



Section 1.8: Infinite Geometric Series

IB Mathematics: Analysis and Approaches SL & HL

60-question worksheet: 50 core questions plus 10 AA HL-only challenges, with SL 1.3 finite-series skills used throughout.

Learning goal	Determine whether a geometric series converges, use $ r < 1$, calculate and interpret S_∞ , and connect infinite sums with the finite geometric-series ideas from SL 1.3.
Syllabus link	IB Mathematics AA SL/HL: SL 1.8 — sum of infinite convergent geometric sequences; use of $ r < 1$ and modulus notation. This worksheet deliberately revisits SL 1.3 geometric sequences and finite sums where they support SL 1.8.
Structure	Section 1: 15 short-response questions. Section 2: 15 extended-response questions. Section 3: 20 original IB-style Paper 1 questions. Section 4: 10 additional AA HL-exclusive challenge questions.
Calculator	Primarily Paper 1 / non-calculator practice. Questions marked GDC are designed for technology. Give exact answers unless a decimal approximation is requested.

Instructions

Before using $S_\infty = \frac{u_1}{1-r}$, first check that $|r| < 1$. Show clear working. Use the finite sum S_n whenever the number of terms is finite. Unless another accuracy is stated, give non-exact numerical answers to **3 significant figures**.

1. Short response — core fluency (15 questions)

1. State the condition on the common ratio r for an infinite geometric series to have a finite sum.

2. Determine whether $12 + 6 + 3 + \cdots$ has a sum to infinity. State the common ratio.

3. Determine whether $5 - 7.5 + 11.25 - \cdots$ has a sum to infinity. Give a reason.



4. Find the sum to infinity of $18 + 6 + 2 + \dots$.

5. Find the sum to infinity of $20 - 10 + 5 - \dots$.

6. A geometric series has first term 7 and common ratio 0.2. Find S_{∞} .

7. A geometric series has first term 12 and common ratio $-\frac{1}{4}$. Find S_{∞} .

8. A convergent geometric series has $S_{\infty} = 30$ and $r = 0.4$. Find u_1 .

9. A convergent geometric series has $S_{\infty} = 20$ and $u_1 = 8$. Find r .

10. A convergent geometric series has second term $u_2 = 9$ and $S_{\infty} = 36$. Find u_1 and r .

11. Find $3 + 1 + \frac{1}{3} + \frac{1}{9} + \dots$.

12. Find $6 - 2 + \frac{2}{3} - \frac{2}{9} + \dots$.

13. The common ratio of a geometric series is $\frac{k}{3}$. Find the values of k for which the series converges.

14. A geometric series has $u_1 = 16$ and $r = \frac{1}{2}$. Find the amount still to be added after the first five terms, that is, $S_{\infty} - S_5$.



15. A geometric series has first term 5 and common ratio $-\frac{1}{5}$. Find its sum to infinity.

2. Extended response — mixed SL 1.8 and SL 1.3 skills (15 questions)

16. A geometric sequence has $u_3 = 12$ and $r = \frac{1}{2}$. Find u_1 and hence find the sum to infinity.

17. A geometric sequence has $u_4 = -2$ and $r = -\frac{1}{2}$.

- (a) Find u_1 .
(b) Determine whether the corresponding series converges.
(c) Hence find S_∞ .

18. A geometric sequence has $u_3 = 8$ and $u_6 = 1$, with $r > 0$.

- (a) Find r and u_1 .
(b) Hence determine whether the corresponding infinite series converges, and find S_∞ if it does.

19. A convergent geometric series has $u_2 = 6$ and $S_\infty = 24$. Find u_1 and r .

20. A geometric series has common ratio $r = 2k - 1$. Find the values of k for which the series converges.

21. A geometric sequence has $u_1 = 81$ and $r = \frac{1}{3}$. Find the sum of all terms from u_4 onwards:

$$u_4 + u_5 + u_6 + \cdots$$

22.

GDC

A geometric series has $u_1 = 25$ and $r = 0.6$. Find the least value of n for which $S_\infty - S_n < 0.05$.



23. A geometric series has $u_1 = 40$ and $r = -\frac{1}{2}$.

- (a) Find S_∞ .
 - (b) Find S_6 .
 - (c) Hence find $S_\infty - S_6$ and comment on its sign.
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24. Express the recurring decimal $0.363636\dots$ as a fraction by writing it as an infinite geometric series.

25. A ball is dropped from a height of 2.4 m. Each rebound reaches 75% of the previous maximum height. Find the total vertical distance travelled by the ball over all time. Do not use a memorized bouncing-ball formula; form the required geometric series.

26. The perimeter of the first square in a pattern is 32 cm. Each new square has side length $\frac{3}{4}$ of the preceding square. Find the total of the perimeters of all the squares in the infinite pattern.

27. The area of the first shaded region in a self-similar pattern is 36 cm^2 . Each subsequent shaded region has one quarter of the area of the preceding region. Find the total shaded area.

28. A geometric series has $u_1 = 10$ and $r = \frac{3}{5}$. Find

$$u_5 + u_6 + u_7 + \dots$$

29. A convergent geometric series has $u_1 = 12$ and $r = \frac{1}{3}$.

- (a) Find the sum of the odd-indexed terms $u_1 + u_3 + u_5 + \dots$.
 - (b) Find the sum of the even-indexed terms $u_2 + u_4 + u_6 + \dots$.
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30. A convergent geometric series has $S_\infty = 50$ and $S_4 = 46.875$. Given that $r > 0$, find r and u_1 .

3. IB exam-style — original sister questions informed by past papers (20 questions)

Exam-style practice

These questions are original sister questions. They reflect recurring SL 1.8 structures from IB AA sequence-and-series papers: deciding convergence, parameter conditions, sums to infinity, tails such as $|S_\infty - S_n|$, recurring decimals, bouncing-ball distance and contextual infinite sums. Questions marked

GDC

are intended for technology.

31. The first three terms of a sequence are $k + 2$, k and 1, where $k \in \mathbb{R}$.

- Find the two values of k for which the three terms form a geometric sequence. [3 marks]
- For one of these values the corresponding infinite geometric series does not have a sum to infinity. Identify this value of k and state why. [2 marks]
- For the other value of k , find S_∞ . [2 marks]

32. A ball is dropped from a height of 6 m. After each impact it rebounds to $\frac{2}{3}$ of its previous maximum height.

- Find the maximum height reached after the fourth bounce. [2 marks]
- Find the total vertical distance travelled immediately before the fourth impact with the ground. [3 marks]
- Find the total vertical distance travelled if the motion continues indefinitely. [2 marks]

33.

GDC

For $c > 0$, the sum of the first n terms of a geometric series is

$$S_n = \sum_{j=0}^{n-1} 5(\log_2 c)^j.$$

- Find the range of values of c for which S_n converges as $n \rightarrow \infty$. [3 marks]
- Let $c = \sqrt{2}$. Find S_∞ . [1 marks]
- Hence find the least value of n such that $|S_\infty - S_n| < 0.01$. [3 marks]



34. Consider the infinite geometric series

$$3x + 2x^2 + \frac{4}{3}x^3 + \cdots, \quad x > 0.$$

- (a) Find the common ratio in terms of x . [1 marks]
 - (b) Find the range of values of x for which the series converges. [2 marks]
 - (c) Given that $S_\infty = 9$, find x . [2 marks]
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35.

GDC

The sum of the first n terms of a geometric series is

$$S_n = \sum_{j=1}^n \frac{3}{4} \left(\frac{5}{6}\right)^j.$$

- (a) Find the first term. [1 marks]
 - (b) Find S_∞ . [2 marks]
 - (c) Find the least value of n for which $S_\infty - S_n < 0.001$. [3 marks]
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36. Maya deposits \$7200 into a fund. At the end of the first year she adds no further interest; instead, she plans to add another amount equal to 60% of her original deposit, then each later yearly addition is 60% of the previous addition. Treat the initial deposit as the first term of the geometric series.

- (a) Show that, even if Maya continues making deposits indefinitely, the total amount she can deposit is less than \$20000. [3 marks]
 - (b) Suppose instead that her first deposit is \$ x and the same 60% pattern continues. Find the least whole-dollar value of x required for the total after six deposits to be at least \$20000. [4 marks]
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37. An infinite geometric series has first term $u_1 = a$ and second term $u_2 = a(3 - a)$, where $a > 0$.

- (a) Find the common ratio in terms of a . [1 marks]
 - (b) Find the values of a for which the sum to infinity exists. [3 marks]
 - (c) Given that $S_\infty = 3$, find a . [2 marks]
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38. Express the recurring decimal

$$0.24545454 \dots$$

as a fraction in simplest form by using an infinite geometric series.

[5 marks]



39. A positive geometric series has $S_{\infty} = 40$ and $S_3 = 35$.

- (a) Find the common ratio. [3 marks]
- (b) Hence find the first term. [2 marks]
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40. A convergent geometric series has $S_{\infty} = 12$ and first term 18.

- (a) Find the common ratio. [2 marks]
- (b) Find S_4 . [2 marks]
-

41. A geometric series has first term 18 and common ratio $\frac{1}{3}$.

- (a) Find $u_1 + u_3 + u_5 + \dots$. [3 marks]
- (b) Find $u_2 + u_4 + u_6 + \dots$. [2 marks]
- (c) Hence verify that the two answers add to S_{∞} . [1 marks]
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42. A geometric sequence has first term 81 and common ratio $\frac{1}{3}$. Find

$$u_5 + u_6 + u_7 + \dots$$

[3 marks]

43. An infinite pattern is made from equilateral triangles. The first triangle has side length 9 cm and each subsequent triangle has side length half that of the preceding triangle. Find the total of the perimeters of all triangles. [4 marks]

44.

GDC

A decorative path is made from rods of lengths

$$30 \text{ cm}, \quad 18 \text{ cm}, \quad 10.8 \text{ cm}, \dots$$

continuing indefinitely.

- (a) Find the total length of all rods. [2 marks]
- (b) Find the least value of n for which the total length of all rods after the first n rods is less than 0.5 cm. [4 marks]



45. For a convergent geometric series, show that

$$S_{\infty} - S_n = S_{\infty} r^n.$$

Hence, for a series with $u_1 = 9$ and $r = \frac{2}{3}$, find the least value of n for which

$$S_{\infty} - S_n < 0.05.$$

[6 marks]

46. Consider the infinite geometric series

$$8 + 4q + 2q^2 + \dots$$

(a) Find the values of q for which the series converges. [2 marks]

(b) Find S_{∞} in terms of q . [2 marks]

(c) Given that $S_{\infty} = 16$, find q . [2 marks]

47.

GDC

A geometric series is

$$50 + 25 + 12.5 + \dots$$

Find the least number of terms whose sum exceeds 99% of the sum to infinity. [5 marks]

48. A geometric sequence satisfies $u_2 = 15$ and $u_5 = \frac{15}{8}$.

(a) Find the common ratio. [2 marks]

(b) Find the first term. [1 marks]

(c) Hence find S_{∞} . [2 marks]

49. A convergent geometric series has $S_{\infty} = 50$ and $S_4 = 46.875$. Given that $r > 0$, find r and u_1 . [5 marks]



50.

GDC

A charity receives \$500 in the first month of a campaign. Each subsequent monthly donation is 80% of the previous month's donation.

- (a) Find the total amount the charity would receive if the pattern continued indefinitely. [2 marks]
- (b) Find the first month by the end of which the cumulative donations exceed \$2400. [3 marks]

AA HL extension

These questions are exclusive to AA HL students by difficulty. They remain within SL 1.8 and its SL 1.3 foundation, but use the less scaffolded parameter, convergence and remainder structures seen in AA HL sequence-and-series papers.

4. AA HL exclusive challenge (10 additional questions)

51. An infinite geometric series has first term a and second term $a(2 - a)$, where $a > 0$.

- (a) Find the values of a for which the series converges. [3 marks]
- (b) Given that $S_{\infty} = 4$, find a . [3 marks]

52. Express the recurring decimal

$$0.25373737 \dots$$

as a fraction by separating the non-recurring part and writing the recurring part as an infinite geometric series. [5 marks]

53. Consider the series

$$\ln x + p \ln x + \frac{1}{5} \ln x + \dots, \quad x > 1,$$

where $p \neq 0$.

- (a) Suppose the series is geometric.
- (i) Show that $p = \pm \frac{1}{\sqrt{5}}$. [2 marks]
- (ii) Given that $p > 0$ and $S_{\infty} = \frac{5 + \sqrt{5}}{2}$, find x . [3 marks]
- (b) Now suppose that the displayed terms instead begin an arithmetic sequence.
- (i) Find p and write the common difference in the form $k \ln x$, where $k \in \mathbb{Q}$. [3 marks]
- (ii) The sum of the first n terms is $\ln(x^{-8})$. Find n . [4 marks]



54.

GDC

For $c > 0$,

$$S_n = \sum_{j=0}^{n-1} 7(\log_3 c)^j.$$

- (a) Find the range of values of c for which S_n converges as $n \rightarrow \infty$. [3 marks]
- (b) Let $c = \sqrt{3}$. Find the least value of n for which $|S_\infty - S_n| < 0.02$. [4 marks]
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55. The first three terms of a sequence are $k + 2$, k and 1.

- (a) Find the two values of k for which the sequence is geometric. [3 marks]
- (b) For each value of k , state whether a sum to infinity exists. [2 marks]
- (c) For the convergent case, find S_∞ . For the non-convergent case, find S_{2m} , where $m \in \mathbb{Z}^+$. [3 marks]
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56. A particle starts at the origin. It travels 6 cm to a turning point and returns to the origin. It then travels 3 cm to a turning point on the opposite side and returns, then 1.5 cm, and so on. The maximum distances from the origin form a geometric sequence. Find the total distance travelled if the motion continues indefinitely. [5 marks]

57. For a convergent geometric series, the sum of the odd-indexed terms is 20 and the sum of the even-indexed terms is 12.

- (a) Find r . [2 marks]
- (b) Find u_1 . [2 marks]
- (c) Find S_∞ . [1 marks]
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58. A positive geometric series has sum to infinity 30. The sum of the squares of all its terms is 180. Find the first term and common ratio. [7 marks]

59. Two convergent geometric series have the same first term $a > 0$. Their common ratios are r and $-r$, where $0 < r < 1$. Their sums to infinity are 30 and 18, respectively. Find a and r . [6 marks]



60. In a positive convergent geometric series, the sum of all terms after the first n terms is four times u_{n+2} , for every positive integer n .

(a) Find the common ratio. [4 marks]

(b) Given that $S_{\infty} = 30$, find the first term. [2 marks]

Answer key and solution outlines — Section 1

1. $|r| < 1$.
2. $r = \frac{1}{2}$, so $|r| < 1$; the series converges.
3. $r = -\frac{3}{2}$, so $|r| > 1$; no finite sum to infinity.
4. $r = \frac{1}{3}$, so $S_{\infty} = \frac{18}{1 - 1/3} = 27$.
5. $r = -\frac{1}{2}$, so $S_{\infty} = \frac{20}{1 + 1/2} = \frac{40}{3}$.
6. $S_{\infty} = \frac{7}{0.8} = \frac{35}{4} = 8.75$.
7. $S_{\infty} = \frac{12}{1 + 1/4} = \frac{48}{5}$.
8. $u_1 = S_{\infty}(1 - r) = 30(0.6) = 18$.
9. $8 = 20(1 - r)$, so $r = \frac{3}{5}$.
10. $u_1 = 36(1 - r)$ and $u_1 r = 9$. Thus $36r(1 - r) = 9$, so $(2r - 1)^2 = 0$. Hence $r = \frac{1}{2}$ and $u_1 = 18$.
11. $r = \frac{1}{3}$, so $S_{\infty} = \frac{3}{1 - 1/3} = \frac{9}{2}$.
12. $r = -\frac{1}{3}$, so $S_{\infty} = \frac{6}{1 + 1/3} = \frac{9}{2}$.
13. $\left|\frac{k}{3}\right| < 1 \Rightarrow |k| < 3$, so $-3 < k < 3$.
14. $S_{\infty} = 32$ and $S_5 = 16 \frac{1 - (1/2)^5}{1 - 1/2} = 31$, so the remainder is 1.
15. $S_{\infty} = \frac{5}{1 + 1/5} = \frac{25}{6}$.

Answer key and solution outlines — Section 2

16. $12 = u_1(1/2)^2$, so $u_1 = 48$ and $S_{\infty} = \frac{48}{1/2} = 96$.
17. $-2 = u_1(-1/2)^3$, so $u_1 = 16$. Since $|r| = 1/2 < 1$, it converges and $S_{\infty} = \frac{16}{1 + 1/2} = \frac{32}{3}$.
18. $\frac{u_6}{u_3} = r^3 = \frac{1}{8}$, so $r = \frac{1}{2}$. Then $8 = u_1(1/2)^2$, giving $u_1 = 32$. Since $|r| < 1$, the series converges and $S_{\infty} = \frac{32}{1/2} = 64$.
19. $u_1 = 24(1 - r)$ and $u_1 r = 6$, so $24r(1 - r) = 6$. Thus $(2r - 1)^2 = 0$, giving $r = \frac{1}{2}$, $u_1 = 12$.
20. $|2k - 1| < 1 \Rightarrow -1 < 2k - 1 < 1 \Rightarrow 0 < k < 1$.
21. $u_4 = 81(1/3)^3 = 3$. The required tail is $\frac{3}{1 - 1/3} = \frac{9}{2}$.
22. $S_{\infty} = \frac{25}{0.4} = 62.5$. The remainder after n terms is $62.5(0.6)^n$. Solve $62.5(0.6)^n < 0.05$; the least integer is $\boxed{14}$.
23. (a) $S_{\infty} = \frac{40}{1 + 1/2} = \frac{80}{3}$. (b) $S_6 = 40 \frac{1 - (1/2)^6}{1 + 1/2} = \frac{105}{4}$. (c) $S_{\infty} - S_6 = \frac{5}{12} > 0$; after an even number of terms this alternating partial sum lies below the limit.
24. $0.36 + 0.0036 + \dots$ has $u_1 = \frac{36}{100}$, $r = \frac{1}{100}$, so $S_{\infty} = \frac{36/100}{99/100} = \frac{4}{11}$.
25. First rebound height = $2.4(0.75) = 1.8$. Total distance = $2.4 + 2(1.8 + 1.8(0.75) + \dots) =$



$$2.4 + 2\frac{1.8}{0.25} = 16.8 \text{ m.}$$

26. Perimeters form $32 + 24 + 18 + \dots$ with $r = \frac{3}{4}$, so total $= \frac{32}{1/4} = 128 \text{ cm.}$

27. $36 + 9 + \frac{9}{4} + \dots$ has $r = \frac{1}{4}$, so total $= \frac{36}{3/4} = 48 \text{ cm}^2.$

28. $u_5 = 10(3/5)^4 = \frac{162}{125}$. Tail $= \frac{u_5}{1 - 3/5} = \frac{81}{25}.$

29. (a) Odd-indexed terms have first term 12 and ratio $1/9$: sum $= \frac{12}{1 - 1/9} = \frac{27}{2}$. (b) Even-indexed terms have first term 4 and ratio $1/9$: sum $= \frac{4}{1 - 1/9} = \frac{9}{2}.$

30. $S_\infty - S_4 = S_\infty r^4$, so $50 - 46.875 = 50r^4$. Thus $r^4 = \frac{1}{16}$ and, since $r > 0$, $r = \frac{1}{2}$. Hence $u_1 = 50(1 - 1/2) = 25.$

Answer key and solution outlines — Section 3

31. Geometric condition: $k^2 = (k+2)(1)$, so $k^2 - k - 2 = 0$ and $k = 2$ or -1 . For $k = -1$, $r = -1$, so $|r| = 1$ and no sum to infinity exists. For $k = 2$, the terms are $4, 2, 1, \dots$, so $r = \frac{1}{2}$ and $S_\infty = 8.$

32. Rebound heights are $4, \frac{8}{3}, \frac{16}{9}, \frac{32}{27}, \dots$ (a) $\frac{32}{27} \text{ m.}$ (b) $6 + 2\left(4 + \frac{8}{3} + \frac{16}{9}\right) = \frac{206}{9} \text{ m} \approx 22.9 \text{ m.}$ (c) $6 + 2\left(\frac{4}{1 - 2/3}\right) = 30 \text{ m.}$

33. (a) $|\log_2 c| < 1 \Rightarrow -1 < \log_2 c < 1$, so $\frac{1}{2} < c < 2$. (b) If $c = \sqrt{2}$, then $r = \frac{1}{2}$ and $S_\infty = 10$. (c) $10(1/2)^n < 0.01$; least $n = 10$.

34. (a) $r = \frac{2x}{3}$. (b) Since $x > 0$, convergence requires $0 < \frac{2x}{3} < 1$, so $0 < x < \frac{3}{2}$. (c) $\frac{3x}{1 - 2x/3} = 9$, giving $x = 1$.

35. (a) $u_1 = \frac{3}{4} \cdot \frac{5}{6} = \frac{5}{8}$. (b) $r = \frac{5}{6}$, so $S_\infty = \frac{5/8}{1/6} = \frac{15}{4}$. (c) $\frac{15}{4}\left(\frac{5}{6}\right)^n < 0.001$; least $n = 46$.

36. (a) Infinite total $= \frac{7200}{1 - 0.6} = 18000 < 20000$, so the target cannot be reached. (b) $x \frac{1 - 0.6^6}{1 - 0.6} \geq 20000$, so $x \geq 8391.1\dots$; least whole-dollar value 8392.

37. (a) $r = 3 - a$. (b) $|3 - a| < 1 \Rightarrow 2 < a < 4$. (c) $\frac{a}{1 - (3 - a)} = \frac{a}{a - 2} = 3$, so $a = 3$.

38. $0.245454\dots = 0.24 + 0.0054 + 0.000054 + \dots = \frac{6}{25} + \frac{54/10000}{1 - 1/100} = \frac{27}{110}.$

39. $S_3 = S_\infty(1 - r^3)$, so $35 = 40(1 - r^3)$ and $r^3 = \frac{1}{8}$. Thus $r = \frac{1}{2}$ and $u_1 = 40(1 - 1/2) = 20$.

40. $18 = 12(1 - r)$, so $r = -\frac{1}{2}$. Then $S_4 = 18 \frac{1 - (-1/2)^4}{1 - 1/2} = \frac{45}{4}.$

41. Odd-indexed terms have first term 18 and ratio $1/9$, so their sum is $\frac{18}{1 - 1/9} = \frac{81}{4}$. Even-indexed terms have first term 6 and ratio $1/9$, so their sum is $\frac{27}{4}$. Total $= 27 = S_\infty$.

42. $u_5 = 81(1/3)^4 = 1$, so the required tail is $\frac{1}{1 - 1/3} = \frac{3}{2}.$

43. Perimeters are $27, 13.5, 6.75, \dots$, so total $= \frac{27}{1 - 1/2} = 54 \text{ cm.}$

44. (a) $r = 0.6$, so total $= \frac{30}{0.4} = 75 \text{ cm.}$ (b) Tail after n rods is $75(0.6)^n$. The least n with this below 0.5 is $n = 10$.

45. From $S_n = S_\infty(1 - r^n)$, subtraction gives $S_\infty - S_n = S_\infty r^n$. Here $S_\infty = 27$, so $27(2/3)^n < 0.05$; least $n = 16$.
46. (a) $r = q/2$, so $|q| < 2$. (b) $S_\infty = \frac{8}{1 - q/2} = \frac{16}{2 - q}$. (c) $\frac{16}{2 - q} = 16$, so $q = 1$.
47. $S_\infty = 100$ and $S_n = 100(1 - 2^{-n})$. Need $S_n > 99$, hence $2^{-n} < 0.01$. The least value is $n = 7$.
48. $r^3 = \frac{u_5}{u_2} = \frac{1}{8}$, so $r = \frac{1}{2}$. Hence $u_1 = 30$ and $S_\infty = 60$.
49. $S_\infty - S_4 = S_\infty r^4$, so $3.125 = 50r^4$ and $r^4 = 1/16$. Since $r > 0$, $r = \frac{1}{2}$. Thus $u_1 = 25$.
50. (a) $S_\infty = \frac{500}{1 - 0.8} = 2500$. (b) $2500(1 - 0.8^n) > 2400$; the least integer is $n = 15$.

Answer key and solution outlines — Section 4: AA HL exclusive

51. $r = 2 - a$. Convergence requires $|2 - a| < 1$, so $1 < a < 3$. Also $S_\infty = \frac{a}{1 - (2 - a)} = \frac{a}{a - 1} = 4$, giving $a = \frac{4}{3}$.
52. $0.253737 \dots = 0.25 + 0.0037 + 0.000037 + \dots = \frac{1}{4} + \frac{37/10000}{1 - 1/100} = \frac{628}{2475}$.
53. (a)(i) Geometric condition gives $p^2 = 1/5$, so $p = \pm 1/\sqrt{5}$. (ii) With $p > 0$, $S_\infty = \frac{5 + \sqrt{5}}{4} \ln x$; equating to $\frac{5 + \sqrt{5}}{2}$ gives $x = e^2$. (b)(i) Arithmetic condition $2p \ln x = (1 + 1/5) \ln x$ gives $p = 3/5$. Hence $d = (p - 1) \ln x = -\frac{2}{5} \ln x$. (ii) $S_n = \frac{n}{2} \left(2 \ln x + (n - 1)(-\frac{2}{5} \ln x) \right) = \frac{n(6 - n)}{5} \ln x$. Set this equal to $-8 \ln x$ to obtain $n^2 - 6n - 40 = 0$, so $n = 10$.
54. (a) $|\log_3 c| < 1 \Rightarrow \frac{1}{3} < c < 3$. (b) With $c = \sqrt{3}$, $r = 1/2$ and $S_\infty = 14$. Need $14(1/2)^n < 0.02$, giving least $n = 10$.
55. $k^2 = k + 2$, so $k = 2$ or -1 . For $k = 2$, $r = 1/2$ and $S_\infty = 8$. For $k = -1$, $r = -1$, so there is no sum to infinity; the terms alternate $1, -1, 1, -1, \dots$, hence $S_{2m} = 0$.
56. Each excursion to a turning point and back contributes twice its maximum distance. Total $= 2(6 + 3 + 1.5 + \dots) = 2 \cdot \frac{6}{1 - 1/2} = 24$ cm.
57. The even-indexed sum is r times the odd-indexed sum, so $r = 12/20 = 3/5$. The odd-indexed series has ratio $r^2 = 9/25$, so $20 = \frac{u_1}{1 - 9/25}$ and $u_1 = 64/5$. Hence $S_\infty = 20 + 12 = 32$.
58. Let the first term be a . Then $\frac{a}{1 - r} = 30$ and $\frac{a^2}{1 - r^2} = 180$. Therefore $30 \frac{a}{1 + r} = 180$, so $\frac{a}{1 + r} = 6$. With $a = 30(1 - r)$, solve to obtain $r = \frac{2}{3}$ and $a = 10$.
59. $\frac{a}{1 - r} = 30$ and $\frac{a}{1 + r} = 18$. Dividing gives $\frac{1 + r}{1 - r} = \frac{5}{3}$, hence $r = \frac{1}{4}$. Then $a = 30(3/4) = \frac{45}{2}$.
60. The tail after n terms is $S_\infty r^n$, while $u_{n+2} = S_\infty(1 - r)r^{n+1}$. Thus $1 = 4r(1 - r)$, so $(2r - 1)^2 = 0$ and $r = \frac{1}{2}$. Hence $u_1 = S_\infty(1 - r) = 15$.