

Binomial Theorem

IB Mathematics: Analysis and Approaches SL/HL — SL 1.9

60-question worksheet: finite binomial expansions, Pascal's triangle, coefficients and $\binom{n}{r}$

Learning goal	Expand $(a + b)^n$ for positive integer n , identify specific terms and coefficients, and use Pascal's triangle and $\binom{n}{r}$ efficiently.
Syllabus link	IB Mathematics AA SL/HL: SL 1.9 — The binomial theorem . This worksheet remains within finite positive-integer expansions.
Structure	Section 1: 15 short response; Section 2: 15 extended response; Section 3: 20 IB-style Paper 1 questions; Section 4: 10 AA HL challenge questions.
Calculator / technology	Sections 1–2 are mixed. Section 3 is designed as non-calculator Paper 1 practice. Use technology only when a question explicitly asks for it; Q15 is a deliberate nCr technology-table task.

Instructions. Give exact answers unless a question states otherwise. Show sufficient working for coefficient and term questions. For $\binom{n}{r}$, be comfortable using both

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$

and calculator/GDC technology. In Section 4, the difficulty is aimed at AA HL students, but the mathematical content still comes from SL 1.9 rather than the AHL 1.10 extension to fractional or negative indices.

Section 1 — Short response

Questions 1–15

- Expand $(x + 3)^4$.
- Expand $(2x - 1)^5$.
- Find the coefficient of x^3 in the expansion of $(1 + 2x)^7$.
- Write the row of Pascal's triangle that gives the coefficients in the expansion of $(a + b)^6$.
- Evaluate $\binom{10}{3}$ using the factorial formula.
- Find all values of r such that $\binom{8}{r} = 56$.



7. State the general $(r + 1)$ th term in the expansion of $(a + b)^n$.
8. Find the fifth term in the expansion of $(2x + 3)^8$.
9. Find the coefficient of x^4 in $(3 + x)^6$.
10. Find the sum of the coefficients in the expansion of $(2x - 3)^5$ without expanding the expression.
11. Find the coefficient of x^5 in $(2 + x)^7$.
12. Given that $\binom{n}{2} = 45$, where $n \in \mathbb{N}$, find n .
13. Find the middle term in the expansion of $(1 + x)^8$.
14. Find the coefficient of x^2 in $(1 - 3x)^5$.
15. Use a table of $\binom{12}{r}$ values generated with technology to find all values of r for which $\binom{12}{r} = 495$.

Section 2 — Extended response**Questions 16–30**

16. Find the first four terms, in ascending powers of x , in the expansion of $(2 - 3x)^6$.
17. The coefficient of x^2 in $(1 + kx)^6$ is 240, where $k > 0$. Find k .
18. The first three terms of $(1 + ax)^n$, in ascending powers of x , are
- $$1 + 12x + 54x^2 + \dots,$$
- where $a > 0$ and $n \in \mathbb{N}$. Find a and n .
19. Find the term independent of x in the expansion of

$$\left(x^2 + \frac{2}{x}\right)^6.$$



20. Find the coefficient of x^5 in the expansion of

$$\left(x^3 - \frac{2}{x}\right)^7.$$

21. In the expansion of $(1+x)^n$, the coefficient of x^3 is equal to the coefficient of x^2 . Find n .

22. By substituting a suitable value of x into the binomial expansion of $(1+x)^n$, show that

$$\sum_{r=0}^n \binom{n}{r} = 2^n.$$

23. By considering $(1+1)^n$ and $(1-1)^n$, show that, for $n \geq 1$, the sum of the even-indexed binomial coefficients equals the sum of the odd-indexed binomial coefficients, and find their common value.

24. In the expansion of $(1+2x)^8$, let A be the coefficient of x^3 and B the coefficient of x^4 . Find A , B , and the ratio $B:A$.

25. The expansion of $(a+bx)^4$, where $a > 0$, begins

$$16 - 96x + 216x^2 + \dots$$

Find a and b , then find the coefficient of x^3 .

26. In the expansion of $(1+x)^n$, the coefficient of x^2 is six times the coefficient of x . Find n .

27. Explain why the coefficients of x^5 and x^7 in $(1+x)^{12}$ are equal, and find their common value.

28. Find the greatest coefficient in the expansion of $(1+x)^{12}$, and state the power of x to which it belongs.

29. Find the term containing x^7 in the expansion of $(2x-1)^{10}$.

30. Given that

$$\binom{n}{2} + \binom{n}{1} = 66,$$

find n .

Section 3 — IB exam-style / Paper 1

Questions 31–50 ■ non-calculator

31. Consider the expansion of $(1 + 2x)^6$.

(a) Write down the first four terms in ascending powers of x .

[3 marks]

(b) Find the coefficient of x^4 .

[1 marks]

32. The coefficient of x in $(2 + ax)^5$ is 240, where $a > 0$.

(a) Find a .

[2 marks]

(b) Hence find the coefficient of x^2 .

[2 marks]

33.

(a) Write the coefficients in the 8th row of Pascal's triangle (the row corresponding to $(a + b)^7$).

[2 marks]

(b) Hence write the first four terms of $(x - 2)^7$, in descending powers of x .

[3 marks]

34. The expansion of $(1 + ax)^n$ begins

$$1 + 12x + 60x^2 + \dots,$$

where $a > 0$ and $n \in \mathbb{N}$. Find a and n .

[5 marks]

35. Find the term independent of x in

$$\left(x^2 + \frac{3}{x}\right)^9.$$

Show clearly how you identify the required term.

[5 marks]

36. Consider $\left(2x^3 - \frac{1}{x}\right)^5$.

(a) Determine which term contains x^3 .

[2 marks]

(b) Hence find the coefficient of x^3 .

[2 marks]

37.

(a) Using the factorial definition, show that $\binom{n}{r} = \binom{n}{n-r}$.

[2 marks]

(b) Hence solve $\binom{11}{r} = 330$.

[2 marks]

38. Let $P(x) = (1 + 2x)^7$.

(a) Find $P(1)$ and $P(-1)$.

[2 marks]

(b) Hence find the sum of the coefficients of the even powers of x , and the sum of the coefficients of the odd powers of x .

[4 marks]

39. In $(1 + x)^n$, the coefficient of x^3 is four times the coefficient of x^2 .

(a) Find n . [3 marks]

(b) Hence find the coefficient of x^4 . [1 marks]

40. The coefficient of x^4 in $(1 + x)^n$ is 715, where $n > 4$.

(a) Find n . [3 marks]

(b) Hence write down the coefficient of x^9 . [1 marks]

41. Consider $(1 - x)^8$.

(a) Find the coefficient of x^5 . [2 marks]

(b) Find the sum of all the positive coefficients in the expansion. [3 marks]

42. Consider the expansion of $(2 + x)^9$.

(a) Find the coefficient of x^3 . [2 marks]

(b) Find the ratio $\frac{\text{coefficient of } x^4}{\text{coefficient of } x^3}$. [2 marks]

(c) Hence find the coefficient of x^4 . [1 marks]

43. In the expansion of $(1 + mx)^7$, where $m \neq 0$, the coefficients of x^2 and x^3 are equal.

(a) Find m . [4 marks]

(b) Find the coefficient of x^2 . [1 marks]

44. Let

$$P(x) = (1 + x)^6 + (1 - x)^6.$$

(a) Explain why all odd powers of x cancel. [2 marks]

(b) Find $P(x)$ in expanded form. [3 marks]

45. In the expansion of $(3 + x)^8$, let A be the coefficient of x^2 and B the coefficient of x^6 .

(a) Find A and B . [3 marks]

(b) Hence find $A : B$ in simplest form. [1 marks]

46. The coefficient of x^3 in $(1 + kx)^8$ is -1512 .

(a) Find k . [3 marks]

(b) Hence find the coefficient of x^4 . [2 marks]

47. Find the term independent of x in

$$\left(x + \frac{2}{x^2}\right)^6.$$

Explain why only one term is independent of x .

[5 marks]

48. In the expansion of $(a + x)^5$, where $a > 0$, the coefficient of x^4 is 15.

(a) Find a .

[2 marks]

(b) Find the coefficient of x^2 .

[2 marks]

(c) Write down the constant term.

[1 marks]

49. The coefficients of x^2 and x^3 in $(1 + x)^n$ are 45 and 120, respectively.

(a) Find n .

[2 marks]

(b) Find the coefficient of x^4 .

[1 marks]

(c) Find the central coefficient of the expansion.

[1 marks]

50. In the expansion of $(1 + x)^8$, the coefficient of x^r is 56.

(a) Find all possible values of r .

[2 marks]

(b) Explain why two values occur.

[1 marks]

(c) Find the sum of the corresponding terms when $x = 2$.

[2 marks]

Section 4 — AA HL challenge

Questions 51–60 ■ harder SL 1.9 content only

HL note. This section is exclusive to HL students by *difficulty*, not by syllabus content. It deliberately avoids fractional and negative binomial indices, which belong to AHL 1.10.

51. In the expansion of $(1 + x)^n$,

$$\binom{n}{4} = 5 \binom{n}{2}.$$

Find n , then find the greatest coefficient in the expansion.

[6 marks]

52. Find the coefficient of x^6 in

$$(1 + x)^8(1 - x)^5$$

without fully expanding either factor.

[6 marks]

53. Without expanding, evaluate

$$\sum_{r=0}^{10} (-1)^r \binom{10}{r} 2^{10-r} 3^r.$$

State clearly which binomial expansion you use.

[4 marks]



54. By comparing the coefficient of x^k on both sides of

$$(1+x)^m(1+x)^n = (1+x)^{m+n},$$

show that, for $0 \leq k \leq \min(m, n)$,

$$\sum_{r=0}^k \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}.$$

[7 marks]

55. Three consecutive coefficients in the expansion of $(1+x)^n$ are in the ratio

$$6 : 15 : 20.$$

The first of these is the coefficient of x^r . Find r and n .

[7 marks]

56. Find the term independent of x in

$$\left(x^2 + \frac{1}{x}\right)^{12}.$$

[5 marks]

57. Consider

$$Q(x) = \frac{(1+x)^{10} - (1-x)^{10}}{2x}.$$

Find the coefficient of x^4 in $Q(x)$, explaining why only odd powers from the numerator are relevant.

[6 marks]

58. The first three terms, in ascending powers of x , of $(a+bx)^n$ are

$$32 + 240x + 720x^2 + \dots,$$

where $a > 0$, $b > 0$ and $n \in \mathbb{N}$. Find a , b and n .

[8 marks]

59. The coefficient of x^r in $(2+x)^{12}$ is denoted by c_r .

(a) Show that

$$\frac{c_{r+1}}{c_r} = \frac{12-r}{2(r+1)}.$$

[3 marks]

(b) Hence determine the greatest coefficient in the expansion.

[5 marks]

60. Use the identity

$$r \binom{n}{r} = n \binom{n-1}{r-1}$$

to show that

$$\sum_{r=0}^n r \binom{n}{r} = n2^{n-1}.$$

Hence evaluate $\sum_{r=0}^8 r \binom{8}{r}$.

[7 marks]

Answer key and solution outlines

Questions 1–60

Section 1 — Short response

1. $x^4 + 12x^3 + 54x^2 + 108x + 81$.
2. $32x^5 - 80x^4 + 80x^3 - 40x^2 + 10x - 1$.
3. $\binom{7}{3}(2)^3 = 35 \cdot 8 = 280$.
4. 1, 6, 15, 20, 15, 6, 1.
5. $\binom{10}{3} = \frac{10!}{3!7!} = 120$.
6. $r = 3$ or $r = 5$, since $\binom{8}{3} = \binom{8}{5} = 56$.
7. $T_{r+1} = \binom{n}{r} a^{n-r} b^r$, for $r = 0, 1, \dots, n$.
8. The fifth term has $r = 4$: $\binom{8}{4}(2x)^4(3)^4 = 90720x^4$.
9. $\binom{6}{4}3^2 = 15 \cdot 9 = 135$.
10. Substitute $x = 1$: $(2 - 3)^5 = -1$.
11. $\binom{7}{5}2^2 = 21 \cdot 4 = 84$.
12. $n(n - 1)/2 = 45 \Rightarrow n = 10$.
13. There are 9 terms, so the middle term is $T_5 = \binom{8}{4}x^4 = 70x^4$.
14. $\binom{5}{2}(-3)^2 = 10 \cdot 9 = 90$.
15. $r = 4$ or $r = 8$.

Section 2 — Extended response

16. $(2 - 3x)^6 = 64 - 576x + 2160x^2 - 4320x^3 + \dots$.
17. $\binom{6}{2}k^2 = 240 \Rightarrow 15k^2 = 240 \Rightarrow k = 4$ since $k > 0$.
18. $na = 12$ and $\binom{n}{2}a^2 = 54$. Dividing gives $\frac{n-1}{2}a = \frac{54}{12} = \frac{9}{2}$. With $a = 12/n$, $6(n - 1)/n = 9/2$, so $n = 4$, $a = 3$.
19. General power of x : $x^{2(6-r)}x^{-r} = x^{12-3r}$. Set $12 - 3r = 0 \Rightarrow r = 4$. Constant term $= \binom{6}{4}2^4 = 240$.
20. General power is $x^{3(7-r)}x^{-r} = x^{21-4r}$. Set $21 - 4r = 5 \Rightarrow r = 4$. Coefficient $= \binom{7}{4}(-2)^4 = 560$.
21. $\binom{n}{3} = \binom{n}{2} \Rightarrow \frac{n-2}{3} = 1 \Rightarrow n = 5$.
22. Put $x = 1$: $(1 + 1)^n = \sum_{r=0}^n \binom{n}{r}$, hence the sum is 2^n .
23. Let E and O be the sums of even- and odd-indexed coefficients. From $x = 1$, $E + O = 2^n$. From $x = -1$, $E - O = 0$. Thus $E = O = 2^{n-1}$.
24. $A = \binom{8}{3}2^3 = 448$, $B = \binom{8}{4}2^4 = 1120$, so $B : A = 5 : 2$.
25. $a^4 = 16$ and $a > 0 \Rightarrow a = 2$. Then $4a^3b = -96 \Rightarrow b = -3$. Coefficient of x^3 : $4ab^3 = 4(2)(-27) = -216$.
26. $\binom{n}{2} = 6\binom{n}{1} \Rightarrow n(n - 1)/2 = 6n \Rightarrow n = 13$.
27. By symmetry, $\binom{12}{5} = \binom{12}{7}$. Their common value is 792.
28. The central coefficient is greatest: $\binom{12}{6} = 924$, attached to x^6 .
29. For x^7 , choose three factors of -1 : $T_4 = \binom{10}{3}(2x)^7(-1)^3 = -15360x^7$.
30. $\binom{n}{2} + \binom{n}{1} = \frac{n(n-1)}{2} + n = \frac{n(n+1)}{2} = 66 \Rightarrow n = 11$.

Section 3 — IB exam-style / Paper 1

31. (a) $1 + 12x + 60x^2 + 160x^3 + \dots$. (b) $\binom{6}{4}2^4 = 240$.
32. (a) Coefficient of x : $5(2^4)a = 80a = 240 \Rightarrow a = 3$. (b) $\binom{5}{2}2^3(3^2) = 720$.
33. (a) 1, 7, 21, 35, 35, 21, 7, 1. (b) $x^7 - 14x^6 + 84x^5 - 280x^4 + \dots$.
34. $na = 12$, $\binom{n}{2}a^2 = 60$. Therefore $\frac{n-1}{2}a = 5$. Substitute $a = 12/n$: $6(n-1)/n = 5 \Rightarrow n = 6$, $a = 2$.
35. Power $x^{2(9-r)-r} = x^{18-3r}$. Set exponent 0: $r = 6$. Constant term $= \binom{9}{6}3^6 = 84 \cdot 729 = 61236$.
36. (a) Power $x^{3(5-r)-r} = x^{15-4r}$. Set $15 - 4r = 3 \Rightarrow r = 3$, so it is T_4 . (b) Coefficient $= \binom{5}{3}2^2(-1)^3 = -40$.
37. (a) $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n!}{(n-r)!r!} = \binom{n}{n-r}$. (b) $\binom{11}{4} = 330$, so $r = 4$ or 7.
38. (a) $P(1) = 3^7 = 2187$, $P(-1) = (-1)^7 = -1$. (b) If E and O are the even/odd coefficient sums, $E + O = 2187$, $E - O = -1$. Thus $E = 1093$, $O = 1094$.
39. (a) $\binom{n}{3} = 4\binom{n}{2} \Rightarrow \frac{n-2}{3} = 4 \Rightarrow n = 14$. (b) $\binom{14}{4} = 1001$.
40. (a) $\binom{13}{4} = 715$, hence $n = 13$. (b) By symmetry $\binom{13}{9} = \binom{13}{4} = 715$.
41. (a) $\binom{8}{5}(-1)^5 = -56$. (b) Positive coefficients are the even-power coefficients. Their sum is $2^7 = 128$.
42. (a) $\binom{9}{3}2^6 = 5376$. (b) Ratio $= \frac{\binom{9}{4}2^5}{\binom{9}{3}2^6} = \frac{3}{4}$. (c) 4032.
43. (a) $\binom{7}{2}m^2 = \binom{7}{3}m^3 \Rightarrow 21m^2 = 35m^3$. Since $m \neq 0$, $m = 3/5$. (b) $21(3/5)^2 = 189/25$.
44. (a) Odd-power terms have opposite signs in the two expansions, so they cancel. (b) $P(x) = 2 + 30x^2 + 30x^4 + 2x^6$.
45. $A = \binom{8}{2}3^6 = 20412$, $B = \binom{8}{6}3^2 = 252$. Thus $A : B = 81 : 1$.
46. (a) $\binom{8}{3}k^3 = 56k^3 = -1512 \Rightarrow k^3 = -27 \Rightarrow k = -3$. (b) $\binom{8}{4}(-3)^4 = 5670$.
47. Power $x^{6-r}x^{-2r} = x^{6-3r}$. Set exponent 0 $\Rightarrow r = 2$. Constant term $= \binom{6}{2}2^2 = 60$. The linear equation $6 - 3r = 0$ has a unique integer solution.
48. (a) Coefficient of x^4 is $5a = 15 \Rightarrow a = 3$. (b) $\binom{5}{2}a^3 = 10 \cdot 27 = 270$. (c) $a^5 = 243$.
49. (a) $\binom{n}{2} = 45 \Rightarrow n = 10$, consistent with $\binom{10}{3} = 120$. (b) $\binom{10}{4} = 210$. (c) $\binom{10}{5} = 252$.
50. (a) $\binom{8}{3} = \binom{8}{5} = 56$, so $r = 3, 5$. (b) Symmetry $\binom{n}{r} = \binom{n}{n-r}$. (c) $56(2^3) + 56(2^5) = 2240$.

Section 4 — AA HL challenge

51. $\binom{n}{4} = 5\binom{n}{2} \Rightarrow \frac{(n-2)(n-3)}{12} = 5$. Hence $(n-2)(n-3) = 60$, so $n = 9$. Greatest coefficients are the two central ones: $\binom{9}{4} = \binom{9}{5} = 126$.
52. Write $(1+x)^8(1-x)^5 = (1+x)^3(1-x^2)^5$. For x^6 , contributions are $j + 2k = 6$ with $0 \leq j \leq 3$: $(j, k) = (0, 3), (2, 2)$. Coefficient $= -\binom{5}{3} + \binom{5}{2}\binom{3}{2} = -10 + 30 = 20$.
53. This is exactly the expansion of $(2-3)^{10}$. Therefore the sum is $(-1)^{10} = 1$.
54. The coefficient of x^k in the product is obtained by choosing x^r from $(1+x)^m$ and x^{k-r} from $(1+x)^n$, giving $\sum_{r=0}^k \binom{m}{r}\binom{n}{k-r}$. On the right, the coefficient is $\binom{m+n}{k}$. Equating coefficients proves the identity.
55. Let the three coefficients be C_r, C_{r+1}, C_{r+2} . Then

$$\frac{n-r}{r+1} = \frac{15}{6} = \frac{5}{2}, \quad \frac{n-r-1}{r+2} = \frac{20}{15} = \frac{4}{3}.$$

Thus $2n = 7r + 5$ and $3n = 7r + 11$. Solving gives $r = 1$, $n = 6$.

56. Power $x^{2(12-r)-r} = x^{24-3r}$. Set exponent 0 $\Rightarrow r = 8$. Constant term $= \binom{12}{8} = 495$.

57. In the numerator, even-power terms cancel and odd-power terms double. To obtain x^4 after division by $2x$, use the x^5 term of the numerator. Its coefficient after cancellation/doubling is $2\binom{10}{5}$, so after dividing by $2x$ the required coefficient is $\binom{10}{5} = 252$.

Section 4 — AA HL challenge (continued)

58. Let the constant, x , x^2 coefficients be C_0, C_1, C_2 . Then

$$\frac{C_2 C_0}{C_1^2} = \frac{n-1}{2n} = \frac{720 \cdot 32}{240^2} = \frac{2}{5}.$$

Hence $5(n-1) = 4n \Rightarrow n = 5$. Then $a^5 = 32 \Rightarrow a = 2$, and $5a^4b = 240 \Rightarrow b = 3$.

59. (a) $c_r = \binom{12}{r} 2^{12-r}$, so

$$\frac{c_{r+1}}{c_r} = \frac{12-r}{r+1} \cdot \frac{1}{2} = \frac{12-r}{2(r+1)}.$$

(b) The ratio exceeds 1 while $12-r > 2r+2$, i.e. $r < 10/3$. Thus coefficients increase up to c_4 and then decrease. Greatest coefficient:

$$c_4 = \binom{12}{4} 2^8 = 495 \cdot 256 = 126720.$$

60. Since $r \binom{n}{r} = n \binom{n-1}{r-1}$,

$$\sum_{r=0}^n r \binom{n}{r} = n \sum_{r=1}^n \binom{n-1}{r-1} = n \sum_{s=0}^{n-1} \binom{n-1}{s} = n 2^{n-1}.$$

For $n = 8$, the value is $8 \cdot 2^7 = 1024$.