2 + 8x^3$. It terminates because p is a non-negative integer.

29. Put $y = \sum_{n=0}^{\infty} a_n x^n$. Then $a_0 = 3$, $a_1 = 0$, and $(n+1)a_{n+1} = 2a_{n-1}$ for $n \geq 1$. Thus $a_2 = 3$, $a_3 = 0$, $a_4 = 3/2$, $a_5 = 0$, $a_6 = 1/2$: $y = 3 + 3x^2 + \frac{3}{2}x^4 + \frac{1}{2}x^6 + \dots$. The sketch is symmetric about the y -axis, with a minimum at $(0, 3)$, and rises to $493/128$ at both endpoints.

30. Write $y = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \dots$. Since $a_0 = 2$, comparison gives $a_1 = 2$, $2a_2 = 1 + a_1 = 3$, $3a_3 = a_2 = 3/2$, $4a_4 = a_3 = 1/2$. Hence $y = 2 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{8}x^4 + \dots$.

3. IB Paper 1-style questions | Questions 31–50

31. (a) $f^{(n)}(0) = (-3)^n$, so $f = 1 - 3x + \frac{9}{2}x^2 - \frac{9}{2}x^3 + \frac{27}{8}x^4 + \dots$. (b) Substitute $x = 1/10$: $e^{-3/10} \approx 1 - 3/10 + 9/200 - 9/2000 + 27/80000 = 59267/80000$. (c) $f^{(5)}(0) = -243$.

32. (a) Substitution of $u = -2x$ gives $-2x - 2x^2 - \frac{8}{3}x^3 - 4x^4 + \dots$. (b) The first two terms cancel. Divide by x^3 : $-8/3 - 4x + \dots \rightarrow -8/3$.

33. (a) Multiply $(1 + 2x + 2x^2 + \frac{4}{3}x^3 + \frac{2}{3}x^4)(1 - x^2/2 + x^4/24)$, retaining powers through 4. This gives $f = 1 + 2x + \frac{3}{2}x^2 + \frac{1}{3}x^3 - \frac{7}{24}x^4 + \dots$. (b) $f^{(3)}(0) = 2$ and $f^{(4)}(0) = -7$. (c) The quotient is $1/3 - 7x/24 + \dots$, so the limit is $1/3$.

34. (a) Multiply $(x - x^2/2 + x^3/3 + \dots)(2x - \frac{4}{3}x^3 + \dots)$. Thus $g = 2x^2 - x^3 - \frac{2}{3}x^4 + \dots$, since the x^4 coefficient is $2/3 - 4/3 = -2/3$. (b) Divide by x^2 to get $2 - x - 2x^2/3 + \dots \rightarrow 2$.

35. Put $u = 2x - \frac{4}{3}x^3 + \dots$. Then $u^2 = 4x^2 - \frac{16}{3}x^4 + \dots$, $u^3 = 8x^3 + \dots$, and $u^4 = 16x^4 + \dots$. Substitution into $1 + u + u^2/2 + u^3/6 + u^4/24$ yields $f = 1 + 2x + 2x^2 - 2x^4 + \dots$. The cubic terms cancel. Hence $f^{(3)}(0) = 0$, $f^{(4)}(0) = -48$.

36. With $u = \sin x = x - x^3/6 + \dots$, we have $u^2 = x^2 - x^4/3 + \dots$, $u^3 = x^3 + \dots$ and $u^4 = x^4 + \dots$. Thus $u - u^2/2 + u^3/3 - u^4/4$ gives $f = x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \dots$. The requested quotient tends to $1/6$.

**3. IB Paper 1-style questions | Questions 31–50 (continued)**

37. (a) Multiply $1 + x/2 - x^2/8 + x^3/16 - 5x^4/128$ by $1 + x/2 + 3x^2/8 + 5x^3/16 + 35x^4/128$. The result is $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4 + \dots$. (b) $(1+x)/(1-x) = 11/9$ gives $x = 1/10$. The estimate is $1 + 0.1 + 0.005 + 0.0005 + 0.0000375 = 1.1055375$. (c) The curves are much closer at 0.1; the graph shows a substantial underestimate at 0.8.

38. The series begins $1 - 2kx + 3k^2x^2 - 4k^3x^3 + \dots$. Thus $-4k^3 = 4(3k^2)$, and division by $4k^2 \neq 0$ gives $k = -3$. The expansion is $1 + 6x + 27x^2 + 108x^3 + \dots$, valid for $-1/3 < x < 1/3$.

39. Use $F = \exp\{p \ln(1+x) - px\}$. The exponent is $-px^2/2 + px^3/3 + \dots$; its square starts at x^4 , so $e^u = 1 + u + \dots$ gives the stated terms. Then $-p/2 = -3$, so $p = 6$ and $F^{(3)}(0) = 3!(p/3) = 12$.

40. Factor out x from numerator and denominator: $g = (2 + 2x + \frac{4}{3}x^2 + \frac{2}{3}x^3 + \dots)/(1 - x^2/6 + \dots)$. The reciprocal denominator is $1 + x^2/6 + \dots$. Multiplication gives $g = 2 + 2x + \frac{5}{3}x^2 + x^3 + \dots$. Hence $g(0) = 2$ and $g'(0) = 2$. The numerator must initially be expanded through x^4 .

41. Let $y = a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5 + \dots$. Comparison in $y' = 2 + 2y^2$ gives $a_1 = 2$, $2a_2 = 0$, $3a_3 = 2a_1^2 = 8$, $4a_4 = 4a_1a_2 = 0$, $5a_5 = 2(2a_1a_3 + a_2^2) = 64/3$. Hence $y = 2x + \frac{8}{3}x^3 + \frac{64}{15}x^5 + \dots$. Dividing the remaining terms by x^3 gives the limit $8/3$.

42. $y'(0) = -1$. Differentiating gives $y'' = 1 - 2yy'$, $y^{(3)} = -2[(y')^2 + yy'']$, $y^{(4)} = -2[3y'y'' + yy^{(3)}]$. Substitution yields $y''(0) = 3$, $y^{(3)}(0) = -8$, $y^{(4)}(0) = 34$. Therefore $y = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 + \dots$.

43. $y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$, whereas $xy = \sum_{n=1}^{\infty} a_{n-1}x^n$. The constant coefficient gives $2a_2 = 0$. With $a_0 = 1$, $a_1 = 2$, the recurrence gives $a_3 = 1/6$, $a_4 = 1/6$, $a_5 = 0$, $a_6 = 1/180$. Hence $y = 1 + 2x + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \frac{1}{180}x^6 + \dots$.

44. Comparing coefficients of x^n gives $(n+1)a_{n+1} - na_n = 2a_n$, including $n = 0$. Therefore the recurrence follows. Since $a_0 = 1$, the coefficients are 1, 2, 3, 4, 5. Thus $y = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots = (1-x)^{-2}$ for $-1 < x < 1$.

45. (a) Substitute $u = t^2$ into $\sin u$: $t^2 - t^6/6 + t^{10}/120 + \dots$. (b) $A \approx [t^3/3 - t^7/42 + t^{11}/1320]_0^1 = 1/3 - 1/42 + 1/1320 = 2867/9240$. This is an approximation, not an exact value of the integral.

46. $\ln(1+t^2) = t^2 - t^4/2 + t^6/3 - t^8/4 + \dots$. Integrating from 0 gives $H = x^3/3 - x^5/10 + x^7/21 - x^9/36 + \dots$. After cancellation and division, the quotient tends to $-1/10$.

47. (a) $f = (1 - \cos 2x)/2$ and $\cos 2x = 1 - 2x^2 + \frac{2}{3}x^4 - \frac{4}{45}x^6 + \dots$. Thus $f = x^2 - x^4/3 + 2x^6/45 + \dots$. (b) Differentiate: $2x - \frac{4}{3}x^3 + \frac{4}{15}x^5 + \dots$. (c) $2 \sin x \cos x = \sin 2x$ has exactly this series.

48. (a) $P_2 = 1 - x^2/2$ and $P_4 = 1 - x^2/2 + x^4/24$. (b) $P_2(1) = 0.5$, $P_4(1) = 13/24 \approx 0.5416667$. The errors are approximately 0.0403 and 0.0014, respectively; P_4 is closer. (c) A nonconstant polynomial is unbounded and not periodic. These polynomials approximate the curve locally near 0.

49. Multiplication gives $F = (1 + a - b)x + (1/2 + a)x^2 + (1/6 + a/2)x^3 + \dots$. Thus $1/2 + a = 0$ and $1 + a - b = 0$, so $a = -1/2$, $b = 1/2$. The cubic coefficient becomes $1/6 - 1/4 = -1/12$, giving the limit $-1/12$.

50. (a) $y'' = e^x - y' = e^x - (e^x - y) = y$, and $y'(0) = 1$. (b) The derivative values at 0 alternate 0, 1, 0, 1, 0, 1, ... so $y = x + x^3/6 + x^5/120 + \dots$. (c) Subtract $\sin x = x - x^3/6 + x^5/120 + \dots$ to obtain $y - \sin x = x^3/3 + \dots$; the limit is $1/3$. (d) Termwise integration gives $\int_0^a y \, dx \approx a^2/2 + a^4/24 + a^6/720$.