Integration by substitution and integration by parts

IB Mathematics: Analysis and Approaches HL

Fundamentals · guided practice · detailed solutions

Learning goal	Rewrite an integral correctly and apply substitution or integration by parts.
Syllabus link	SL 5.10–5.11; AHL 5.16. Focused review of the foundations.
Structure	3 sections; 12 core questions and 7 optional follow-up questions.
Technology	No calculator needed. Use radians for trigonometric functions.

Instructions. Work on separate paper. Show your substitutions and all key steps. Give exact answers. Include $+C$ for indefinite integrals. Study each worked example before the practice. Detailed solutions begin on page 8.

Suggested 90-minute route

5 min	Recall: Questions 1–2.
25 min	Straightforward substitution: Example A and Questions 3–6.
25 min	HL substitution: Example B and Questions 7 and 9.
30 min	Integration by parts: Example C and Questions 11–13.
5 min	Check and reflect: Question 17.

Questions 8, 10, 14–16 and 18–19 provide extra practice. Time estimates include reading the examples.

1 Straightforward u-substitution — Start with what we already know

1. Recall the basic integrals.

Find each indefinite integral.

(a) $\displaystyle\int x^3\,\mathrm{d}x$ (b) $\displaystyle\int \frac{1}{x}\,\mathrm{d}x$ (c) $\displaystyle\int \mathrm{e}^x\,\mathrm{d}x$

(d) $\displaystyle\int \sin x\,\mathrm{d}x$ (e) $\displaystyle\int \cos x\,\mathrm{d}x$

2. Prepare the substitution.

For each expression, find $\dfrac{\mathrm{d}u}{\mathrm{d}x}$, then write $\mathrm{d}u$ in terms of x and $\mathrm{d}x$.

(a) $u = x^2 + 1$ (b) $u = 3x - 2$ (c) $u = \sin x$

The substitution routine

Choose u, usually the expression inside a power, root, exponential or denominator. Find $\mathrm{d}u$, then rewrite the integral entirely in u. Integrate and return to x.

Recall: $\displaystyle\int u^n\,\mathrm{d}u = \frac{u^{n+1}}{n+1} + C$ for $n \neq -1$; $\displaystyle\int \frac{1}{u}\,\mathrm{d}u = \ln|u| + C$.

1 Straightforward u-substitution — From an inner function to a simpler integral

Worked example A | Adjust the constant factor

Find $\int 4x(x^2+3)^2\,\mathrm{d}x$.

Solution

Let $u = x^2+3$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$, so $4x\,\mathrm{d}x = 2\,\mathrm{d}u$.

$$\begin{aligned}\int 4x(x^2+3)^2\,\mathrm{d}x &= 2\int u^2\,\mathrm{d}u \\ &= \frac{2u^3}{3} + C \\ &= \boxed{\frac{2}{3}(x^2+3)^3 + C}.\end{aligned}$$

Check: differentiating the answer gives $\frac{2}{3}\cdot 3(x^2+3)^2\cdot 2x = 4x(x^2+3)^2$.

3. Guided practice: account for the factor.

Find $\int 6x(x^2+1)^4\,\mathrm{d}x$, using $u = x^2+1$.

(a) Find $\mathrm{d}u$, then express $6x\,\mathrm{d}x$ in terms of $\mathrm{d}u$.

(b) Rewrite the integral entirely in terms of u.

(c) Integrate, then give your answer in terms of x.

4. Choose the inner function.

Find each integral. In each case, state u and $\mathrm{d}u$ first.

(a) $\int xe^{x^2}\,\mathrm{d}x$

(b) $\int \cos(3x-2)\,\mathrm{d}x$

5. Recognize the logarithm.

Find $\int \dfrac{2x}{x^2+5}\,\mathrm{d}x$.

Hint: choose the denominator as u. Which basic integral will you obtain?

6. Change the limits as well.

Use $u = x^2+1$ to evaluate $\int_0^1 2x(x^2+1)^2\,\mathrm{d}x$.

(a) Find the new limits when $x=0$ and $x=1$.

(b) Write and evaluate the definite integral in u.

An integral written in u needs limits expressed as values of u.

2 HL substitution — Rewrite every remaining factor

The next step

A substitution may leave another x-term to replace. Rearrange the given substitution so that **every factor**, including $\mathrm{d}x$, is written using the new variable. The more involved substitutions in this section are provided.

Worked example B | Replace the remaining x

Find $\int x\sqrt{x+1}\,\mathrm{d}x$, for $x > -1$, using $u = x + 1$.

Solution

Rearrange and differentiate the substitution:

$$u = x + 1 \quad \Rightarrow \quad x = u - 1, \qquad \mathrm{d}x = \mathrm{d}u.$$

The root becomes $\sqrt{u}$, but we must also replace the x in front.

$$\begin{aligned}\int x\sqrt{x+1}\,\mathrm{d}x &= \int (u-1)u^{1/2}\,\mathrm{d}u \\ &= \int \left(u^{3/2} - u^{1/2}\right)\,\mathrm{d}u \\ &= \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C \\ &= \boxed{\frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C}.\end{aligned}$$

7. Guided HL substitution.

Use $u = x + 2$ to find $\int x\sqrt{x+2}\,\mathrm{d}x$, for $x > -2$.

(a) Express x in terms of u, and write $\mathrm{d}x$ in terms of $\mathrm{d}u$.

(b) Show that the integral becomes $\int (u-2)u^{1/2}\,\mathrm{d}u$.

(c) Expand the integrand, integrate, and give your answer in terms of x.

8. Rewrite a leftover power. Optional follow-up

Use $u = x^2 + 1$ to find $\int \dfrac{x^3}{x^2+1}\,\mathrm{d}x$.

(a) Write $x^3\,\mathrm{d}x = x^2(x\,\mathrm{d}x)$. Express x^2 and $x\,\mathrm{d}x$ in terms of u and $\mathrm{d}u$.

(b) Show that the integral becomes $\dfrac{1}{2}\int \left(1 - \dfrac{1}{u}\right)\,\mathrm{d}u$.

(c) Integrate and return to x.

Checkpoint: have all the variables changed?

After making a substitution, the new integral should contain only the new variable. For example, $\int x\sqrt{u}\,\mathrm{d}u$ is not ready to integrate: the remaining x still needs to be replaced.

2 HL substitution — A guided trigonometric substitution

Why a trigonometric substitution can help

For a root of the form $\sqrt{a^2 - x^2}$, with $a > 0$, the substitution $x = a\sin u$ lets us use

$$1 - \sin^2 u = \cos^2 u.$$

Choose $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, so $\cos u > 0$. On this interval,

$$\sqrt{\cos^2 u} = |\cos u| = \cos u.$$

Here x is given in terms of u, so differentiate $x = a\sin u$ to find $\mathrm{d}x$.

9. One step at a time.

Use $x = 2\sin u$, where $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, to find

$$\int \frac{1}{\sqrt{4 - x^2}}\,\mathrm{d}x, \qquad -2 < x < 2.$$

(a) Differentiate $x = 2\sin u$, then write $\mathrm{d}x$ in terms of u and $\mathrm{d}u$.

(b) Show that

$$\sqrt{4 - x^2} = \sqrt{4 - 4\sin^2 u} = 2\cos u.$$

(c) Substitute both results into the integral. Show that it becomes $\displaystyle\int 1\,\mathrm{d}u$.

(d) Integrate, then express your answer in terms of x.

Hint for (d): $x = 2\sin u$ gives $u = \arcsin(x/2)$. Here arcsin means the inverse sine function.

10. Use the same idea with limits. Optional follow-up

Use $x = 3\sin u$, with $-\dfrac{\pi}{2} < u < \dfrac{\pi}{2}$, to evaluate

$$\int_0^{3/2} \frac{1}{\sqrt{9 - x^2}}\,\mathrm{d}x.$$

(a) Find $\mathrm{d}x$ and simplify $\sqrt{9 - x^2}$.

(b) Find the new limits, using $\sin(\pi/6) = 1/2$.

(c) Write the integral entirely in u, then evaluate it exactly.

Before moving on

Can you explain why the numerator and denominator cancel in Question 9? The cancellation works because $\mathrm{d}x$ **changes too**.

3 Integration by parts — Choose, differentiate, integrate

The formula and the choice

$$\boxed{\int uv' \,\mathrm{d}x = uv - \int u'v \,\mathrm{d}x}$$

Choose u so that differentiating it makes the remaining integral easier. Choose v' so that you can integrate it to find v. Always record all four entries:

$$u = \cdots, \qquad v' = \cdots, \qquad u' = \cdots, \qquad v = \cdots.$$

Here u and v' are the two factors of the integrand. Primes mean derivatives with respect to x. The formula reverses the product rule.

Worked example C | Start with xe^x

Find $\displaystyle\int xe^x \,\mathrm{d}x$.

Solution

Choose $u = x$ and $v' = e^x$. Then $u' = 1$ and $v = e^x$.

$$\begin{aligned}\int xe^x \,\mathrm{d}x &= xe^x - \int e^x \,\mathrm{d}x \\ &= \boxed{xe^x - e^x + C}.\end{aligned}$$

Differentiating x leaves 1, so the remaining integral is simpler. When finding v, use one antiderivative; put $+C$ in the final answer.

11. Guided practice: an algebraic factor and a trigonometric factor.

Find $\displaystyle\int x\cos x \,\mathrm{d}x$.

(a) Take $u = x$ and $v' = \cos x$. Find u' and v.
(b) Substitute into the integration-by-parts formula.
(c) Evaluate the remaining integral and simplify.

Check the sign carefully: $\displaystyle\int \sin x \,\mathrm{d}x = -\cos x + C$.

12. Remember the constant factor.

Find $\displaystyle\int xe^{2x} \,\mathrm{d}x$. Choose u and v' yourself, and state all four entries before integrating.

Hint: the antiderivative of e^{2x} is $\frac{1}{2}e^{2x}$.

13. Make the hidden product visible.

Find $\displaystyle\int \ln x \,\mathrm{d}x$, for $x > 0$.

(a) Write $\ln x = (\ln x)\cdot 1$. Take $u = \ln x$ and $v' = 1$. Find u' and v.
(b) Apply integration by parts. Simplify $u'v$ before integrating.

3 Integration by parts — Extra practice and a final check

For the 90-minute route, go straight to Question 17. Questions 14–16 can be completed afterwards.

14. Evaluate a definite integral. — Optional follow-up

Use integration by parts to evaluate $\int_0^1 xe^x \, dx$.

You may use the antiderivative from Example C. Show the upper-limit value minus the lower-limit value.

15. Choose the logarithm as u. — Optional follow-up

Find $\int x \ln x \, dx$, for $x > 0$.

(a) Take $u = \ln x$ and $v' = x$. Find u' and v.

(b) Use integration by parts and simplify your answer.

16. Apply integration by parts twice. — Optional follow-up

Find $\int x^2 e^x \, dx$.

(a) Take $u = x^2$ and $v' = e^x$. Show that

$$\int x^2 e^x \, dx = x^2 e^x - 2\int xe^x \, dx.$$

(b) Apply integration by parts again to the remaining integral. You may use Example C to check this step.

17. Choose the method, then check your work.

(a) For each integral, choose either substitution or integration by parts. State your substitution, or your choices of u and v'. Do not evaluate the integrals.

(i) $\int 2xe^{x^2} \, dx$ (ii) $\int xe^x \, dx$.

(b) Differentiate your answer to Question 12. Does it give xe^{2x}? Show the cancellation clearly.

(c) Write down one mistake you need to watch for, and one check that would help you catch it.

Quick method check
Substitution: look for an inner expression and its derivative; for a given HL substitution, replace every factor.
By parts: look for a product where differentiating one factor makes the remaining integral easier. Remember that $\ln x = (\ln x) \cdot 1$.

3 Integration by parts — Cyclic integration by parts

When the original integral comes back

Use integration by parts twice. If the original integral reappears, give it a name such as I, collect the I-terms on one side, and solve the resulting equation.

$$\boxed{\int uv' \,\mathrm{d}x = uv - \int u'v \,\mathrm{d}x}$$

For these questions, differentiating the trigonometric factor twice returns to the original trigonometric function with a minus sign.

18. A first cyclic integral. Optional follow-up

Let

$$I = \int \mathrm{e}^x \sin x \,\mathrm{d}x.$$

(a) Take $u = \sin x$ and $v' = \mathrm{e}^x$. Find u' and v, then show that

$$I = \mathrm{e}^x \sin x - J, \qquad J = \int \mathrm{e}^x \cos x \,\mathrm{d}x.$$

(b) Apply integration by parts to J, taking $u = \cos x$ and $v' = \mathrm{e}^x$. Show that $J = \mathrm{e}^x \cos x + I$.

(c) Substitute your result from (b) into (a). Expand the bracket and collect the I-terms on one side.

(d) Hence find $\displaystyle\int \mathrm{e}^x \sin x \,\mathrm{d}x$.

Hint for (c): when I appears on both sides, treat it as an unknown in an equation.

19. A cyclic integral with constant factors. Optional follow-up

Let

$$K = \int \mathrm{e}^{2x} \cos x \,\mathrm{d}x.$$

(a) Take $u = \cos x$ and $v' = \mathrm{e}^{2x}$. Apply integration by parts and express K in terms of $L = \displaystyle\int \mathrm{e}^{2x} \sin x \,\mathrm{d}x$.

(b) Apply integration by parts to L. Substitute back into your first result to show that

$$K = \frac{1}{2}\mathrm{e}^{2x} \cos x + \frac{1}{4}\mathrm{e}^{2x} \sin x - \frac{1}{4}K.$$

(c) Rearrange this equation to find K.

(d) Check your answer by differentiation.

Hint: each time you integrate e^{2x}, remember the factor $\frac{1}{2}$.

Constants. For clarity, omit constants during the repeated steps and combine them into one final $+C$.

Answer key and detailed solutions

Use these after attempting the questions. Every answer keeps the original numbering. Equivalent forms of antiderivatives may differ by a constant.

Section 1 | Recall and straightforward substitution: Questions 1–3

1. Basic antiderivatives

$$\text{(a)} \quad \int x^3\,\mathrm{d}x = \boxed{\frac{x^4}{4} + C}.$$

$$\text{(b)} \quad \int \frac{1}{x}\,\mathrm{d}x = \boxed{\ln|x| + C}, \qquad x \neq 0.$$

$$\text{(c)} \quad \int \mathrm{e}^x\,\mathrm{d}x = \boxed{\mathrm{e}^x + C}.$$

$$\text{(d)} \quad \int \sin x\,\mathrm{d}x = \boxed{-\cos x + C}.$$

$$\text{(e)} \quad \int \cos x\,\mathrm{d}x = \boxed{\sin x + C}.$$

For (d), the minus sign is needed because $\dfrac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$.

2. Differentials

$$\text{(a)} \quad u = x^2 + 1 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 2x, \qquad \mathrm{d}u = 2x\,\mathrm{d}x.$$

$$\text{(b)} \quad u = 3x - 2 \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = 3, \qquad \mathrm{d}u = 3\,\mathrm{d}x.$$

$$\text{(c)} \quad u = \sin x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x, \qquad \mathrm{d}u = \cos x\,\mathrm{d}x.$$

The differential $\mathrm{d}u$ includes $\mathrm{d}x$; it is not just the derivative.

3. Adjust the constant factor

(a) $u = x^2 + 1$, so $\mathrm{d}u = 2x\,\mathrm{d}x$ and $6x\,\mathrm{d}x = 3\,\mathrm{d}u$.

(b) Replace the bracket by u and $6x\,\mathrm{d}x$ by $3\,\mathrm{d}u$:

$$\int 6x(x^2+1)^4\,\mathrm{d}x = 3\int u^4\,\mathrm{d}u.$$

(c) Integrate and substitute back:

$$\begin{aligned} 3\int u^4\,\mathrm{d}u &= \frac{3}{5}u^5 + C \\ &= \boxed{\frac{3}{5}(x^2+1)^5 + C}. \end{aligned}$$

Check: the derivative is $\frac{3}{5}\cdot 5(x^2+1)^4 \cdot 2x = 6x(x^2+1)^4$.

Common mistake: forgetting the scale factor

The substitution in Question 3 gives $6x\,\mathrm{d}x = 3\,\mathrm{d}u$, not $\mathrm{d}u$. The outside coefficient must remain when you integrate.

Section 1 | Straightforward substitution: Questions 4–6

4(a). Exponential of an inner function
Choose $u = x^2$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$, so $x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u$.

$$\begin{aligned}\int xe^{x^2}\,\mathrm{d}x &= \frac{1}{2}\int e^u\,\mathrm{d}u \\ &= \frac{1}{2}e^u + C \\ &= \boxed{\frac{1}{2}e^{x^2} + C}.\end{aligned}$$

4(b). A linear inner function
Choose $u = 3x - 2$. Then $\mathrm{d}u = 3\,\mathrm{d}x$, so $\mathrm{d}x = \frac{1}{3}\,\mathrm{d}u$.

$$\begin{aligned}\int \cos(3x-2)\,\mathrm{d}x &= \frac{1}{3}\int \cos u\,\mathrm{d}u \\ &= \frac{1}{3}\sin u + C \\ &= \boxed{\frac{1}{3}\sin(3x-2) + C}.\end{aligned}$$

5. The logarithm
Choose $u = x^2 + 5$. Then $\mathrm{d}u = 2x\,\mathrm{d}x$.

$$\begin{aligned}\int \frac{2x}{x^2+5}\,\mathrm{d}x &= \int \frac{1}{u}\,\mathrm{d}u \\ &= \ln|u| + C \\ &= \boxed{\ln(x^2+5) + C}.\end{aligned}$$

Since $x^2 + 5 > 0$ for every real x, the final answer does not need absolute-value signs. The power rule cannot be used with exponent -1.

6. A definite integral
Let $u = x^2 + 1$, so $\mathrm{d}u = 2x\,\mathrm{d}x$.
(a) When $x = 0$, $u = 1$. When $x = 1$, $u = 2$.
(b) Use the new limits:

$$\begin{aligned}\int_0^1 2x(x^2+1)^2\,\mathrm{d}x &= \int_1^2 u^2\,\mathrm{d}u \\ &= \left[\frac{u^3}{3}\right]_1^2 \\ &= \frac{8}{3} - \frac{1}{3} \\ &= \boxed{\frac{7}{3}}.\end{aligned}$$

There is no $+C$ in the final value of a definite integral.

Section 2 | HL algebraic substitutions: Questions 7–8

7. Replace the remaining x

(a) With $u = x + 2$,

$$x = u - 2, \qquad \mathrm{d}u = \mathrm{d}x.$$

(b) Replace x, the root and the differential:

$$\int x\sqrt{x+2}\,\mathrm{d}x = \int (u-2)u^{1/2}\,\mathrm{d}u.$$

(c) Expand before using the power rule:

$$\begin{aligned}\int (u-2)u^{1/2}\,\mathrm{d}u &= \int \left(u^{3/2} - 2u^{1/2}\right)\,\mathrm{d}u \\ &= \frac{2}{5}u^{5/2} - \frac{4}{3}u^{3/2} + C \\ &= \boxed{\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2} + C}.\end{aligned}$$

Check:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{5}(x+2)^{5/2} - \frac{4}{3}(x+2)^{3/2}\right) &= (x+2)^{3/2} - 2(x+2)^{1/2} \\ &= \sqrt{x+2}\big((x+2) - 2\big) \\ &= x\sqrt{x+2}.\end{aligned}$$

8. Replace the leftover power (optional)

(a) $u = x^2 + 1$ gives

$$x^2 = u - 1, \qquad \mathrm{d}u = 2x\,\mathrm{d}x, \qquad x\,\mathrm{d}x = \frac{1}{2}\,\mathrm{d}u.$$

(b) Separate $x^3\,\mathrm{d}x$ as $x^2(x\,\mathrm{d}x)$:

$$\begin{aligned}\int \frac{x^3}{x^2+1}\,\mathrm{d}x &= \frac{1}{2}\int \frac{u-1}{u}\,\mathrm{d}u \\ &= \frac{1}{2}\int \left(1 - \frac{1}{u}\right)\,\mathrm{d}u.\end{aligned}$$

(c) Integrate and return to x:

$$\begin{aligned}\frac{1}{2}\int \left(1 - \frac{1}{u}\right)\,\mathrm{d}u &= \frac{1}{2}u - \frac{1}{2}\ln|u| + C \\ &= \frac{1}{2}(x^2+1) - \frac{1}{2}\ln(x^2+1) + C.\end{aligned}$$

Since a constant can be absorbed into C, an equivalent answer is

$$\boxed{\frac{x^2}{2} - \frac{1}{2}\ln(x^2+1) + C}.$$

Section 2 | Trigonometric substitutions: Questions 9–10

9. A guided trigonometric substitution

(a) $x = 2\sin u$ gives $\mathrm{d}x = 2\cos u\,\mathrm{d}u$.

(b) Use the identity and the given range:

$$\begin{aligned}\sqrt{4-x^2} &= \sqrt{4-4\sin^2 u}\\ &= \sqrt{4\cos^2 u}\\ &= 2|\cos u|\\ &= 2\cos u.\end{aligned}$$

The final equality holds because $\cos u > 0$ for $-\frac{\pi}{2} < u < \frac{\pi}{2}$.

(c) Substitute the root and $\mathrm{d}x$:

$$\int \frac{1}{\sqrt{4-x^2}}\,\mathrm{d}x = \int \frac{2\cos u}{2\cos u}\,\mathrm{d}u = \int 1\,\mathrm{d}u.$$

(d) Integrate to obtain $u + C$. Since $\sin u = x/2$,

$$\boxed{\arcsin\left(\frac{x}{2}\right) + C}, \qquad -2 < x < 2.$$

For $-2 < x < 2$, $\arcsin(x/2)$ lies in the chosen interval.

10. The same substitution with limits (optional)

(a) $x = 3\sin u$ gives

$$\mathrm{d}x = 3\cos u\,\mathrm{d}u, \qquad \sqrt{9-x^2} = \sqrt{9\cos^2 u} = 3\cos u.$$

(b) Convert the limits:

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad \sin u = 0 \quad \Rightarrow \quad u = 0,\\ x = \frac{3}{2} \quad &\Rightarrow \quad \sin u = \frac{1}{2} \quad \Rightarrow \quad u = \frac{\pi}{6}.\end{aligned}$$

(c) The factors cancel:

$$\begin{aligned}\int_0^{3/2} \frac{1}{\sqrt{9-x^2}}\,\mathrm{d}x &= \int_0^{\pi/6} \frac{3\cos u}{3\cos u}\,\mathrm{d}u\\ &= \int_0^{\pi/6} 1\,\mathrm{d}u\\ &= [u]_0^{\pi/6}\\ &= \boxed{\frac{\pi}{6}}.\end{aligned}$$

Section 3 | Integration by parts: Questions 11–13

11. An algebraic factor and a trigonometric factor

(a) The four entries are

$$u = x, \qquad v' = \cos x, \qquad u' = 1, \qquad v = \sin x.$$

(b)–(c) Apply the formula:

$$\begin{aligned}\int x\cos x\,\mathrm{d}x &= x\sin x - \int \sin x\,\mathrm{d}x \\ &= x\sin x - (-\cos x) + C \\ &= \boxed{x\sin x + \cos x + C}.\end{aligned}$$

The two minus signs give a plus sign. Differentiation checks the answer: $\sin x + x\cos x - \sin x = x\cos x$.

12. An exponential with a constant factor

Choose

$$u = x, \qquad v' = \mathrm{e}^{2x}, \qquad u' = 1, \qquad v = \frac{1}{2}\mathrm{e}^{2x}.$$

Then

$$\begin{aligned}\int x\mathrm{e}^{2x}\,\mathrm{d}x &= \frac{x}{2}\mathrm{e}^{2x} - \int \frac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \frac{x}{2}\mathrm{e}^{2x} - \frac{1}{4}\mathrm{e}^{2x} + C \\ &= \boxed{\frac{\mathrm{e}^{2x}}{4}(2x-1) + C}.\end{aligned}$$

The factor $\frac{1}{2}$ appears each time e^{2x} is integrated.

13. The hidden product in $\ln x$

(a) Write $\ln x = (\ln x)\cdot 1$, then choose

$$u = \ln x, \qquad v' = 1, \qquad u' = \frac{1}{x}, \qquad v = x.$$

(b) Apply the formula:

$$\begin{aligned}\int \ln x\,\mathrm{d}x &= x\ln x - \int \frac{1}{x}\cdot x\,\mathrm{d}x \\ &= x\ln x - \int 1\,\mathrm{d}x \\ &= \boxed{x\ln x - x + C}, \qquad x > 0.\end{aligned}$$

The derivative is $\ln x + 1 - 1 = \ln x$.

Section 3 | Optional follow-up: Questions 14–16

14. A definite integral (optional)
As in Example C, take $u = x$, $v' = \mathrm{e}^x$, $u' = 1$, $v = \mathrm{e}^x$. An antiderivative is $x\mathrm{e}^x - \mathrm{e}^x$. Hence

$$\begin{aligned}\int_0^1 x\mathrm{e}^x \,\mathrm{d}x &= [x\mathrm{e}^x - \mathrm{e}^x]_0^1 \\ &= (\mathrm{e} - \mathrm{e}) - (0 - 1) \\ &= 0 - (-1) \\ &= \boxed{1}.\end{aligned}$$

15. A logarithm as one factor (optional)
(a) Choose

$$u = \ln x, \qquad v' = x, \qquad u' = \frac{1}{x}, \qquad v = \frac{x^2}{2}.$$

(b) Then

$$\begin{aligned}\int x \ln x \,\mathrm{d}x &= \frac{x^2}{2}\ln x - \int \frac{1}{x}\cdot\frac{x^2}{2}\,\mathrm{d}x \\ &= \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,\mathrm{d}x \\ &= \boxed{\frac{x^2}{2}\ln x - \frac{x^2}{4} + C}, \qquad x > 0.\end{aligned}$$

16. Repeated integration by parts (optional)
(a) Choose

$$u = x^2, \qquad v' = \mathrm{e}^x, \qquad u' = 2x, \qquad v = \mathrm{e}^x.$$

This gives

$$\int x^2\mathrm{e}^x \,\mathrm{d}x = x^2\mathrm{e}^x - 2\int x\mathrm{e}^x \,\mathrm{d}x.$$

(b) For the remaining integral, use $u = x$, $v' = \mathrm{e}^x$, $u' = 1$, $v = \mathrm{e}^x$:

$$\int x\mathrm{e}^x \,\mathrm{d}x = x\mathrm{e}^x - \mathrm{e}^x + C.$$

Therefore

$$\begin{aligned}\int x^2\mathrm{e}^x \,\mathrm{d}x &= x^2\mathrm{e}^x - 2(x\mathrm{e}^x - \mathrm{e}^x) + C \\ &= x^2\mathrm{e}^x - 2x\mathrm{e}^x + 2\mathrm{e}^x + C \\ &= \boxed{\mathrm{e}^x(x^2 - 2x + 2) + C}.\end{aligned}$$

All constants are combined into one final C.

Section 3 | Check and reflect: Question 17

17(a). Choose the method

(i) For $\int 2xe^{x^2}\,dx$, use **substitution**:

$$u = x^2, \qquad du = 2x\,dx.$$

The derivative of the inner function x^2 is present as the factor $2x$.

(ii) For $\int xe^{x}\,dx$, use **integration by parts**:

$$u = x, \qquad v' = e^x.$$

Differentiating x leaves 1, while e^x is easy to integrate.

17(b). Differentiate the answer to Question 12

Use the expanded form

$$F(x) = \frac{x}{2}e^{2x} - \frac{1}{4}e^{2x} + C.$$

Differentiate the product in the first term:

$$\begin{aligned} F'(x) &= \frac{1}{2}e^{2x} + \frac{x}{2}\cdot 2e^{2x} - \frac{1}{4}\cdot 2e^{2x} \\ &= \frac{1}{2}e^{2x} + xe^{2x} - \frac{1}{2}e^{2x} \\ &= \boxed{xe^{2x}}. \end{aligned}$$

The two terms $\frac{1}{2}e^{2x}$ cancel.

17(c). Possible reflections

Your own answer will depend on your work. For example:

- *I may lose a constant factor. I will write the differential explicitly and differentiate my final answer.*
- *I may leave an x in an integral written in u. I will check that every factor has been replaced.*
- *I may forget to change the limits. I will write a small x-to-u table before evaluating a definite integral.*
- *I may lose the minus sign in integration by parts. I will keep brackets around the remaining antiderivative.*

What to review next

If constants or differentials caused difficulty, repeat Questions 3–6. If rewriting the integrand was difficult, revisit Example B and Question 7. If the by-parts setup was difficult, revisit Example C before Questions 11–13.

Section 3 | Cyclic integration by parts: Question 18

18. A first cyclic integral (optional)
For clarity, constants are omitted during the repeated steps and combined into one final $+C$.

(a) Apply integration by parts once.
Choose

$$u = \sin x, \qquad v' = \mathrm{e}^x, \qquad u' = \cos x, \qquad v = \mathrm{e}^x.$$

Then

$$I = \int \mathrm{e}^x \sin x \,\mathrm{d}x = \mathrm{e}^x \sin x - \int \cos x \,\mathrm{e}^x \,\mathrm{d}x \qquad J = \int \mathrm{e}^x \cos x \,\mathrm{d}x.$$
$$= \mathrm{e}^x \sin x - J,$$

(b) Apply integration by parts to J.
Choose

$$u = \cos x, \qquad v' = \mathrm{e}^x, \qquad u' = -\sin x, \qquad v = \mathrm{e}^x.$$

Thus

$$\begin{aligned} J &= \mathrm{e}^x \cos x - \int (-\sin x)\mathrm{e}^x \,\mathrm{d}x \\ &= \mathrm{e}^x \cos x + \int \mathrm{e}^x \sin x \,\mathrm{d}x \\ &= \mathrm{e}^x \cos x + I. \end{aligned}$$

(c) Substitute back and collect the I-terms.

$$\begin{aligned} I &= \mathrm{e}^x \sin x - (\mathrm{e}^x \cos x + I) \\ I &= \mathrm{e}^x \sin x - \mathrm{e}^x \cos x - I \\ 2I &= \mathrm{e}^x (\sin x - \cos x). \end{aligned}$$

The minus sign in front of the bracket changes **both** signs inside.

(d) Divide by 2 and include the constant.

$$\boxed{\int \mathrm{e}^x \sin x \,\mathrm{d}x = \frac{\mathrm{e}^x}{2}(\sin x - \cos x) + C}.$$

Check:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^x}{2}(\sin x - \cos x)\right] &= \frac{\mathrm{e}^x}{2}(\sin x - \cos x) + \frac{\mathrm{e}^x}{2}(\cos x + \sin x) \\ &= \mathrm{e}^x \sin x. \end{aligned}$$

Section 3 | Cyclic integration by parts: Question 19

19. A cyclic integral with constant factors (optional)
Again, combine all constants into one final $+C$.

(a) Apply integration by parts once.
Choose

$$u = \cos x, \qquad v' = \mathrm{e}^{2x}, \qquad u' = -\sin x, \qquad v = \frac{1}{2}\mathrm{e}^{2x}.$$

Then

$$\begin{aligned} K &= \frac{1}{2}\mathrm{e}^{2x}\cos x - \int(-\sin x)\frac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{2}L, \qquad L = \int \mathrm{e}^{2x}\sin x\,\mathrm{d}x. \end{aligned}$$

(b) Apply integration by parts to L, then substitute back.
Choose

$$u = \sin x, \qquad v' = \mathrm{e}^{2x}, \qquad u' = \cos x, \qquad v = \frac{1}{2}\mathrm{e}^{2x}.$$

Therefore

$$\begin{aligned} L &= \frac{1}{2}\mathrm{e}^{2x}\sin x - \int \cos x\,\frac{1}{2}\mathrm{e}^{2x}\,\mathrm{d}x \\ &= \frac{1}{2}\mathrm{e}^{2x}\sin x - \frac{1}{2}K. \end{aligned}$$

Substitute this result into $K = \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{2}L$:

$$\begin{aligned} K &= \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{2}\left(\frac{1}{2}\mathrm{e}^{2x}\sin x - \frac{1}{2}K\right) \\ &= \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{4}\mathrm{e}^{2x}\sin x - \frac{1}{4}K. \end{aligned}$$

(c) Collect the K-terms and solve.

$$\begin{aligned} K + \frac{1}{4}K &= \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{4}\mathrm{e}^{2x}\sin x \\ \frac{5}{4}K &= \frac{1}{2}\mathrm{e}^{2x}\cos x + \frac{1}{4}\mathrm{e}^{2x}\sin x \\ K &= \frac{\mathrm{e}^{2x}}{5}(2\cos x + \sin x) + C. \end{aligned}$$

Hence

$$\boxed{\int \mathrm{e}^{2x}\cos x\,\mathrm{d}x = \frac{\mathrm{e}^{2x}}{5}(2\cos x + \sin x) + C}.$$

(d) Check by differentiation.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\mathrm{e}^{2x}}{5}(2\cos x + \sin x)\right] &= \frac{\mathrm{e}^{2x}}{5}\left[2(2\cos x + \sin x) - 2\sin x + \cos x\right] \\ &= \mathrm{e}^{2x}\cos x. \end{aligned}$$

Syllabus reference. International Baccalaureate, *Mathematics: analysis and approaches guide*, supplied project edition: SL 5.10–5.11, printed p. 65 (PDF p. 71); AHL 5.16, printed p. 68 (PDF p. 74). Original practice questions. This worksheet reviews selected foundations; it is not a complete revision of AHL 5.16.