Graphs of Functions

IB Mathematics: Analysis and Approaches SL & HL

Section SL 2.3 • Drawing, sketching, graphs from context, and sums/differences

Learning goal	Construct clear, correctly labelled graphs from equations, tables, sketches, contexts and technology.
Syllabus link	IB AA SL and AA HL: **SL 2.3** (shared content).
Structure	15 short-response + 15 extended-response + 20 original IB-style Paper 1 questions + **10 AA HL-exclusive challenges**.
Technology	Sections 1 and 3: no GDC. Section 2: GDC where marked. HL section: technology permitted where marked.

Instructions. A *sketch* must have the correct shape and all significant features labelled; a *drawn* graph should use a suitable scale and accurately plotted points. Label both axes and any endpoints, intercepts, turning points or asymptotes that are relevant. Exact coordinates are preferred unless a decimal approximation is requested. The supplied axes are for use or reference; redraw on graph paper if more working space is needed.
GDC means that technology is expected. AA SL students complete Questions 1–50; AA HL students should also complete Questions 51–60. Answers begin on a new page after Question 60.

SECTION 1 SHORT RESPONSE

Questions 1–15 • Core fluency

1. Explain the difference between the IB command terms *sketch* and *draw* for a graph. Give one requirement common to both.

2. For $f(x) = 2x - 3$, explain what is meant by the graph $y = f(x)$, and state the coordinates of two points on it.

3. Sketch $y = x^2 - 4$. Label the vertex and all axis intercepts.

4. Sketch $y = -(x - 1)^2 + 4$. Label its vertex and intercepts with the axes.

5. Draw the graph of $y = x^2$ for $-2 \le x \le 2$ using the table. Use a consistent scale.

x	-2	-1	0	1	2
y	4	1	0	1	4

6. Sketch $f(x) = 2 - x$ for $-1 \le x \le 4$. Clearly mark both endpoints.

7. Sketch $y = |x + 1| - 2$ and label its vertex and x-intercepts.

8. Sketch $y = \dfrac{2}{x}$ for $x \neq 0$. Label the axes and any straight lines approached by the curve.

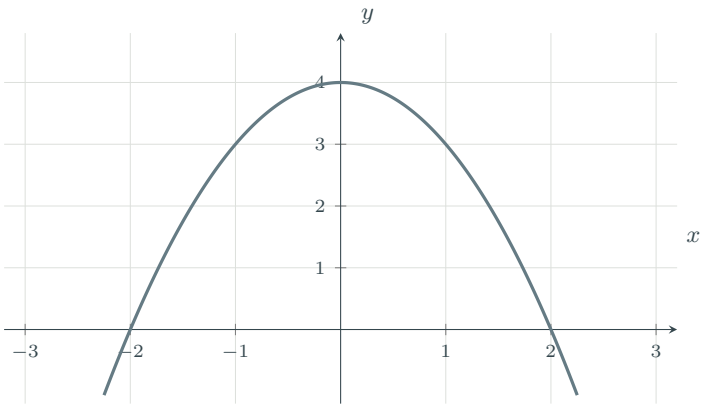
9. Sketch $y = 2^x$ for $-2 \leq x \leq 2$; show $(0, 1)$ and both endpoints.

10. Sketch $y = \sin x$ for $0 \leq x \leq 2\pi$. Label all intercepts and the maximum and minimum points.

11. Sketch $y = \sqrt{x + 1}$ for $-1 \leq x \leq 8$; label the starting point, the y-intercept and the right endpoint.

12. A bicycle-sharing service charges an unlock fee of €3 and then €2 per hour, for $0 \leq t \leq 5$. Sketch the total cost C (euros) against time t (hours). Label the intercept and the endpoint.

13. The graph of $y = f(x)$ is shown. Write down $f(0)$, $f(1)$, the zeros of f, and the interval on which $f(x) > 0$.



14. At $x = 1$, two functions satisfy $f(1) = 3$ and $g(1) = -2$. Find $(f + g)(1)$ and $(f - g)(1)$.

15. Two students sketch $y = x(x - 2)$. Student A draws an upward parabola passing through $(0, 0)$ and $(2, 0)$; Student B draws a downward parabola through the same points. State who is correct and justify using the equation.

SECTION 2 EXTENDED RESPONSE

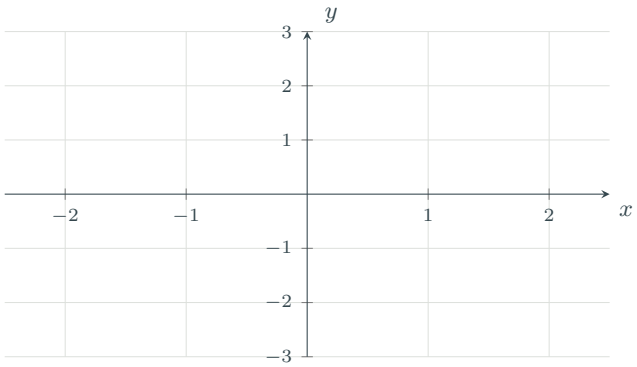
Questions 16–30 • Graphical reasoning and GDC

16. The function $f(x) = x^3 - 3x$ is defined for $-2 \leq x \leq 2$.

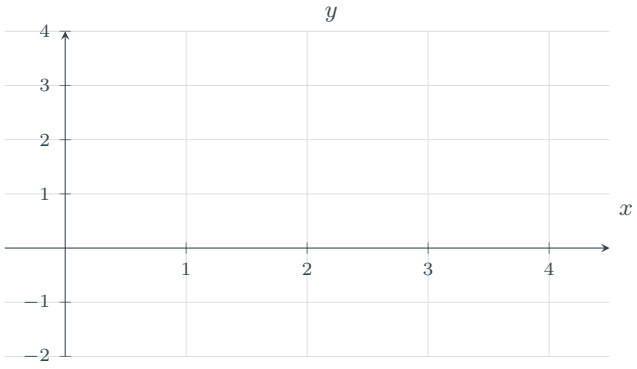
(a) Complete the table for $x = -2, -1, 0, 1, 2$.

(b) Draw a smooth graph of $y = f(x)$ using the table. Label its intercepts and endpoints.

x	-2	-1	0	1	2
$f(x)$			0		

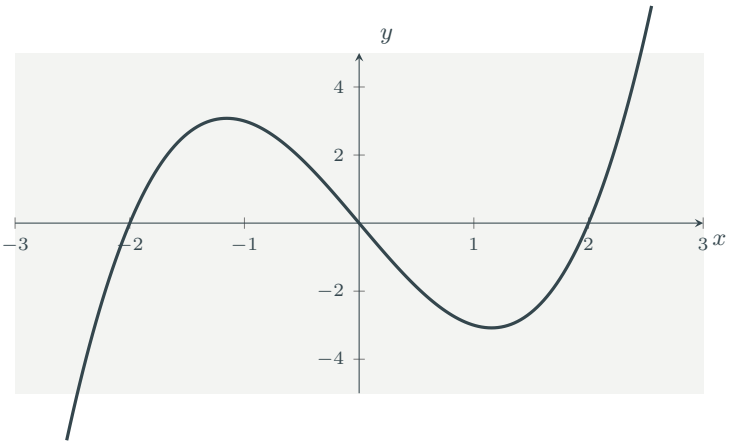


17. Sketch $y = (x-2)^2 - 1$ over $0 \le x \le 4$. Label the vertex, the x-intercepts, and the two endpoints.



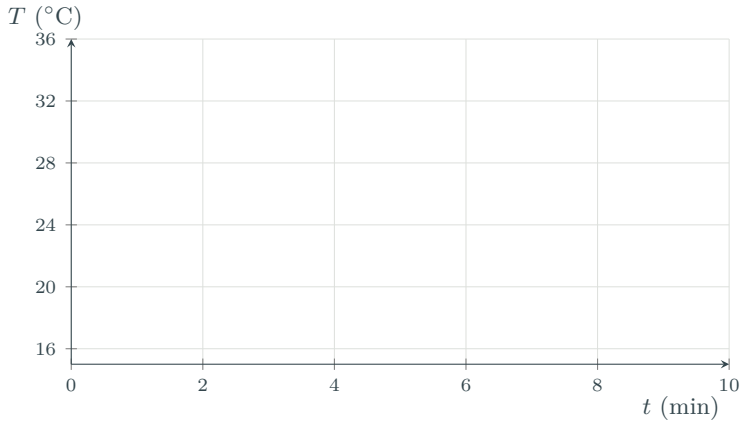
18. A student walks away from home at a constant speed for 4 minutes, rests for 2 minutes, and walks back at twice the outward speed, arriving home 2 minutes later. Sketch distance from home against time for $0 \le t \le 8$. The maximum distance reached is 300 m. Label key points and explain the slopes.

19. A GDC displays $y = x^3 - 4x$ in the window $-3 \le x \le 3$, $-5 \le y \le 5$. The screen image below has no labels on its intercepts. Transfer the curve to a clearly labelled sketch, marking all x-intercepts and the y-intercept. GDC



20. The table shows a measured temperature T in °C over a 10-minute interval. *Draw* the graph using accurately plotted points joined with straight segments (no smoothing). State two limitations of predicting temperatures between readings from this graph.

t (min)	0	2	4	6	8	10
T (°C)	18	22	29	33	31	26



21. Let $f(x) = x^2 - 2$ and $g(x) = x + 1$, for $-3 \leq x \leq 3$. GDC

(a) Graph f and g on the same axes and label both graphs.

(b) Graph $h = f + g$ and $k = f - g$ in a suitable window.

(c) Write down the equations of h and k, and label each y-intercept and vertex on your sketches.

22. Let $f(x) = \sin x$ and $g(x) = \cos x$ for $0 \leq x \leq 2\pi$. GDC

(a) Graph $h(x) = f(x) + g(x)$ and $k(x) = f(x) - g(x)$.

(b) State the x-intercepts of both curves in $[0, 2\pi]$.

(c) Explain why the two curves have the same maximum y-value.

23. The equation $y = \sqrt{9 - x^2}$ is defined for $-3 \leq x \leq 3$.

(a) Draw a table of values at $x = -3, -\sqrt{5}, 0, \sqrt{5}, 3$.

(b) Hence sketch the graph, with the endpoints and maximum labelled.

(c) Describe the shape using a familiar geometrical object.

24. On $-3 \leq x \leq 3$, let $f(x) = |x| - 1$ and $g(x) = 2 - x$. GDC

(a) Graph f and g together.

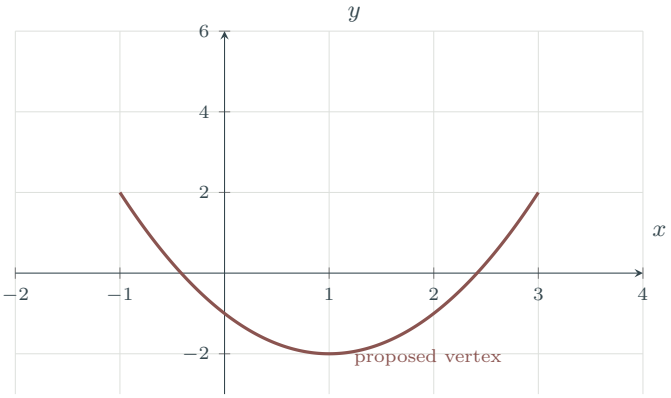
(b) Use the GDC to graph $d(x) = f(x) - g(x)$.

(c) Find the x-intercept of d in this interval and explain what it means for the graphs of f and g.

25. A student graphs $f(x) = 0.02x^3 - x$ in the window $-2 \le x \le 2$, $-2 \le y \le 2$ and reports seeing almost a straight line. GDC

Suggest an improved window that reveals the overall turning behaviour. Explain, with approximate coordinates, what the student missed.

26. A student proposes the following sketch for $y = (x-1)^2 + 2$. Identify *two* mathematical errors and describe how to correct the sketch.



27. Without graphing technology, take $f(x) = x^2$ and $g(x) = -x$ for $-2 \le x \le 2$.

(a) Sketch f and g on one set of axes.

(b) Sketch $p = f + g$ and $q = f - g$ on separate axes.

(c) Label the zeros and vertex of each of p and q.

28. The height (m) of a ball above ground is modelled by $h(t) = 1 + 6t - t^2$, $0 \le t \le 6$ (seconds). GDC

Sketch h against t, labelling the endpoints and the greatest height. Describe what the graph tells you about the ball's movement.

29. A graph is described by the line segments joining $(-2, -1)$ to $(0, 3)$, and $(0, 3)$ to $(3, 0)$; the point at $x = -2$ is *open*, and the point at $x = 3$ is *closed*.

(a) Draw the described graph, including the correct open and filled circles.

(b) State the domain, the range, and the value of $f(0)$.

30. Use technology to sketch $y = \dfrac{x^3}{1 + x^2}$ on $-4 \le x \le 4$. Label the origin and endpoints, state whether the graph lies above or below the x-axis for positive x, and explain why choosing only $-0.2 \le x \le 0.2$ gives a misleading impression. GDC

SECTION 3 IB-STYLE PAPER 1 PRACTICE

Questions 31–50 • No calculator

31. Consider $f(x) = (x+1)^2 - 4$, $x \in \mathbb{R}$.

(a) Write down the coordinates of the vertex. [1 marks]

(b) Find the intercepts with the coordinate axes. [2 marks]

(c) Sketch $y = f(x)$, clearly labelling these features. [2 marks]

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32. A function is defined by $f(x) = (x+2)(x-1)(x-3)$, $x \in \mathbb{R}$.

(a) Write down all zeros of f. [2 marks]

(b) Find the y-intercept. [1 marks]

(c) Sketch a possible graph, showing the correct cubic end-behaviour and labelling the intercepts. [3 marks]

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33. The function $f(x) = |2x - 4| - 3$ is defined for $0 \le x \le 5$.

(a) Find the coordinates of the vertex and endpoints of the graph. [3 marks]

(b) Hence sketch $y = f(x)$ and label its x-intercepts. [3 marks]

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34. The function g is defined by $g(x) = \sqrt{5 - x}$ for $1 \le x \le 5$.

(a) Find the coordinates of both endpoints. [2 marks]

(b) Sketch $y = g(x)$, showing its shape and the endpoints. [2 marks]

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35. Let $f(x) = -\dfrac{3}{2}x + 3$ for $-2 \le x \le 4$.

(a) Find its x- and y-intercepts. [2 marks]

(b) Draw the line segment and show the exact coordinates of its endpoints. [3 marks]

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36. The percentage charge B of a portable battery varies with time t (hours): at $t = 0$, $B = 20$; it rises linearly to 80 at $t = 3$, remains at 80 until $t = 5$, then decreases linearly to 40 at $t = 9$.

(a) Draw the graph of B against t for $0 \le t \le 9$, labelling all four key points. [4 marks]

(b) State the rate of change of B during each of the three intervals. [3 marks]

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37. A continuous function f on $-3 \le x \le 4$ passes through $(-3, -2)$, $(-1, 0)$, $(0, 3)$, $(2, 1)$, $(3, 0)$ and $(4, -1)$. It increases for $-3 \le x \le 0$ and decreases for $0 \le x \le 4$.

(a) Sketch a possible smooth graph, with the two zeros and maximum labelled. [4 marks]

(b) Explain why a sketch with a minimum at $(2, 1)$ would be inconsistent with the information. [2 marks]

38. Consider $f(x) = \dfrac{2}{x}$, $x \neq 0$.

(a) Find $f(-2)$ and $f(1)$. [2 marks]

(b) Sketch the graph, labelling both axes, these points and any asymptotes. [3 marks]

39. The function f is defined by $f(x) = 3^x - 1$ for $-2 \leq x \leq 2$.

(a) Find $f(0)$, $f(1)$ and $f(2)$. [2 marks]

(b) Sketch $y = f(x)$ for the stated domain, with the endpoints labelled. Show the horizontal level approached by the curve as x decreases beyond the displayed interval. [3 marks]

40. Consider $f(x) = x^2 - 3x - 4$ for $-1 \leq x \leq 4$.

(a) Complete a table of $f(x)$ for integer x from -1 to 4. [3 marks]

(b) Draw the graph, using the table and a suitable scale; label its intercepts. [3 marks]

(c) Estimate the vertex from the graph. [1 marks]

41. Let $f(x) = x^2 - 1$ and $g(x) = 2x + 1$. A new function is defined by $h = f + g$.

(a) Show that $h(x) = x^2 + 2x$. [1 marks]

(b) Write $h(x)$ in vertex form. [2 marks]

(c) Sketch $y = h(x)$, labelling its vertex and intercepts. [3 marks]

42. Let $f(x) = x^2 - 4$ and $g(x) = x - 2$. A function is defined by $h = f - g$.

(a) Show that $h(x) = x^2 - x - 2$. [1 marks]

(b) Find the zeros of h. [2 marks]

(c) Sketch $y = h(x)$ and label the vertex. [3 marks]

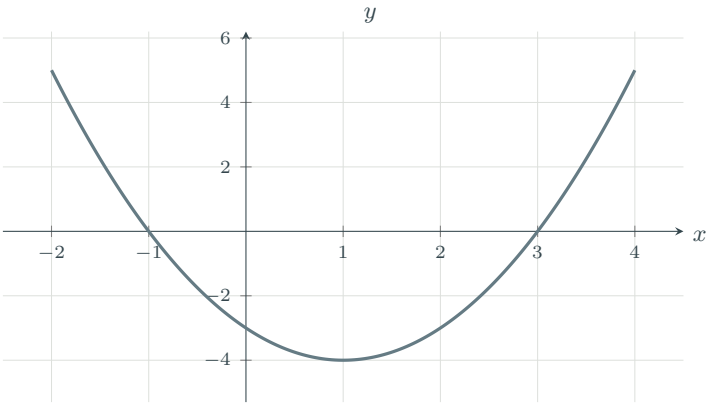
43. The function f is defined by

$$f(x) = \begin{cases} x + 2, & -3 \leq x < 0, \\ 3 - x, & 0 \leq x \leq 3. \end{cases}$$

(a) Sketch $y = f(x)$, showing any open/closed endpoint at $x = 0$. [4 marks]

(b) State $f(0)$ and all x-intercepts. [2 marks]

44. The graph of $y = f(x)$ is shown below, for $-2 \leq x \leq 4$.



(a) Read off the coordinates of the vertex and all intercepts. [3 marks]

(b) Make a fresh sketch, labelling the two endpoints as well. [2 marks]

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45. The function $f(x) = 2\cos x$ is defined for $-\pi \leq x \leq \pi$.

(a) Find the values of $f(-\pi)$, $f(0)$ and $f(\pi)$. [2 marks]

(b) Sketch $y = f(x)$ and show its x-intercepts. [3 marks]

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46. The area of a circular disc is modelled by $A(r) = \pi r^2$, for $0 \leq r \leq 3$, where r is its radius.

(a) Calculate $A(0)$, $A(1)$, $A(2)$ and $A(3)$. [2 marks]

(b) Sketch A against r, labelling the axes and units. Explain why the graph is curved rather than straight. [3 marks]

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47. A cubic function is defined by $f(x) = (x+1)(x-2)^2$.

(a) Write down the zeros of f and their multiplicities. [2 marks]

(b) Find the y-intercept. [1 marks]

(c) Sketch $y = f(x)$, showing that the curve crosses at one zero and touches the axis at the other. [3 marks]

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48. A cyclist is initially at home. They cycle 2 km away in 20 minutes at a constant speed, stop for 10 minutes, and then return home at a constant speed in 15 minutes.

(a) Sketch the cyclist's distance from home d (km) against time t (min), labelling the turning points and endpoint. [4 marks]

(b) Explain why the final line segment is steeper than the first. [2 marks]

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49. Let $f(x) = \sin x$ and $g(x) = -\sin x$, for $0 \leq x \leq 2\pi$.

(a) Show that $(f+g)(x) = 0$. [1 marks]

(b) Find $(f-g)(x)$. [1 marks]

(c) Sketch $y = (f-g)(x)$, labelling all axis intercepts, the maximum and the minimum. [3 marks]

50. Let $f(x) = x + 3$ and $g(x) = x^2$, for $-3 \le x \le 2$. Define $h = f + g$.

(a) Find $h(x)$ and express it in vertex form. [3 marks]

(b) Hence sketch $y = h(x)$, showing the endpoints, the y-intercept and the minimum. [3 marks]

(c) Explain why the graph has no x-intercepts. [2 marks]

SECTION 4 AA HL-EXCLUSIVE CHALLENGE

Questions 51–60 • HL only

AA HL extension of SL 2.3. These problems test harder graph construction and combinations of functions. They develop the shared sketching standard rather than requiring additional AHL calculus methods. A GDC may be used only where explicitly marked.

51. The function $f(x) = x^4 - 5x^2 + 4$ is defined for $-3 \le x \le 3$. GDC

(a) Factorise $f(x)$ into linear factors and label its four x-intercepts.

(b) Use a graph to sketch its shape, showing its symmetry and three turning points (exact coordinates for the two lower points may be left as surds).

52. Let $f(x) = x^2 - 4$ and $g(x) = |x|$. Define $h = f + g$.

(a) Write a piecewise formula for h on $x < 0$ and $x \ge 0$.

(b) Sketch h, identifying its corner and both zeros exactly.

53. Graph $h(x) = \dfrac{x^3}{1 + x^2}$ for $-5 \le x \le 5$. GDC

Label $h(-1)$, $h(0)$, $h(1)$ and the endpoints. Explain how the equation demonstrates the rotational symmetry visible on the graph.

54. Let $f(x) = \sqrt{x + 2}$ and $g(x) = \sqrt{2 - x}$ for $-2 \le x \le 2$. GDC

(a) Sketch $d(x) = f(x) - g(x)$, including its endpoints and its only zero.

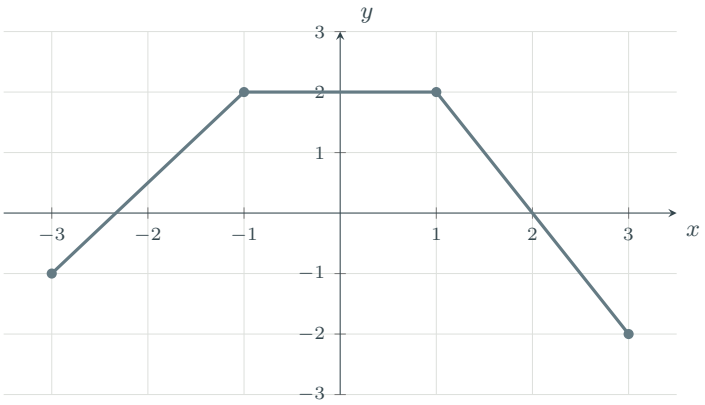
(b) Show algebraically that $d(-x) = -d(x)$.

55. Consider $f(x) = (x + 1)^2(x - 2)(x - 3)$, $x \in \mathbb{R}$.

(a) Identify the zeros at which the graph crosses and touches the x-axis.

(b) Sketch a possible graph, labelling its y-intercept and indicating its sign in each interval separated by the zeros.

56. The graph of $y = f(x)$ consists of straight line segments joining the four vertices shown below, in order. Let $g(x) = \frac{1}{2}x$ and $h = f + g$ on $[-3, 3]$.



(a) Find the coordinates of the four vertices of $y = h(x)$.

(b) Sketch h, correctly joining its vertices with straight segments.

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57. Let $p(x) = \sin(6x) + \dfrac{x}{5}$ for $0 \le x \le 2\pi$. GDC

Choose and justify a GDC window that shows all the oscillations without clipping the graph. Sketch the overall behaviour and label the two endpoints. Explain why $0 \le x \le \pi/3$ shows an incomplete picture of the function.

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58. Let $f(x) = e^x$ and $g(x) = e^{-x}$; define $h = f - g$.

(a) Show that h is an odd function.

(b) Sketch h for $-2 \le x \le 2$, labelling the intercept and the values at $x = \pm 1$.

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59. The function $f(x) = \dfrac{x^2 - 1}{x - 1}$ is defined for $-2 \le x \le 4$, $x \neq 1$.

(a) Simplify the expression for every x in the domain.

(b) Draw the graph, showing the excluded point clearly as an open circle.

(c) State the domain and range of the function.

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60. Let $f(x) = x^2 - 2x + k$ and $g(x) = x + 1$, where k is a constant. The graph of $y = f(x) - g(x)$ touches the x-axis at exactly one point.

(a) Find k and the coordinates of this point.

(b) Sketch $f - g$ for this value of k and explain what this tells you about the two original graphs.

Answer key and solution outlines

IB AA SL/HL • SL 2.3 • Original practice questions 1–60

Section 1 Short response (1–15)

1. *Sketch*: correct qualitative shape and labelled key features, without needing a precise scale. *Draw*: accurate plotting using a suitable scale. Both need clear axes and labels.

2. $y = f(x)$ is the set of all ordered pairs $(x, 2x - 3)$ in the domain. Examples: $(0, -3)$ and $(2, 1)$.

3. Upward-opening parabola; vertex $(0, -4)$; x-intercepts $(-2, 0), (2, 0)$; y-intercept $(0, -4)$.

4. Downward-opening parabola; vertex $(1, 4)$; x-intercepts $(-1, 0), (3, 0)$; y-intercept $(0, 3)$.

5. Correctly plot and join $(-2, 4), (-1, 1), (0, 0), (1, 1), (2, 4)$ smoothly; endpoints filled.

6. Straight line from the filled point $(-1, 3)$ through $(0, 2)$ and $(2, 0)$ to the filled point $(4, -2)$.

7. V-shaped graph opening upward; corner $(-1, -2)$; zeros $(-3, 0)$ and $(1, 0)$; y-intercept $(0, -1)$.

8. Branches in quadrants I and III, through $(1, 2), (2, 1), (-1, -2), (-2, -1)$. Dashed asymptotes $x = 0$ and $y = 0$.

9. Increasing exponential segment through $(0, 1)$, with filled endpoints $(-2, \frac{1}{4})$ and $(2, 4)$.

10. Zeros at $x = 0, \pi, 2\pi$; maximum $(\frac{\pi}{2}, 1)$; minimum $(\frac{3\pi}{2}, -1)$; filled endpoints.

11. Square-root curve starts at $(-1, 0)$, passes through $(0, 1)$ and ends at $(8, 3)$; all marked as appropriate.

12. $C = 3 + 2t$, $0 \leq t \leq 5$; straight line joining $(0, 3)$ and $(5, 13)$. Units: t in hours; C in euros.

13. $f(0) = 4$, $f(1) = 3$, zeros $x = -2, 2$; $f(x) > 0$ for $-2 < x < 2$.

14. $(f + g)(1) = 3 + (-2) = 1$; $(f - g)(1) = 3 - (-2) = 5$.

15. Student A. The leading coefficient of $x^2 - 2x$ is positive, so the parabola opens upward; its zeros are indeed 0 and 2.

Section 2 Extended response (16–30)

16. (a) The missing values, in increasing x order, are $-2, 2, -2, 2$. (b) $f(x) = x(x^2 - 3)$; x-intercepts at $x = -\sqrt{3}, 0, \sqrt{3}$, filled endpoints $(-2, -2)$ and $(2, 2)$. A smooth odd cubic with one local high and one local low between the endpoints.

17. Vertex $(2, -1)$; roots $x = 1, 3$; endpoints $(0, 3)$ and $(4, 3)$. Symmetric upward parabola.

18. Join $(0, 0) \to (4, 300) \to (6, 300) \to (8, 0)$ with straight line segments. Gradients $75, 0, -150$ m/min respectively; the final magnitude is twice the first.

19. $x(x - 2)(x + 2) = 0$, so the three x-intercepts are $(-2, 0), (0, 0), (2, 0)$; the y-intercept is also $(0, 0)$. Copy the smooth S-shaped cubic, not the pixel resolution of the screen.

20. Plot $(0, 18), (2, 22), (4, 29), (6, 33), (8, 31), (10, 26)$ on labelled axes; join consecutive points by straight segments. The true temperature may vary non-linearly between readings; the graph contains no information about sudden fluctuations or measurement error between samples.

21. $h(x) = x^2 + x - 1$ has y-intercept $(0, -1)$ and vertex $(-\frac{1}{2}, -\frac{5}{4})$. $k(x) = x^2 - x - 3$ has y-intercept $(0, -3)$ and vertex $(\frac{1}{2}, -\frac{13}{4})$. Both open upward; each is drawn only on $[-3, 3]$. The GDC should show four labelled curves in total, using separate views if needed.

22. $h = \sin x + \cos x = \sqrt{2}\sin(x + \frac{\pi}{4})$: zeros $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. $k = \sin x - \cos x = \sqrt{2}\sin(x - \frac{\pi}{4})$: zeros $x = \frac{\pi}{4}, \frac{5\pi}{4}$. The maximum of each is $\sqrt{2}$, because both have amplitude $\sqrt{2}$.

23. (a) y-values $0, 2, 3, 2, 0$. (b) Endpoints $(-3, 0)$ and $(3, 0)$; maximum $(0, 3)$. (c) The upper semicircle of $x^2 + y^2 = 9$, radius 3.

24. $d(x) = |x| + x - 3 = \begin{cases} -3, & x < 0, \\ 2x - 3, & x \geq 0. \end{cases}$ Its only x-intercept is $x = \frac{3}{2}$. This means $f(\frac{3}{2}) = g(\frac{3}{2}) = \frac{1}{2}$; the graphs meet at $(\frac{3}{2}, \frac{1}{2})$.

25. A suitable window is $-8 \leq x \leq 8$, $-5 \leq y \leq 5$. The turning points occur when $0.06x^2 - 1 = 0$, i.e. $x = \pm\sqrt{50/3} \approx \pm 4.08$; they are approximately $(-4.08, 2.72)$ and $(4.08, -2.72)$. A narrow x-window hides both.

26. The proposed vertex is at $(1, -2)$ instead of $(1, 2)$, and the proposed curve crosses the x-axis, whereas $(x-1)^2 + 2 \geq 2$. The corrected graph is an upward parabola with minimum $(1, 2)$ and no real x-intercept.

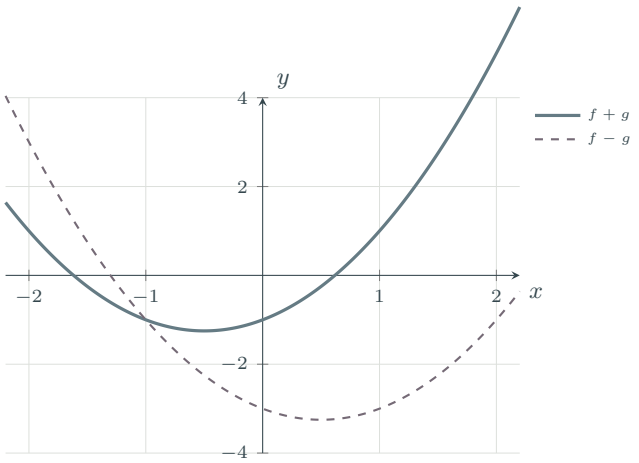
27. $p = x^2 - x = x(x-1)$: roots $0, 1$; vertex $(\frac{1}{2}, -\frac{1}{4})$. $q = x^2 + x = x(x+1)$: roots $-1, 0$; vertex $(-\frac{1}{2}, -\frac{1}{4})$. Both open upward and are restricted to $[-2, 2]$.

28. $h(t) = -(t-3)^2 + 10$. Endpoints $(0, 1), (6, 1)$; maximum $(3, 10)$. The ball rises for $0 < t < 3$, reaches its greatest height at $t = 3$, then descends for $3 < t < 6$; its height remains positive.

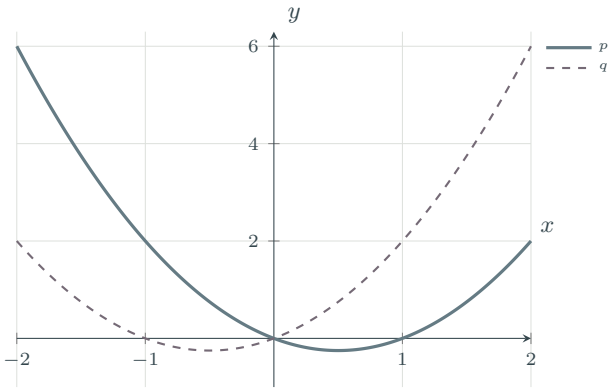
29. (a) Use straight segments with an open circle at $(-2, -1)$ and filled circles at $(0, 3), (3, 0)$. (b) Domain $(-2, 3]$; range $(-1, 3]$; $f(0) = 3$.

30. $h(0) = 0$, $h(4) = \frac{64}{17} \approx 3.76$, $h(-4) = -\frac{64}{17}$. Since numerator and denominator have the same sign as x (the denominator is positive), $h(x) > 0$ for $x > 0$. Near zero, $h(x) \approx x^3$ looks almost horizontal; a wider interval reveals its steadily increasing, almost linear outer portions.

Selected sketch check: Q21



Selected sketch check: Q27



Section 3 IB-style Paper 1 (31–50)

31. (a) $(-1, -4)$. (b) $(0, -3)$ and $(-3, 0), (1, 0)$, since $(x+1)^2 = 4$. (c) Upward-opening parabola through the labelled points.

32. (a) $x = -2, 1, 3$. (b) $f(0) = (2)(-1)(-3) = 6$, so $(0, 6)$. (c) Positive leading cubic coefficient: curve enters from bottom left, crosses each zero, exits top right.

33. (a) Vertex $(2, -3)$; endpoints $(0, 1), (5, 3)$. (b) Solve $|2x - 4| = 3$: $2x - 4 = \pm 3$, giving $x = \frac{1}{2}, \frac{7}{2}$. Join the points with two straight line segments.

34. (a) $g(1) = 2$, $g(5) = 0$; endpoints $(1, 2), (5, 0)$. (b) Decreasing curved arc with negative gradient, becoming steeper towards $(5, 0)$; no extension outside the given domain.

35. (a) y-intercept $(0, 3)$; x-intercept $(2, 0)$. (b) Endpoints $(-2, 6)$ and $(4, -3)$; a straight line segment between them.

36. (a) Join $(0, 20) \to (3, 80) \to (5, 80) \to (9, 40)$ with straight segments. (b) Rise: $(80 - 20)/3 = 20$ percentage points/hour; plateau: 0; decline: $(40 - 80)/(9 - 5) = -10$ percentage points/hour.

37. (a) The highest point is $(0, 3)$; zeros are $(-1, 0)$ and $(3, 0)$. The graph rises from $(-3, -2)$ to $(0, 3)$ and falls to $(4, -1)$, passing through $(2, 1)$. (b) A local minimum at $x = 2$ would require a transition from decreasing to increasing, contradicting the stated decreasing interval.

38. (a) $f(-2) = -1$, $f(1) = 2$. (b) Two hyperbola branches through $(-2, -1), (1, 2)$ with dashed asymptotes

$x = 0$ and $y = 0$.

39. (a) $f(0) = 0$, $f(1) = 2$, $f(2) = 8$. (b) Increasing exponential with left endpoint $(-2, -\frac{8}{9})$ and right endpoint $(2, 8)$, approaching the horizontal asymptote $y = -1$ when extended left.

40. (a) Values for $x = -1, 0, 1, 2, 3, 4$ are $0, -4, -6, -6, -4, 0$. (b) Smooth upward parabola with zeros $(-1, 0), (4, 0)$ and y-intercept $(0, -4)$. (c) Vertex at approximately $(1.5, -6.25)$; in fact $f(x) = (x - \frac{3}{2})^2 - \frac{25}{4}$.

41. (a) $h = x^2 - 1 + 2x + 1 = x^2 + 2x$. (b) $h = (x + 1)^2 - 1$. (c) Vertex $(-1, -1)$; roots $x = -2, 0$; y-intercept $(0, 0)$.

42. (a) $h = (x^2 - 4) - (x - 2) = x^2 - x - 2$. (b) $h = (x - 2)(x + 1)$, giving $x = -1, 2$. (c) $h = (x - \frac{1}{2})^2 - \frac{9}{4}$, so vertex $(\frac{1}{2}, -\frac{9}{4})$, y-intercept $(0, -2)$.

43. (a) First segment joins the filled point $(-3, -1)$ to an *open* point $(0, 2)$; second joins the *filled* point $(0, 3)$ to the filled point $(3, 0)$. (b) $f(0) = 3$; x-intercepts $(-2, 0)$ and $(3, 0)$. The gap at $x = 0$ is intentional.

44. (a) Vertex $(1, -4)$; intercepts $(-1, 0), (3, 0)$ and $(0, -3)$. (b) Same upward-opening parabola with filled endpoints $(-2, 5)$ and $(4, 5)$.

45. (a) $f(-\pi) = -2$, $f(0) = 2$, $f(\pi) = -2$. (b) Cosine arch from $(-\pi, -2)$ up through $(-\frac{\pi}{2}, 0)$ to $(0, 2)$, then through $(\frac{\pi}{2}, 0)$ to $(\pi, -2)$.

46. (a) $0, \pi, 4\pi, 9\pi$ square units at radii $0, 1, 2, 3$. (b) Plot r (units of length) on the horizontal axis and A (square units) vertically. The upward curve is quadratic: increasing r by a fixed amount does not increase A by a fixed amount.

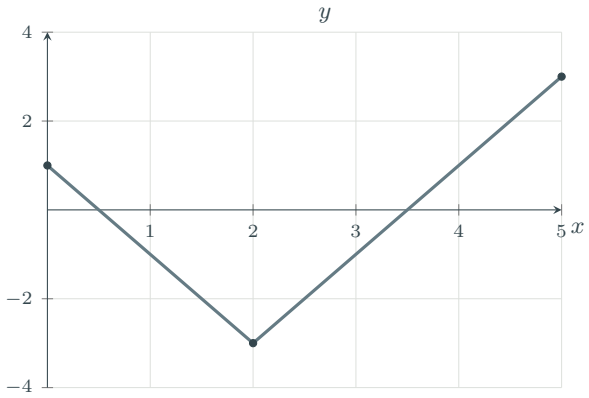
47. (a) $x = -1$ is a simple zero (crossing); $x = 2$ is a double zero (touching). (b) $f(0) = 4$. (c) Cubic curve enters from below at left, crosses at -1, touches the axis at 2 while remaining nonnegative nearby, and leaves towards $+\infty$.

48. (a) Join $(0, 0) \to (20, 2) \to (30, 2) \to (45, 0)$ on labelled t (min), d (km) axes. (b) Outward slope $2/20 = 0.1$ km/min; return slope $-2/15 \approx -0.133$ km/min. The magnitude is greater on the return.

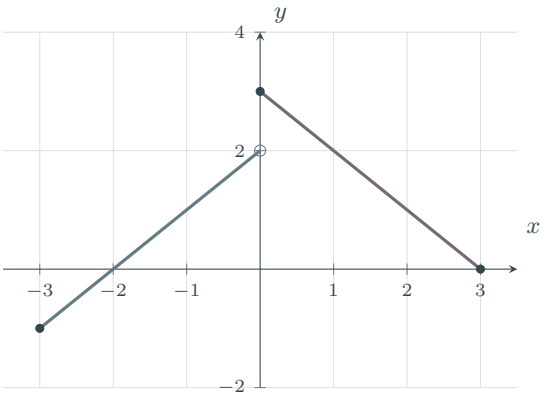
49. (a) $\sin x + (-\sin x) = 0$. (b) $(f - g)(x) = 2 \sin x$. (c) Zeros $(0, 0), (\pi, 0), (2\pi, 0)$; maximum $(\frac{\pi}{2}, 2)$; minimum $(\frac{3\pi}{2}, -2)$.

50. (a) $h = x^2 + x + 3 = (x + \frac{1}{2})^2 + \frac{11}{4}$. (b) Minimum $(-\frac{1}{2}, \frac{11}{4})$, y-intercept $(0, 3)$, endpoints $(-3, 9)$ and $(2, 9)$. (c) As a sum of a square and a positive constant, $h(x) \geq \frac{11}{4} > 0$ for every real x; thus no x-intercepts.

Selected sketch check: Q33



Selected sketch check: Q43



Section 4 AA HL-exclusive challenge (51–60)

51. (a) $f = (x - 2)(x - 1)(x + 1)(x + 2)$, giving zeros $\pm 1, \pm 2$. (b) $f(-x) = f(x)$, so it is even. Stationary points are $(0, 4)$ and $\left(\pm\sqrt{\frac{5}{2}}, -\frac{9}{4}\right)$; the endpoints are $(-3, 40), (3, 40)$. The graph is W-shaped, symmetric about the y-axis.

52. (a) $h(x) = \begin{cases} x^2 - x - 4, & x < 0, \\ x^2 + x - 4, & x \geq 0. \end{cases}$ (b) Both branches rise away from the sharp minimum $(0, -4)$. If $u = |x|$, then $u^2 + u - 4 = 0$, so $u = (\sqrt{17} - 1)/2$ and the zeros are $x = \pm(\sqrt{17} - 1)/2$.

53. $h(-1) = -\frac{1}{2}$, $h(0) = 0$, $h(1) = \frac{1}{2}$; endpoints $(-5, -\frac{125}{26}), (5, \frac{125}{26})$. Since $h(-x) = -\frac{x^3}{1+x^2} = -h(x)$, the graph has rotational symmetry about the origin. It increases through the labelled points.

54. (a) Endpoints $(-2, -2)$ and $(2, 2)$, with unique zero $(0, 0)$. The graph is smooth, strictly increasing and origin-symmetric. (b) $d(-x) = \sqrt{2-x} - \sqrt{2+x} = -[\sqrt{x+2} - \sqrt{2-x}] = -d(x)$.

55. (a) Touches at $x = -1$ (double zero), crosses at $x = 2$ and $x = 3$ (simple zeros). (b) y-intercept $(0, 6)$. Sign is $+$ for $x < -1$ and $-1 < x < 2$, $-$ for $2 < x < 3$, $+$ for $x > 3$; the quartic rises to $+\infty$ at both ends.

56. (a) At the four x-coordinates, add $x/2$ to the given ordinate: $(-3, -\frac{5}{2})$, $(-1, \frac{3}{2})$, $(1, \frac{5}{2})$, $(3, -\frac{1}{2})$. (b) Join these in order with straight segments; no extra curve should appear between vertices because both f and g are piecewise linear on those intervals.

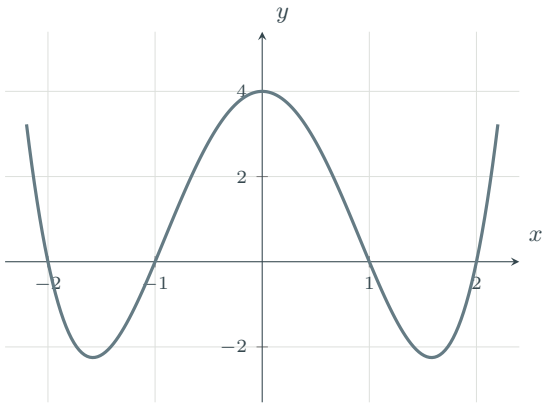
57. Use $0 \le x \le 2\pi$, $-1.5 \le y \le 2.5$, showing all six sinusoidal cycles superposed on a rising trend. Endpoints $(0, 0)$ and $(2\pi, \frac{2\pi}{5})$. The shorter interval $[0, \frac{\pi}{3}]$ displays just one of the six oscillations and hides the accumulating upward drift.

58. (a) $h(-x) = e^{-x} - e^{x} = -h(x)$, so h is odd. (b) $h(0) = 0$, $h(1) = e - e^{-1} \approx 2.35$, $h(-1) = -(e - e^{-1}) \approx -2.35$; the graph is increasing and origin-symmetric.

59. (a) $f(x) = \frac{(x-1)(x+1)}{x-1} = x + 1$ for $x \neq 1$ only. (b) The graph is the line segment $y = x + 1$ from filled $(-2, -1)$ to filled $(4, 5)$, with an open circle at $(1, 2)$. (c) Domain $[-2, 1) \cup (1, 4]$; range $[-1, 2) \cup (2, 5]$.

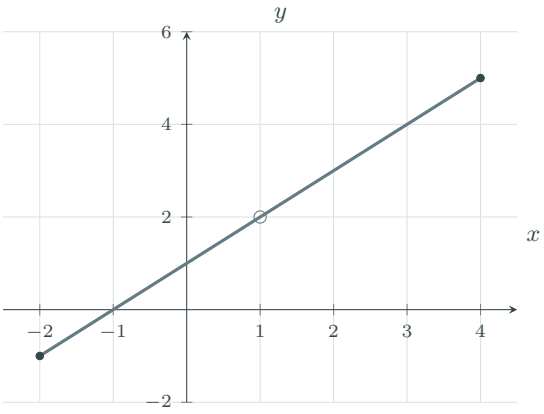
60. (a) $f - g = x^2 - 3x + k - 1 = (x - \frac{3}{2})^2 + k - \frac{13}{4}$. Touching at one point requires $k = \frac{13}{4}$, and the point on $y = f - g$ is $(\frac{3}{2}, 0)$. (b) It is an upward parabola with vertex $(\frac{3}{2}, 0)$. The original curves meet at $(\frac{3}{2}, \frac{5}{2})$ and are tangent there.

HL sketch check: Q51



Detail view near the four roots.

HL sketch check: Q59



Open circle at the excluded input.

Graph-check reminder. Equivalent sketches are acceptable when the shape, domain, endpoints, intercepts, asymptotes and labelled features agree with the answer. No particular artistic style is required.