

Arithmetic Sequences and Series

IB Mathematics Analysis and Approaches SL and HL

60 questions | Terms, sums, sigma notation and applications

Learning goal	Recognise arithmetic structure, find terms and sums, and build and interpret suitable models.
Syllabus link	SL 1.2: arithmetic sequences and series, sigma notation, applications, simple interest and approximately arithmetic data. The HL challenges deepen shared content; no additional AHL methods are required.
Structure	15 short + 15 extended + 20 IB-style + 10 HL-exclusive questions.
Technology	Non-calculator unless marked  GDC (Questions 30, 40, 43 and 49). Give exact answers unless an approximation is requested.

Instructions. Show your method; identify the first term and common difference even when using technology. Unless stated otherwise, u_n is the n th term, $S_n = u_1 + \dots + u_n$, and $n \in \mathbb{Z}^+$. Distinguish an individual term from a cumulative total. Check neighbouring integers in threshold questions, state units, and give money to the nearest cent when needed. Work on separate paper.

Section 1: Short response

1. For $-7, -2, 3, \dots$, state the common difference and find u_{12} .

2. An arithmetic sequence has $u_1 = 19$ and $d = -3$. Find u_9 .

3. For $u_n = 5 - 2n$, state the first term and common difference, and find u_{15} .

4. The table gives an arithmetic sequence. Find p and u_9 .

n	1	2	3	4
u_n	3	7	p	15

5. Given $u_4 = 17$ and $u_{10} = 47$, find u_1 and d .

6. Find the index of the term 58 in $7, 10, 13, \dots$

7. Determine whether 200 occurs in the arithmetic sequence $5, 12, 19, \dots$

8. Insert three arithmetic means between -6 and 18 .

9. Find S_{15} for the sequence with $u_1 = 4$ and $d = 2$.

10. An arithmetic series has 24 terms, first term -8 and last term 84 . Find its sum.



11. Evaluate $\sum_{k=1}^{12} (3k - 5)$.

12. Evaluate $\sum_{k=4}^{14} (2k + 1)$.

13. An arithmetic sequence has $u_9 = 31$ and $d = 3$. Find u_1 .

14. The expressions x , $3x - 4$ and $2x + 1$ are consecutive terms of an arithmetic sequence in that order. Find x .

15. An arithmetic sequence has $u_1 = \sqrt{2}$ and $d = 3\sqrt{2}$. Find u_8 and S_8 .

Section 2: Extended response

16. An arithmetic sequence begins 9, 13, 17, ...

- (a) Find the index of the term 85.
- (b) Find the sum up to and including that term.

17. An arithmetic sequence has $u_5 = 28$ and $u_{12} = 77$.

- (a) Find d and u_1 .
- (b) Find the sum of its first twelve terms.

18. The sequence $u_n = -35 + 4(n - 1)$ is arithmetic.

- (a) Find its first positive term and its index.
- (b) Find the sum up to and including that term.

19. The sum of the first n terms of a sequence is $S_n = 3n^2 - 2n$, with $S_0 = 0$.

- (a) Use $u_n = S_n - S_{n-1}$ to show that $u_n = 6n - 5$.
- (b) State the first term and common difference, and find u_8 .

20. Find the sum of all integers from 14 to 33 inclusive. Write the sum in sigma notation in two ways, one beginning with index $k = 1$.

21. Find the sum of all multiples of 7 between 100 and 500.

22. Consider $\sum_{k=5}^{18} (7 - 2k)$.

- (a) State the first term, last term and number of terms.
- (b) Evaluate the sum.

23. The expressions $2p - 1$, $p + 5$ and $4p - 5$ are consecutive arithmetic terms.

- Find p and the common difference.
- Find the sum of the first 20 terms of the sequence beginning with these terms.

24. An arithmetic sequence begins 15, 13, 11, ...

- Find the positive integer n for which $S_n = 0$.
- Find the sum of all its positive terms.

25. An arithmetic sequence has $u_6 = 18$ and $S_{10} = 130$.

- Find its first term and common difference.
- Find S_{20} .

26. An investment of €5400 earns 3% simple interest per year, added at each year-end.

- Write a formula for the balance B_n after n complete years, with $B_0 = 5400$.
- Find B_6 and the first year-end at which the balance exceeds €7000.

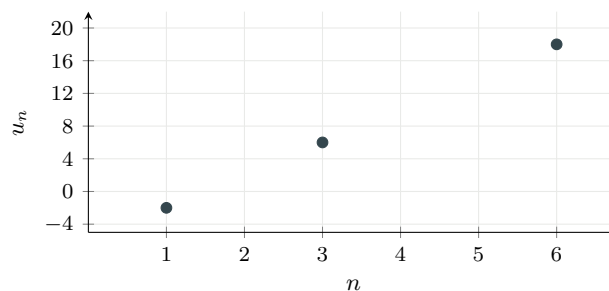
27. A venue records the following attendance at four monthly events.

Event n	1	2	3	4
Attendance	204	215	225	236

- Explain why an arithmetic sequence is only an approximate model.
- Use the first and last observations to estimate d , retaining its exact value. Predict attendance at event 10.
- State one limitation of this extrapolation.

28. The plotted points belong to an arithmetic sequence.

- Determine u_1 and d .
- Find u_{14} .
- Explain why a graph of the sequence contains separate points rather than every point on a line.



29. Two arithmetic sequences are $u_n = 8 + 5(n - 1)$ and $v_n = 32 + 2(n - 1)$.

- Find the least n for which $u_n > v_n$.
- Find the least n for which $\sum_{k=1}^n u_k > \sum_{k=1}^n v_k$.

- 30.**  *GDC* A project receives €380 in month 1 and €45 more each month thereafter.
- (a) Write its cumulative funding after n months in sigma notation and as a simplified quadratic expression.
- (b) Find the least n for which cumulative funding reaches €18 000. Check adjacent integers.

Section 3: Original IB exam-style questions

- 31.** An arithmetic sequence has second term 11 and seventh term 36.
- (a) Find d and u_1 . [3 marks]
- (b) Find u_{20} . [2 marks]
- (c) Find S_{20} . [2 marks]

- 32.** The first and sixth terms of an arithmetic sequence are 42 and 12 respectively.
- (a) Find its fourteenth term. [3 marks]
- (b) The sum of its first n terms is zero. Find the positive integer n . [3 marks]

- 33.** An arithmetic sequence satisfies $u_8 = 23$ and $u_5 + u_{13} = 56$.
- (a) Find the first term and common difference. [5 marks]
- (b) Find S_{12} . [2 marks]

- 34.** The first three terms of an arithmetic sequence are t , $4t - 5$ and $2t + 10$.
- (a) Find t . [3 marks]
- (b) Show that $S_n = \frac{n(7n+1)}{2}$. [3 marks]
- (c) Find n when $S_n = 355$. [3 marks]

- 35.** For an arithmetic sequence, $u_7 = 18$ and $S_{15} = 180$.
- (a) Find u_1 and d . [4 marks]
- (b) Find the index k for which $u_k = 0$. [2 marks]

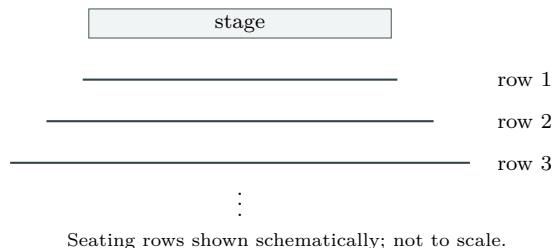
- 36.** The partial sums of an arithmetic sequence are $S_n = pn^2 + qn$, where p and q are constants. It is given that $S_4 = 60$ and $S_7 = 168$.
- (a) Find p and q . [4 marks]
- (b) Find u_n and hence u_8 . [3 marks]

- 37.** An arithmetic sequence has $u_n = 21 - 3n$.
- (a) Find the first negative term and its index. [3 marks]
- (b) Find the maximum value of S_n . State every positive integer n at which this maximum is attained. [4 marks]

- 38.** Consider the arithmetic terms $3k + 1$.
- (a) Evaluate $\sum_{k=2}^{15} (3k + 1)$. [3 marks]
- (b) Find the positive integer m such that $\sum_{k=1}^m (3k + 1) = 424$. [4 marks]

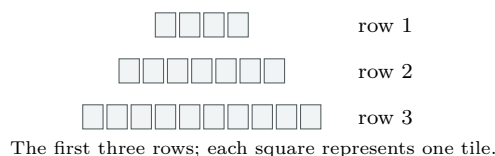
39. A lecture theatre has 16 rows. The first contains 20 seats and each subsequent row has 4 more seats.

- Find the number of seats in the final row. [2 marks]
- Find the seating capacity. [2 marks]
- All seats are sold at €15 each. Find the total revenue. [2 marks]



40.  *GDC* A mosaic has 4 tiles in the top row and 3 more tiles in every subsequent row.

- Show that an n -row mosaic requires $\frac{n(3n+5)}{2}$ tiles. [3 marks]
- Find the number of tiles needed for ten rows. [1 mark]
- A designer has 500 tiles. Find the greatest number of complete rows and the number of unused tiles. [4 marks]



41. An account starts with €8000. At the end of every month it receives interest equal to 0.25% of the original €8000. There are no other deposits or withdrawals.

- Find the interest paid each month. [2 marks]
- Write the balance B_n after n complete months and find the balance after one year. [3 marks]
- Find the least number of complete months after which the balance is strictly greater than €9500. [3 marks]

42. Job A pays €32 000 in year 1 with a fixed annual increase of €1800. Job B pays €35 000 in year 1 with a fixed annual increase of €1200.

- Find the first year in which job A pays a higher annual salary. [3 marks]
- Find the total paid by each job over the first ten years. [3 marks]
- Find the first year by whose end total earnings from job A exceed total earnings from job B. [4 marks]

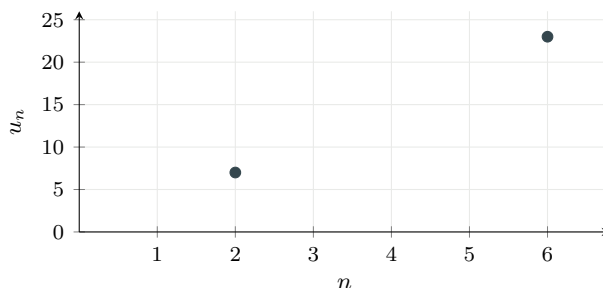
43.  *GDC* A sensor produces the following readings at equal time intervals.

Reading n	1	2	3	4	5
Value	152	161	171	180	191

- Explain why the readings do not form an exact arithmetic sequence. [1 mark]
- Use the first and last readings to estimate a common difference and predict reading 9. [3 marks]
- The actual ninth reading is 224. State the prediction error and comment on extending the model far beyond the observed readings. [2 marks]

44. The graph shows two terms of an arithmetic sequence with general term $u_n = an + b$.

- (a) Find a and b . [3 marks]
- (b) Show that $S_n = n(2n + 1)$. [3 marks]
- (c) Find n when $S_n = 210$. [3 marks]



45. A finite arithmetic series has first term 12, last term 104 and sum 1392.

- (a) Find its number of terms. [3 marks]
- (b) Find its common difference. [2 marks]
- (c) Write the series in sigma notation with lower limit $k = 1$. [2 marks]

46. For an arithmetic sequence, $u_3 = 12$ and $S_{15} - S_4 = 440$.

- (a) Explain why $S_{15} - S_4 = 11u_{10}$. [2 marks]
- (b) Find the first term and common difference. [4 marks]
- (c) Find S_{20} . [2 marks]

47. An arithmetic sequence is defined by $u_n = 2n + 1$. A new sequence consists of its odd-indexed terms $u_1, u_3, u_5, \dots$

- (a) Write the first three terms of the new sequence and state its common difference. [2 marks]
- (b) Find the sum of its first ten terms. [3 marks]
- (c) Find, in terms of m , $\sum_{k=1}^m u_{2k} - \sum_{k=1}^m u_{2k-1}$. [3 marks]

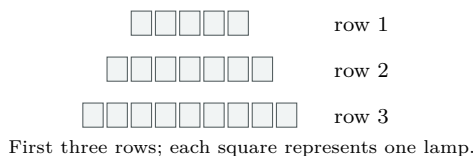
48. Some arithmetic means are inserted between 2 and 26. The sum of all the terms, including the two endpoints, is 140.

- (a) Find the number of inserted terms. [3 marks]
- (b) Find the common difference. [2 marks]
- (c) Find the fifth term of the complete sequence. [2 marks]

49.  *GDC* A library recorded 18 400 loans in 2026. A model predicts an increase of 425 loans each year. The year 2026 corresponds to $n = 1$.

- (a) Write the predicted annual loans u_n . [2 marks]
- (b) Find the predicted total over the twelve years from 2026 to 2037 inclusive. [3 marks]
- (c) Find the first calendar year in which the model predicts at least 23 000 loans. [3 marks]
- (d) State one reason why the prediction may not be reliable. [1 mark]

- 50.** A stepped display has 5 lamps in its top row and 2 more in each following row.
- (a) Show that the total number in n rows is $n(n + 4)$. [3 marks]
- (b) Find the number of lamps in a fifty-row display. [1 mark]
- (c) A contractor has 720 lamps. Find the maximum number of complete rows, justifying your answer. [3 marks]



Section 4: AA HL exclusive challenge

- 51.** The expressions $x^2 - 2x$, $x^2 + 3$ and $2x^2 + 2x - 3$ are the first three terms of an arithmetic sequence.
- (a) Find all possible values of x . [3 marks]
- (b) For each value, find u_{10} . [3 marks]
- (c) Determine which value of x gives a sequence in which every term is positive. [2 marks]

- 52.** An arithmetic sequence has first term a and non-zero common difference d . For a positive integer n , $S_{2n} = 3S_n$.
- (a) Prove that $2a = (n + 1)d$. [4 marks]
- (b) In the case $n = 5$ and $S_5 = 75$, find a and d . [3 marks]

- 53.** An arithmetic sequence has $S_n = 2n^2 + 5n$, including $S_0 = 0$.
- (a) Find u_n . [2 marks]
- (b) Find a simplified expression for $\sum_{k=1}^m u_{3k-1}$. [4 marks]
- (c) Evaluate this sum when $m = 8$. [1 mark]

- 54.** An arithmetic sequence has $u_5 = 15$ and $u_{17} = -9$.
- (a) Find its first term and common difference. [3 marks]
- (b) Find the positive integer N for which $S_N = 0$. [3 marks]
- (c) Find $\sum_{k=1}^N |u_k|$. [4 marks]

- 55.** In an arithmetic sequence, $u_p = q$ and $u_q = p$, where p and q are distinct positive integers.
- (a) Prove that the common difference is -1 . [3 marks]
- (b) Find u_n in terms of p , q and n . [2 marks]
- (c) Hence find the index of the zero term and the sum up to and including that term. [3 marks]

- 56.** An arithmetic sequence satisfies $S_{12} = 4S_6$ and $S_8 = 576$.
- (a) Find the first term and common difference. [5 marks]
- (b) Prove that S_n is a perfect square for every positive integer n . [3 marks]



57. An arithmetic sequence has common difference 3. For a positive integer n , $u_1 - u_2 + u_3 - u_4 + \cdots + u_{2n-1} - u_{2n} = -60$ and $S_{2n} = 2540$.

- (a) Find n . [3 marks]
(b) Find the first term. [3 marks]
(c) For an arbitrary arithmetic sequence with difference d , prove that the analogous alternating sum through term $2n$ is $-nd$. [2 marks]

58. For $u_k = 5k - 2$, define $T_n = u_{n+1} + u_{n+2} + \cdots + u_{3n}$.

- (a) Show that $T_n = n(20n + 1)$. [4 marks]
(b) Find the positive integer n for which $T_n = 726$. [3 marks]

59. Two arithmetic sequences are $u_n = 5n - 1$ and $v_n = 3n + 8$.

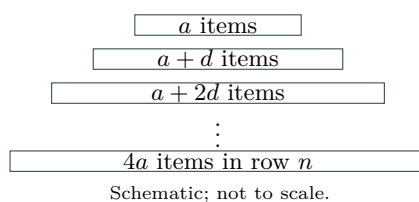
- (a) Find the positive integer N for which the sums of their first N terms are equal. [4 marks]
(b) Find the first three values that occur in both sequences. The indices need not be equal. [3 marks]
(c) Find the sum of all common values not exceeding 200. [4 marks]

n	1	2	3	4
u_n	4	9	14	19

n	1	2	3	4
v_n	11	14	17	20

60. A display contains n rows. The first row has a items, each following row has three more items, and the last row has $4a$ items. There are 455 items altogether.

- (a) Form two equations connecting a and n . [3 marks]
(b) Find a and n . [4 marks]
(c) Find the total number of items in the final seven rows. [3 marks]



Answer key and solution outlines

Equivalent forms and correct alternative methods are accepted. The IB-style questions are original practice items, not official IB examination questions.

Section 1: Short response

1. $d = -2 - (-7) = 5$; $u_{12} = -7 + 11(5) = 48$.
2. $u_n = u_1 + (n - 1)d$, so $u_9 = 19 + 8(-3) = -5$.
3. $u_1 = 3$, $d = -2$ and $u_{15} = 5 - 30 = -25$.
4. $d = 7 - 3 = 4$, so $p = 11$ and $u_9 = 3 + 8(4) = 35$.
5. $u_{10} - u_4 = 6d = 30$, so $d = 5$ and $u_1 = 17 - 3(5) = 2$.
6. $58 = 7 + 3(n - 1)$ gives $n - 1 = 17$, hence $n = 18$.
7. $200 = 5 + 7(n - 1)$ gives $n = 202/7$, not a positive integer. Therefore 200 is not a term.
8. There are four equal steps: $d = [18 - (-6)]/4 = 6$. The inserted terms are 0, 6, 12.
9. $S_n = \frac{n}{2}[2u_1 + (n - 1)d]$, so $S_{15} = \frac{15}{2}[8 + 14(2)] = 270$.
10. $S_{24} = \frac{24}{2}(-8 + 84) = 912$.
11. The first and last terms are -2 and 31 . There are 12 terms, so the sum is $6(-2 + 31) = 174$.
12. There are $14 - 4 + 1 = 11$ terms, from 9 to 29. The sum is $\frac{11}{2}(9 + 29) = 209$.
13. $u_9 = u_1 + 8d$, so $u_1 = 31 - 8(3) = 7$.
14. Equal successive differences give $(3x - 4) - x = (2x + 1) - (3x - 4)$. Thus $2x - 4 = 5 - x$, so $x = 3$.
15. $u_8 = \sqrt{2} + 7(3\sqrt{2}) = 22\sqrt{2}$; $S_8 = 4(\sqrt{2} + 22\sqrt{2}) = 92\sqrt{2}$.

Section 2: Extended response

16. (a) $85 = 9 + 4(n - 1)$ gives $n = 20$. (b) $S_{20} = 10(9 + 85) = 940$.
17. (a) $7d = 77 - 28 = 49$, so $d = 7$ and $u_1 = 28 - 4(7) = 0$. (b) $S_{12} = 6(0 + 77) = 462$.
18. (a) $-35 + 4(n - 1) > 0$ gives $n > 9.75$, so $u_{10} = 1$ is first positive. (b) $S_{10} = 5(-35 + 1) = -170$.
19. (a) $u_n = (3n^2 - 2n) - [3(n - 1)^2 - 2(n - 1)] = 6n - 5$. (b) $u_1 = 1$, $d = 6$ and $u_8 = 43$.
20. There are $33 - 14 + 1 = 20$ terms. Sum = $10(14 + 33) = 470$. Two forms are $\sum_{k=14}^{33} k$ and $\sum_{k=1}^{20} (k + 13)$.
21. The first and last are 105 and 497. Since $497 = 105 + 7(n - 1)$, there are 57 terms. The sum is $\frac{57}{2}(105 + 497) = 17\,157$.
22. (a) First term = -3 , last term = -29 , number of terms = $18 - 5 + 1 = 14$. (b) The sum is $7(-3 - 29) = -224$.
23. (a) $(p + 5) - (2p - 1) = (4p - 5) - (p + 5)$ gives $6 - p = 3p - 10$, so $p = 4$. The first terms are 7, 9, 11; $d = 2$. (b) $S_{20} = 10[14 + 19(2)] = 520$.
24. (a) $S_n = \frac{n}{2}[30 - 2(n - 1)] = n(16 - n)$. Since $n > 0$, $S_n = 0$ gives $n = 16$. (b) The positive terms end with $u_8 = 1$, so their sum is $4(15 + 1) = 64$.
25. (a) $a + 5d = 18$ and $5(2a + 9d) = 130$, so $2a + 9d = 26$. Subtracting this from $2a + 10d = 36$ gives $d = 10$ and $a = -32$. (b) $S_{20} = 10[-64 + 19(10)] = 1260$.
26. (a) Annual interest = $5400(0.03) = 162$, so $B_n = 5400 + 162n$. (b) $B_6 = 6372$. The condition $5400 + 162n > 7000$ gives $n > 800/81$, so $n = 10$; $B_9 = 6858$, $B_{10} = 7020$.
27. (a) Differences 11, 10, 11 are unequal. (b) $d = (236 - 204)/3 = 32/3$. Hence $u_{10} = 204 + 9(32/3) = 300$. (c) Attendance may fluctuate or reach the venue capacity; a constant increase may not continue.
28. (a) $d = [6 - (-2)]/(3 - 1) = 4$, so $u_1 = -2$. (b) $u_{14} = -2 + 13(4) = 50$. (c) The index is a positive integer, so only discrete values of n are allowed.

29. (a) $5n + 3 > 2n + 30$ gives $n > 9$, so $n = 10$. (b) The sum difference is $\frac{n}{2}[16 + 5(n - 1) - 64 - 2(n - 1)] = \frac{3n}{2}(n - 17)$. For positive n it is positive when $n > 17$, so $n = 18$.
30. (a) $S_n = \sum_{k=1}^n [380 + 45(k - 1)] = \frac{n}{2}[760 + 45(n - 1)] = 22.5n^2 + 357.5n$. (b) $S_{21} = 17\,430 < 18\,000$ and $S_{22} = 18\,755 \geq 18\,000$, so $n = 22$.

Section 3: Original IB exam-style questions

31. (a) $5d = 36 - 11 = 25$, so $d = 5$ and $u_1 = 11 - 5 = 6$. (b) $u_{20} = 6 + 19(5) = 101$. (c) $S_{20} = 10(6 + 101) = 1070$.
32. (a) $d = (12 - 42)/5 = -6$, so $u_{14} = 42 + 13(-6) = -36$. (b) $S_n = \frac{n}{2}[84 - 6(n - 1)] = 3n(15 - n)$. Since $n > 0$, $n = 15$.
33. (a) $a + 7d = 23$ and $2a + 16d = 56$. Doubling the first gives $2a + 14d = 46$; subtraction gives $2d = 10$. Thus $d = 5$, $a = -12$. (b) $S_{12} = 6[-24 + 11(5)] = 186$.
34. (a) $3t - 5 = 15 - 2t$, so $t = 4$. (b) The sequence is 4, 11, 18, ...; $S_n = \frac{n}{2}[8 + 7(n - 1)] = \frac{n(7n+1)}{2}$. (c) $7n^2 + n - 710 = (n - 10)(7n + 71) = 0$. The positive solution is $n = 10$.
35. (a) $a + 6d = 18$ and $\frac{15}{2}(2a + 14d) = 180$, hence $a + 7d = 12$. Subtraction gives $d = -6$, then $a = 54$. (b) $54 - 6(k - 1) = 0$, so $k = 10$.
36. (a) $16p + 4q = 60$ gives $4p + q = 15$; $49p + 7q = 168$ gives $7p + q = 24$. Therefore $p = 3$, $q = 3$. (b) $u_n = S_n - S_{n-1} = 3[n^2 - (n - 1)^2] + 3 = 6n$. Thus $u_8 = 48$.
37. (a) $21 - 3n < 0$ gives $n > 7$, so $u_8 = -3$ is first negative. (b) Terms u_1 to u_6 are positive, $u_7 = 0$, and later terms are negative. Thus $S_6 = S_7 = \frac{6}{2}(18 + 3) = 63$ is the maximum, attained at $n = 6$ and $n = 7$.
38. (a) There are 14 terms, from 7 to 46, giving $7(7 + 46) = 371$. (b) The first term is 4, so $S_m = \frac{m}{2}[8 + 3(m - 1)] = \frac{m(3m+5)}{2}$. Thus $3m^2 + 5m - 848 = (m - 16)(3m + 53) = 0$, giving $m = 16$.
39. (a) $u_{16} = 20 + 15(4) = 80$. (b) $S_{16} = 8(20 + 80) = 800$ seats. (c) Revenue = $800(15) = \text{€}12\,000$.
40. (a) $S_n = \frac{n}{2}[8 + 3(n - 1)] = \frac{n(3n+5)}{2}$. (b) $S_{10} = 175$. (c) $S_{17} = 476 \leq 500$ and $S_{18} = 531 > 500$, so 17 rows, leaving $500 - 476 = 24$ tiles.
41. (a) $8000(0.0025) = \text{€}20$. (b) $B_n = 8000 + 20n$, so $B_{12} = \text{€}8240$. (c) $8000 + 20n > 9500$ gives $n > 75$. Therefore 76 months; $B_{75} = 9500$ and $B_{76} = 9520$.
42. (a) The annual difference is $-3000 + 600(n - 1)$, positive when $n > 6$, so year 7. (b) $S_{10}^A = 5[64\,000 + 9(1800)] = 401\,000$; $S_{10}^B = 5[70\,000 + 9(1200)] = 404\,000$. (c) The cumulative difference is $\frac{n}{2}[-6000 + 600(n - 1)] = 300n(n - 11)$. It first becomes positive at $n = 12$.
43. (a) The differences are 9, 10, 9, 11. (b) $d = (191 - 152)/4 = 9.75$, so $u_9 = 152 + 8(9.75) = 230$. (c) The prediction exceeds the actual value by 6. Long extrapolation is uncertain because the rate need not remain constant.
44. (a) $2a + b = 7$ and $6a + b = 23$, so $a = 4$ and $b = -1$. (b) $u_1 = 3$, $d = 4$, giving $S_n = \frac{n}{2}[6 + 4(n - 1)] = n(2n + 1)$. (c) $2n^2 + n - 210 = (n - 10)(2n + 21) = 0$, so $n = 10$.
45. (a) $1392 = \frac{n}{2}(12 + 104) = 58n$, so $n = 24$. (b) $104 = 12 + 23d$, hence $d = 4$. (c) $\sum_{k=1}^{24} [12 + 4(k - 1)]$.
46. (a) The eleven terms $u_5, \dots, u_{15}$ have mean $\frac{1}{2}(u_5 + u_{15}) = u_{10}$, so their sum is $11u_{10}$. (b) $u_{10} = 440/11 = 40$; $7d = 40 - 12 = 28$, giving $d = 4$, $u_1 = 4$. (c) $S_{20} = 10[8 + 19(4)] = 840$.
47. (a) The terms are 3, 7, 11; common difference 4. (b) $u_{2k-1} = 4k - 1$, so the sum is $\frac{10}{2}(3 + 39) = 210$. (c) $u_{2k} - u_{2k-1} = 2$ for each pair, so the difference of the sums is $2m$.
48. (a) If there are n terms in total, $140 = \frac{n}{2}(2 + 26) = 14n$, so $n = 10$; 8 terms were inserted. (b) $d = (26 - 2)/9 = 8/3$. (c) $u_5 = 2 + 4(8/3) = 38/3$.
49. (a) $u_n = 18\,400 + 425(n - 1)$. (b) $S_{12} = 6[36\,800 + 11(425)] = 248\,850$. (c) $n - 1 \geq 4600/425 = 10.8235\dots$, so $n = 12$, the year $2025 + 12 = 2037$. (d) Population, borrowing habits or capacity may change.
50. (a) $S_n = \frac{n}{2}[10 + 2(n - 1)] = n(n + 4)$. (b) $S_{50} = 50(54) = 2700$. (c) $S_{24} = 24(28) = 672 \leq 720$, while $S_{25} = 25(29) = 725 > 720$, so at most 24 rows.

Section 4: AA HL exclusive challenge

51. (a) Equal differences give $3 + 2x = x^2 + 2x - 6$, hence $x^2 = 9$ and $x = \pm 3$. (b) For $x = 3$, $a = 3$, $d = 9$, so $u_{10} = 84$. For $x = -3$, $a = 15$, $d = -3$, so $u_{10} = -12$. (c) Only $x = 3$: it has a positive first term and positive common difference. For $x = -3$, $u_6 = 0$ and later terms are negative.

52. (a) Using the sum formula, $n[2a + (2n - 1)d] = \frac{3n}{2}[2a + (n - 1)d]$. Cancel n and multiply by 2: $4a + (4n - 2)d = 6a + (3n - 3)d$, so $2a = (n + 1)d$. (b) For $n = 5$, $a = 3d$. Then $75 = \frac{5}{2}(6d + 4d) = 25d$, giving $d = 3$, $a = 9$.

53. (a) $u_n = S_n - S_{n-1} = 4n + 3$. (b) $u_{3k-1} = 12k - 1$, an arithmetic sequence with first term 11 and difference 12. Its sum is $\frac{m}{2}[22 + 12(m - 1)] = m(6m + 5)$. (c) $8(48 + 5) = 424$.

54. (a) $12d = -24$, so $d = -2$ and $a = 23$. (b) $S_N = \frac{N}{2}[46 - 2(N - 1)] = N(24 - N)$, so $N = 24$. (c) The first twelve terms are 23, 21, ..., 1, with sum $6(23 + 1) = 144$. The last twelve are $-1, -3, \dots, -23$, giving the same absolute sum. Total = 288.

55. (a) $u_q - u_p = (q - p)d = p - q$. Since $q \neq p$, division gives $d = -1$. (b) $u_n = u_p + (n - p)d = q - (n - p) = p + q - n$. (c) The zero term has index $p + q$; the first term is $p + q - 1$. The required sum is $\frac{(p+q)(p+q-1)}{2}$.

56. (a) $12a + 66d = 4(6a + 15d)$ gives $6d = 12a$, so $d = 2a$. Then $S_8 = \frac{8}{2}[2a + 7(2a)] = 64a = 576$, yielding $a = 9$, $d = 18$. (b) $S_n = \frac{n}{2}[18 + 18(n - 1)] = 9n^2 = (3n)^2$.

57. (a) Each consecutive pair contributes -3 , so $-3n = -60$ and $n = 20$. (b) $2540 = S_{40} = 20[2a + 39(3)]$ gives $2a + 117 = 127$, hence $a = 5$. (c) Each pair $u_{2k-1} - u_{2k} = -d$; there are n pairs, so the sum is $-nd$.

58. (a) There are $3n - (n + 1) + 1 = 2n$ terms. The endpoints are $5n + 3$ and $15n - 2$, so $T_n = \frac{2n}{2}(20n + 1) = n(20n + 1)$. (b) $20n^2 + n - 726 = (n - 6)(20n + 121) = 0$, giving $n = 6$.

59. (a) The difference of sums is $\sum_{k=1}^N (2k - 9) = N(N + 1) - 9N = N(N - 8)$, so $N = 8$. (b) $5n - 1 = 3m + 8$ gives $5n - 3m = 9$. Thus n must be a positive multiple of 3; the common values are 14, 29, 44, ... (c) They form an arithmetic sequence with $a = 14$, $d = 15$. Its largest term at most 200 is 194, at index 13. Sum = $\frac{13}{2}(14 + 194) = 1352$.

60. (a) $a + 3(n - 1) = 4a$, so $a = n - 1$. Also $\frac{n}{2}(a + 4a) = 455$, hence $na = 182$. (b) $n(n - 1) = 182$ gives $(n - 14)(n + 13) = 0$. Thus $n = 14$ and $a = 13$. (c) The final seven rows are rows 8 to 14, with $u_8 = 13 + 7(3) = 34$ and $u_{14} = 52$. Their total is $\frac{7}{2}(34 + 52) = 301$.