Integration: methods and solids

IB Mathematics: Analysis and Approaches HL

Sections 5.16 and 5.17 | 14 original practice questions

Learning goal	Select an integration method and connect a region to its area or volume.
Syllabus link	AHL 5.16: substitution and integration by parts, including repeated parts. AHL 5.17: areas with respect to y; revolution about either coordinate axis.
Structure	6 core questions, 5 mixed questions and 3 optional challenges; full worked solutions follow.
Technology	Non-calculator throughout. An AA formula booklet may be used.

Instructions. Work on separate paper. Give exact answers, use radians and include $+C$ for indefinite integrals. For a volume, state the axis, radius or radii, integration variable and limits before evaluating. Diagrams are guides, not scale measurements.

Choose a timed route

60 minutes: Q1 and Q3 (10 min); Q5 and Q7 (15 min); Q8 and Q10 (25 min); discuss and correct solutions (10 min).
90 minutes: Follow the first 50 minutes above, then Q9 (10 min), Q12 (15 min), Q14(a,b) (10 min), and a final review (5 min).
Other questions provide alternatives or follow-up practice. Timing includes discussion; skip a routine part if it is already secure.

1 Core fluency — Questions 1–6

1. Choose before calculating

State a suitable first method for each integral. Give the substitution or the functions to differentiate and integrate; do not evaluate.

(a) $\displaystyle\int \frac{2x}{(1+x^2)^3}\,\mathrm{d}x$.

(b) $\displaystyle\int x\ln x\,\mathrm{d}x$, for $x>0$.

(c) $\displaystyle\int x^2e^{2x}\,\mathrm{d}x$.

2. Change the limits too

Using $u=1+x^2$, evaluate $\displaystyle\int_0^{\sqrt{3}} x\sqrt{1+x^2}\,\mathrm{d}x$.

1 Core fluency continued — No calculator

3. One application of parts

Find $\int x\cos(2x)\,\mathrm{d}x$. Verify your answer by differentiation.

4. A logarithm is still a product

For $x > 0$, find:

(a) $\int \ln(2x)\,\mathrm{d}x$;

(b) $\int_1^2 x\ln x\,\mathrm{d}x$.

5. Horizontal strips

The region R is bounded by $x = 4 - y^2$, the y-axis and the lines $y = 0$ and $y = 2$.

(a) Sketch R and draw one horizontal strip. State its length.

(b) Find the area of R by integrating with respect to y.

6. Radius, not height

The region under $y = \sqrt{2x+1}$, above the x-axis and between $x = 0$ and $x = 4$ is rotated through 2π about the x-axis.

(a) Write an integral for the volume, stating the radius of a cross-sectional disc.

(b) Evaluate the volume. Explain why $\pi\int_0^4 \sqrt{2x+1}\,\mathrm{d}x$ is incorrect.

2 Mixed skills Questions 7–11

7. A substitution that changes the appearance

Consider $I = \displaystyle\int_0^{\ln 3} \frac{e^{2x}}{\sqrt{1+e^x}}\,\mathrm{d}x$.

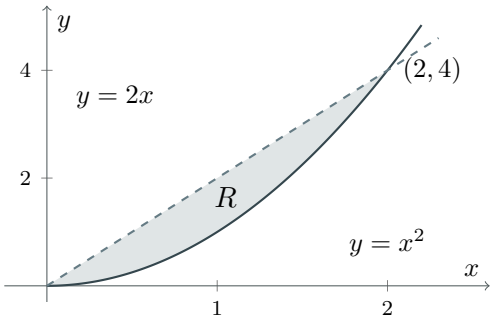
(a) Use $u = \sqrt{1+e^x}$ to show that $I = 2\displaystyle\int_{\sqrt{2}}^{2}(u^2-1)\,\mathrm{d}u$.

(b) Evaluate I exactly.

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8. Same region, different solids

The region R lies between $y = x^2$ and $y = 2x$ in the first quadrant.



(a) Find the intersection points and the area of R.

(b) Find the volume when R is rotated about the x-axis.

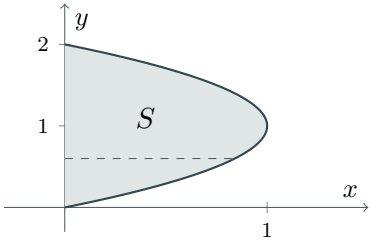
(c) Express both boundaries as x in terms of y. Hence find the volume when R is rotated about the y-axis, using washers perpendicular to that axis.

(d) Explain why squaring the difference of the two radii gives the wrong cross-sectional area.

2 Mixed skills continued — Set up the geometry first

9. A curved boundary expressed in terms of y

The region S is bounded by $x = y(2 - y)$ and the y-axis, for $0 \leq y \leq 2$.



(a) Find its area using horizontal strips.

(b) Find the volume when S is rotated about the y-axis.

(c) A student squares the area found in (a) and multiplies by π. Explain why this does not give the volume.

10. Repeated parts meets volume

Let $f(x) = xe^x$ on $0 \leq x \leq 1$.

(a) Use integration by parts twice to find $\int x^2 e^{2x}\,\mathrm{d}x$.

(b) The region under $y = f(x)$ and above the x-axis on this interval is rotated about the x-axis. Find its volume.

11. Recover a boundary from a volume

The region under $y = e^{-x}$, above the x-axis, from $x = 0$ to $x = a$ $(a > 0)$, is rotated about the x-axis.

(a) Show that its volume is $V = \frac{\pi}{2}(1 - e^{-2a})$.

(b) Find a when $V = \frac{3\pi}{8}$.

(c) Explain why no finite positive a produces $V = \frac{\pi}{2}$.

3 Optional challenges Questions 12–14

12. A substitution followed by parts

Let $I = \displaystyle\int_0^{\sqrt{e-1}} x\ln(1+x^2)\,\mathrm{d}x$.

(a) Use $u = 1 + x^2$ to transform the integral, including its limits.

(b) Evaluate I exactly using integration by parts.

(c) The region under $y = \sqrt{x\ln(1+x^2)}$ on the same interval is rotated about the x-axis. Find its volume and explain why the radius is real throughout the interval.

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13. Parts comes back to itself

Let $I = \displaystyle\int_0^{\pi/2} e^x\cos x\,\mathrm{d}x$ and $J = \displaystyle\int_0^{\pi/2} e^x\sin x\,\mathrm{d}x$.

(a) Use integration by parts to obtain two simultaneous equations in I and J.

(b) Hence find I exactly.

(c) The region under $y = \sqrt{e^x\cos x}$ on $0 \le x \le \pi/2$ is rotated about the x-axis. Find the volume.

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14. A shape comparison with a parameter

Let $a > 0$. The curves $x = y^2$ and $x = ay$ enclose a region T in the first quadrant.

(a) Find their intersections and the area of T by integrating with respect to y.

(b) Find the volume V_y generated by rotating T about the y-axis.

(c) Rewrite the boundaries in terms of x and find the volume V_x generated by rotating T about the x-axis. Use washers, not cylindrical shells.

(d) Find the positive value of a for which $V_x = V_y$.

Answer key and worked solutions — Core fluency

1 Core fluency: Questions 1–6

1. (a) Substitution: $u = 1 + x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$.
(b) Parts: differentiate $\ln x$ and integrate x.
(c) Repeated parts: differentiate x^2 and integrate e^{2x}, then apply parts to the remaining integral involving xe^{2x}.

2. The limits change from $x = 0, \sqrt{3}$ to $u = 1, 4$:

$$\frac{1}{2}\int_1^4 u^{1/2}\,\mathrm{d}u = \left[\frac{u^{3/2}}{3}\right]_1^4 = \boxed{\frac{7}{3}}.$$

3. Take $u = x$, $\mathrm{d}v = \cos(2x)\,\mathrm{d}x$, so $\mathrm{d}u = \mathrm{d}x$ and $v = \frac{1}{2}\sin(2x)$:

$$\int x\cos(2x)\,\mathrm{d}x = \frac{x}{2}\sin(2x) - \frac{1}{2}\int \sin(2x)\,\mathrm{d}x = \boxed{\frac{x}{2}\sin(2x) + \frac{1}{4}\cos(2x) + C}.$$

Differentiation gives $\frac{1}{2}\sin(2x) + x\cos(2x) - \frac{1}{2}\sin(2x) = x\cos(2x)$.

4. (a) Differentiate $\ln(2x)$ and integrate 1:

$$\int \ln(2x)\,\mathrm{d}x = x\ln(2x) - \int x\frac{1}{x}\,\mathrm{d}x = \boxed{x\ln(2x) - x + C}.$$

(b) With $u = \ln x$, $v = x^2/2$,

$$\int_1^2 x\ln x\,\mathrm{d}x = \left[\frac{x^2}{2}\ln x - \frac{x^2}{4}\right]_1^2 = \boxed{2\ln 2 - \frac{3}{4}}.$$

5. (a) The boundary runs from $(4, 0)$ to $(0, 2)$ in the first quadrant. A horizontal strip runs from $x = 0$ to $x = 4 - y^2$; its length is $4 - y^2$.
(b) $A = \displaystyle\int_0^2 (4 - y^2)\,\mathrm{d}y = \left[4y - \frac{y^3}{3}\right]_0^2 = \boxed{\frac{16}{3}}$ square units.

6. (a) Axis: x-axis; disc radius $r = \sqrt{2x + 1}$; variable x; limits $0, 4$.

$$V = \pi\int_0^4 r^2\,\mathrm{d}x = \pi\int_0^4 (2x + 1)\,\mathrm{d}x.$$

(b) $V = \pi[x^2 + x]_0^4 = \boxed{20\pi}$ cubic units. Disc area is πr^2, not πr; the proposed integrand fails to square the radius.

Answer key and worked solutions Mixed skills: Questions 7–9

2 Substitution and geometry

7. (a) $u^2 = 1 + e^x$, so $e^x = u^2 - 1$ and $e^x \mathrm{d}x = 2u\mathrm{d}u$.

$$\frac{e^{2x}}{\sqrt{1+e^x}}\mathrm{d}x = \frac{u^2-1}{u}(2u\,\mathrm{d}u) = 2(u^2-1)\,\mathrm{d}u.$$

At $x = 0$, $u = \sqrt{2}$; at $x = \ln 3$, $u = 2$.

(b) $I = \left[\frac{2}{3}u^3 - 2u\right]_{\sqrt{2}}^{2} = \frac{4}{3} - \left(-\frac{2\sqrt{2}}{3}\right) = \boxed{\frac{4+2\sqrt{2}}{3}}$.

8. (a) $x^2 = 2x \Rightarrow x = 0, 2$, giving $(0, 0)$ and $(2, 4)$. The line is above the parabola for $0 < x < 2$:

$$A = \int_0^2 (2x - x^2)\,\mathrm{d}x = \left[x^2 - \frac{x^3}{3}\right]_0^2 = \boxed{\frac{4}{3}}.$$

(b) About the x-axis: outer radius $2x$, inner radius x^2; integrate in x from 0 to 2:

$$V_x = \pi \int_0^2 \left((2x)^2 - (x^2)^2\right)\mathrm{d}x = \pi\left[\frac{4}{3}x^3 - \frac{1}{5}x^5\right]_0^2 = \boxed{\frac{64\pi}{15}}.$$

(c) $x = \sqrt{y}$ is the right boundary; $x = y/2$ is the left boundary on $0 \le y \le 4$. About the y-axis, these are the outer and inner radii:

$$V_y = \pi \int_0^4 \left((\sqrt{y})^2 - \left(\frac{y}{2}\right)^2\right)\mathrm{d}y = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \boxed{\frac{8\pi}{3}}.$$

(d) A washer is a large disc with a smaller disc removed: $\pi(R^2 - r^2)$. The expression $\pi(R - r)^2$ instead treats the radial thickness as the radius of a solid disc.

9. (a) On $0 \le y \le 2$, the horizontal width is $2y - y^2 \ge 0$:

$$A = \int_0^2 (2y - y^2)\,\mathrm{d}y = \left[y^2 - \frac{y^3}{3}\right]_0^2 = \boxed{\frac{4}{3}}.$$

(b) About the y-axis: radius $r = 2y - y^2$, variable y, limits $0, 2$:

$$V = \pi \int_0^2 (2y - y^2)^2\,\mathrm{d}y = \pi\left[\frac{4}{3}y^3 - y^4 + \frac{1}{5}y^5\right]_0^2 = \boxed{\frac{16\pi}{15}}.$$

(c) Square the radius at each height, then integrate the disc areas. Squaring the total area does not do this; it would also give units of length to the fourth power, rather than cubic units.

Answer key and worked solutions Mixed skills and first challenge

2 Mixed skills: Questions 10–11

10. (a) First take $u = x^2$, $v = \frac{1}{2}e^{2x}$:

$$\int x^2 e^{2x}\,\mathrm{d}x = \frac{x^2}{2}e^{2x} - \int xe^{2x}\,\mathrm{d}x.$$

A second application gives

$$\int xe^{2x}\,\mathrm{d}x = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$$

Hence $\int x^2 e^{2x}\,\mathrm{d}x = \boxed{e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) + C}$.

(b) About the x-axis: radius xe^x, variable x, limits $0, 1$:

$$V = \pi\int_0^1 (xe^x)^2\,\mathrm{d}x = \pi\left[e^{2x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right)\right]_0^1 = \boxed{\frac{\pi}{4}(e^2 - 1)}.$$

11. (a) About the x-axis: radius e^{-x}, variable x, limits $0, a$:

$$V = \pi\int_0^a e^{-2x}\,\mathrm{d}x = \pi\left[-\frac{1}{2}e^{-2x}\right]_0^a = \frac{\pi}{2}(1 - e^{-2a}).$$

(b) $1 - e^{-2a} = 3/4 \Rightarrow e^{-2a} = 1/4$. Thus $-2a = -\ln 4$ and $\boxed{a = \ln 2}$.
(c) For every finite a, $e^{-2a} > 0$, so $V < \pi/2$. The volume approaches $\pi/2$ as a increases without bound; equality is never attained for finite a.

3 First challenge: Question 12

12. (a) $u = 1 + x^2$, $\mathrm{d}u = 2x\,\mathrm{d}x$. The new limits are 1 and e:

$$I = \frac{1}{2}\int_1^e \ln u\,\mathrm{d}u.$$

(b) Parts gives $\int \ln u\,\mathrm{d}u = u\ln u - u + C$. Therefore

$$I = \frac{1}{2}[u\ln u - u]_1^e = \frac{1}{2}(0 - (-1)) = \boxed{\frac{1}{2}}.$$

(c) About the x-axis, the radius is $\sqrt{x\ln(1+x^2)}$; the limits in x are $0, \sqrt{e-1}$. Thus $V = \pi I = \boxed{\pi/2}$ cubic units. Since $x \geq 0$ and $\ln(1+x^2) \geq 0$, the radicand is non-negative throughout.

Answer key and worked solutions Further challenges

3 Further challenges: Questions 13–14

13. (a) Differentiate the trigonometric factor and integrate e^x in both integrals:

$$I = [e^x \cos x]_0^{\pi/2} + \int_0^{\pi/2} e^x \sin x \, \mathrm{d}x = -1 + J,$$

$$J = [e^x \sin x]_0^{\pi/2} - \int_0^{\pi/2} e^x \cos x \, \mathrm{d}x = e^{\pi/2} - I.$$

(b) Substitute the second equation into the first:

$$I = -1 + e^{\pi/2} - I \quad \Rightarrow \quad \boxed{I = \frac{e^{\pi/2} - 1}{2}}.$$

(c) About the x-axis: radius $\sqrt{e^x \cos x}$, variable x, limits $0, \pi/2$. The radicand is non-negative on this interval.

$$V = \pi \int_0^{\pi/2} e^x \cos x \, \mathrm{d}x = \boxed{\frac{\pi}{2}(e^{\pi/2} - 1)}.$$

14. (a) $y^2 = ay \Rightarrow y = 0, a$, so the intersection points are $(0,0)$ and (a^2, a). For $0 < y < a$, the right boundary is $x = ay$ and the left boundary is $x = y^2$:

$$A = \int_0^a (ay - y^2) \, \mathrm{d}y = \left[\frac{ay^2}{2} - \frac{y^3}{3}\right]_0^a = \boxed{\frac{a^3}{6}}.$$

(b) About the y-axis: outer radius ay, inner radius y^2, variable y, limits $0, a$:

$$V_y = \pi \int_0^a (a^2y^2 - y^4) \, \mathrm{d}y = \pi \left[\frac{a^2y^3}{3} - \frac{y^5}{5}\right]_0^a = \boxed{\frac{2\pi a^5}{15}}.$$

(c) On $0 \le x \le a^2$, the upper curve is $y = \sqrt{x}$ and the lower is $y = x/a$. About the x-axis, use outer radius $\sqrt{x}$, inner radius x/a:

$$V_x = \pi \int_0^{a^2} \left(x - \frac{x^2}{a^2}\right) \mathrm{d}x = \pi \left[\frac{x^2}{2} - \frac{x^3}{3a^2}\right]_0^{a^2} = \boxed{\frac{\pi a^4}{6}}.$$

(d) Since $a > 0$, divide $\pi a^4/6 = 2\pi a^5/15$ by πa^4:

$$\frac{1}{6} = \frac{2a}{15} \quad \Rightarrow \quad \boxed{a = \frac{5}{4}}.$$

Review prompts. What told you to use substitution or parts? Where did you change limits? Which boundary gave the outer radius? Why are areas and volumes non-negative in your final answers?