

# Composite and Inverse Functions

IB Mathematics AA SL & HL — Worksheet 2.5

60 questions: composition, domain and range, inverses and identity functions

|                      |                                                                                                                         |
|----------------------|-------------------------------------------------------------------------------------------------------------------------|
| <b>Learning goal</b> | Combine and reverse functions, tracking allowed inputs and outputs.                                                     |
| <b>Syllabus link</b> | AA SL 2.5, shared by SL and HL. Inverse-graph reflection revisits SL 2.2. The HL challenges deepen this same content.   |
| <b>Structure</b>     | 1–15 short response; 16–30 extended response; 31–50 Paper 1 style; 51–60 AA HL exclusive challenge; answer key follows. |
| <b>Calculator</b>    | No calculator, except Questions 25, 26 and 28, marked <b>GDC</b> . Section 3 is entirely non-calculator.                |

**Instructions.** Work on separate paper, using the supplied axes for sketches. Show your reasoning and give exact answers unless an approximation is requested. Unless a domain is stated, use the maximal real domain. In  $f \circ g$ , apply  $g$  first. Mark allocations appear in square brackets.

## Section 1: Short response

Shared SL & HL

1. Let  $f(x) = 3x - 4$  and  $g(x) = x^2 + 2$ .

(a) Find  $f(g(2))$ .

(b) Find  $g(f(2))$ .

2. Let  $f(x) = 2x + 5$  and  $g(x) = x - 3$ .

(a) Find and simplify  $(f \circ g)(x)$ .

(b) Find and simplify  $(g \circ f)(x)$ .

3. The functions  $f$  and  $g$  have domain  $\{-1, 0, 1, 2, 3\}$ . Their values are shown.

(a) Find  $f(g(0))$ .

(b) Find  $g(f(1))$ .

(c) Find  $f(f(2))$ .

| $x$    | -1 | 0  | 1  | 2 | 3 |
|--------|----|----|----|---|---|
| $f(x)$ | 2  | 3  | -1 | 1 | 0 |
| $g(x)$ | 1  | -1 | 3  | 0 | 2 |

4. The diagram defines a function  $h$ . Let  $f(x) = x^2$  and  $g(x) = x - 5$ .

(a) Write  $h$  as a composition of  $f$  and  $g$ .

(b) Find an expression for  $h(x)$ .

(c) Find  $h(-2)$ .



**Section 1: Short response (continued)****Shared SL & HL**

5. Let  $f(x) = 5 - 2x$  and  $g(x) = x^2 - 1$ .

(a) Find  $(f \circ g)(x)$ .

(b) Solve  $(f \circ g)(x) = 0$ .

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6. Let  $f(x) = \sqrt{x+4}$  and  $g(x) = 3x - 1$ .

(a) Find  $(f \circ g)(x)$ .

(b) State the domain of  $f \circ g$ .

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7. Let  $f(x) = \frac{1}{x-2}$  and  $g(x) = x + 5$ .

(a) Find  $(f \circ g)(x)$ .

(b) State the value excluded from its domain.

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8. Let  $f(x) = x^2$ ,  $-3 \leq x \leq 1$ , and  $g(x) = x + 2$ ,  $-2 \leq x \leq 2$ .

(a) Find the domain of  $f \circ g$ .

(b) Find its range.

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9. Let  $f(x) = 4x - 7$ ,  $x \in \mathbb{R}$ .

(a) Find  $f^{-1}(x)$ .

(b) Find  $f^{-1}(9)$ .

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10. A one-to-one function  $f$  has domain  $[-2, 5]$  and range  $[1, 15]$ .

(a) State the domain of  $f^{-1}$ .

(b) State the range of  $f^{-1}$ .

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11. Let  $f(x) = x^3 + 6$ ,  $x \in \mathbb{R}$ .

(a) Find  $f^{-1}(x)$ .

(b) Find  $f^{-1}(-2)$ .

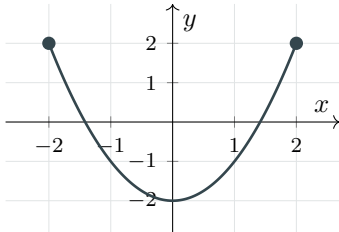
## Section 1: Short response (continued)

Shared SL &amp; HL

12. The graph shows  $f(x) = x^2 - 2$ ,  $-2 \leq x \leq 2$ .

(a) Explain why  $f$  has no inverse function on this domain.

(b) State the two largest subintervals of the given domain, each having an endpoint at 0, on which  $f$  is one-to-one.



Use the graph to justify your answer.

13. Let  $f(x) = (x + 2)^2 - 1$ ,  $x \geq -2$ .

(a) Find  $f^{-1}(x)$ .

(b) State the domain of  $f^{-1}$ .

14. The identity function on  $\mathbb{R}$  is denoted by  $I$ . Let  $f(x) = 3x + 2$ .

(a) Write an expression for  $I(x)$ .

(b) Find  $(I \circ f)(x)$ .

(c) Find  $(f \circ I)(x)$ .

15. Let  $f(x) = \frac{x - 6}{5}$  and  $g(x) = 5x + 6$ , both with domain  $\mathbb{R}$ .

(a) Find  $(f \circ g)(x)$ .

(b) Find  $(g \circ f)(x)$ .

(c) Explain what these results show about  $f$  and  $g$ .

## Section 2: Extended response

## Shared SL &amp; HL

16. Let  $f(x) = 2x + 1$  and  $g(x) = x^2 - 2$ .

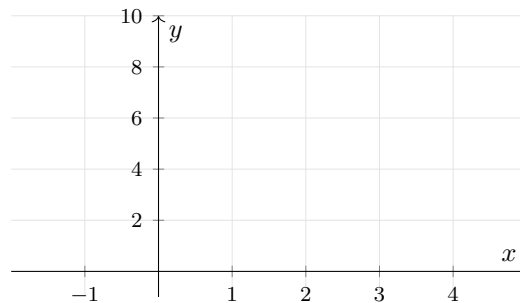
- Find and simplify  $(f \circ g)(x)$ .
- Find and simplify  $(g \circ f)(x)$ .
- Solve  $(f \circ g)(x) = (g \circ f)(x)$ .

17. Let  $f(x) = \sqrt{x-1}$  and  $g(x) = 5-x$ , on their maximal real domains.

- Find an expression for  $(f \circ g)(x)$ .
- Find the domain of  $f \circ g$ .
- Find an expression for  $(g \circ f)(x)$ .
- Find the domain of  $g \circ f$ .

18. Let  $f(x) = x^2$ ,  $-2 \leq x \leq 3$ , and  $g(x) = x - 1$ ,  $-1 \leq x \leq 4$ . Let  $h = f \circ g$ .

- Find an expression for  $h(x)$ .
- Find the domain of  $h$ .
- Find the range of  $h$ .
- Sketch  $y = h(x)$  on the supplied axes, labelling the vertex and both endpoints.



19. Let  $f(x) = \frac{2x+1}{x-3}$ ,  $x \neq 3$ .

- Find  $f^{-1}(x)$ .
- State the domain of  $f^{-1}$ .
- State the range of  $f^{-1}$ .
- State the equations of the asymptotes of  $y = f^{-1}(x)$ .

20. Let  $f(x) = (x-4)^2 + 2$ ,  $x \leq 4$ .

- Find  $f^{-1}(x)$ .
- Explain how the restriction on  $f$  determines your choice of square-root branch.
- State the domain of  $f^{-1}$ .
- State the range of  $f^{-1}$ .

## Section 2: Extended response (continued)

Shared SL &amp; HL

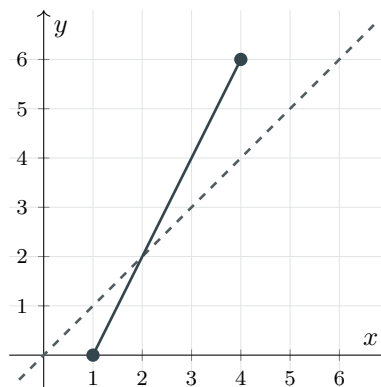
21. Functions  $f$  and  $g$  are bijections of  $\{0, 1, 2, 3, 4\}$  onto itself.

- Find  $f^{-1}(4)$ .
- Find  $g^{-1}(3)$ .
- Find  $(f \circ g)(2)$ .
- Find  $(g^{-1} \circ f^{-1})(1)$ .
- Explain how parts (c) and (d) illustrate reversing a composite process.

| $x$    | 0 | 1 | 2 | 3 | 4 |
|--------|---|---|---|---|---|
| $f(x)$ | 2 | 4 | 0 | 3 | 1 |
| $g(x)$ | 1 | 3 | 4 | 0 | 2 |

22. The solid segment represents  $f(x) = 2x - 2$ ,  $1 \leq x \leq 4$ . The dashed line is  $y = x$ .

- State the range of  $f$ .
- Find  $f^{-1}(x)$ .
- Add the graph of  $f^{-1}$  to the axes, labelling its endpoints.



23. A shop gives a 20% discount on an item priced at  $p$  dollars, then adds a delivery charge of \$6. Let  $h(p)$  be the final bill, where  $p \geq 0$ .

- Find an expression for  $h(p)$ .
- Find an expression for  $h^{-1}(x)$ .
- Find the original price when the final bill is \$62.
- Find how much smaller the bill would be if the discount were applied after adding delivery.



24. Let  $f(x) = ax + b$  and  $g(x) = 3x - 2$ . It is given that  $(f \circ g)(x) = 6x + 7$  for all real  $x$ .

- Find  $a$  and  $b$ .
- Find  $(g \circ f)(x)$ .
- Find  $f^{-1}(x)$ .

## Section 2: Extended response (continued)

Shared SL & HL

**25.** **GDC** Let  $f(x) = x^3 + x + 1$  and  $g(x) = 2x - 3$ , both on  $\mathbb{R}$ . The function  $f$  is strictly increasing.

- Find an expression for  $(g \circ f)(x)$ .
- Use a GDC to solve  $(g \circ f)(x) = 10$ . Give your answer to 3 significant figures.
- Hence find  $f^{-1}(6.5)$  to 3 significant figures.

**26.** **GDC** Let  $f(x) = 1.4x + 0.8$ ,  $x \in \mathbb{R}$ , and  $g(x) = x^2 + 0.5$ ,  $x \geq 0$ .

- Find an expression for  $(f \circ g)(x)$ .
- Find an expression for  $(g \circ f)(x)$ .
- Use a GDC to find the value of  $x$  for which the two composites are equal and both defined. Give 3 significant figures.
- Find the common output to 3 significant figures.

**27.** Let  $f(x) = \sqrt{2x+3} - 1$ , on its maximal real domain.

- State the domain of  $f$ .
- State its range.
- Find  $f^{-1}(x)$ .
- State the domain of  $f^{-1}$ .
- Explain why using your inverse formula at  $x = -2$  does not give an inverse-function value.

**28.** **GDC** Let  $f(x) = x^3 + 2x$ ,  $x \in \mathbb{R}$ . The function is strictly increasing, so it has an inverse.

- Use a GDC to find  $f^{-1}(7)$  to 3 significant figures.
- Write the coordinates of the corresponding point on  $y = f^{-1}(x)$ , to 3 significant figures where necessary.
- Use  $f(1) = 3$  and  $f(2) = 12$  to explain why the value in part (a) lies between 1 and 2.

**29.** Let  $f(x) = x^2$ ,  $x \geq 0$ , and  $g(x) = \sqrt{x}$ ,  $x \geq 0$ .

- Show that  $(f \circ g)(x) = x$  on  $[0, \infty)$ .
- Show that  $(g \circ f)(x) = x$  on  $[0, \infty)$ .
- Explain why  $\sqrt{x^2} = x$  would be false for some inputs if the restriction  $x \geq 0$  were removed.

**30.** Let  $f(x) = 5 - x$ ,  $x \in \mathbb{R}$ .

- Find  $f^{-1}(x)$ .
- Find  $(f \circ f)(x)$ .
- Find the point where  $y = f(x)$  meets  $y = x$ .
- Explain why  $f$  is self-inverse but is not the identity function.

**Section 3: IB exam-style / Paper 1****Shared SL & HL**

Original examination-style practice. Use no calculator. These questions are not reproduced from an IB examination.

**31.** Let  $f(x) = 3x - 2$  and  $g(x) = x^2$ , where  $x \in \mathbb{R}$ .

[Maximum mark: 6]

- (a) Find and simplify  $(f \circ g)(x)$ . [2]
- (b) Find and simplify  $(g \circ f)(x)$ . [2]
- (c) Solve  $(f \circ g)(x) = 10$ . [2]

**32.** The functions  $f$  and  $g$  are defined by  $f(x) = \sqrt{x+2}$  and  $g(x) = 2x - 5$ , on their maximal real domains.

[Maximum mark: 6]

- (a) Find  $(f \circ g)(x)$ . [2]
- (b) State the domain of  $f \circ g$ . [2]
- (c) Solve  $(f \circ g)(x) = 5$ . [2]

**33.** A function  $f$  is defined by  $f(x) = ax + b$ , where  $a \neq 0$ . It is given that  $f(2) = 7$  and  $f^{-1}(1) = -1$ .

[Maximum mark: 6]

- (a) Write down two equations in  $a$  and  $b$ . [2]
- (b) Find  $a$  and  $b$ . [2]
- (c) Find  $f^{-1}(x)$ . [2]

**34.** Let  $f(x) = (x - 1)^2$ ,  $x \geq 1$ .

[Maximum mark: 7]

- (a) State the range of  $f$ . [1]
- (b) Find  $f^{-1}(x)$ . [3]
- (c) Solve  $f^{-1}(x) = 4$ . [2]
- (d) Explain why  $f$  would not have an inverse function if its domain were  $\mathbb{R}$ . [1]

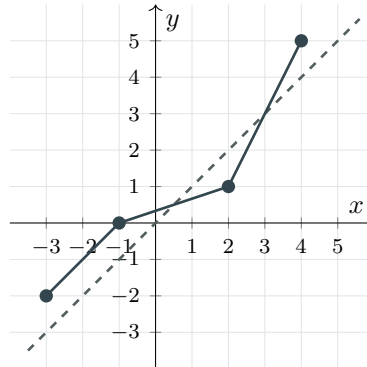
## Section 3: IB exam-style / Paper 1 (continued)

## Shared SL &amp; HL

35. The solid graph of  $f$  consists of line segments joining  $(-3, -2)$ ,  $(-1, 0)$ ,  $(2, 1)$  and  $(4, 5)$ . The dashed line is  $y = x$ .

[Maximum mark: 6]

- Find  $f^{-1}(1)$ . [1]
- State the domain of  $f^{-1}$ . [1]
- State the range of  $f^{-1}$ . [1]
- Sketch  $f^{-1}$  on the same axes, labelling its endpoints and the two points where its gradient changes. [3]



36. Let  $f(x) = \frac{3}{x+2}$ ,  $x \neq -2$ .

[Maximum mark: 8]

- Find  $f^{-1}(x)$ . [3]
- State the domain of  $f^{-1}$ . [1]
- Show that  $(f \circ f)(x) = \frac{3(x+2)}{2x+7}$  wherever the composition is defined. [2]
- State all values excluded from the domain of  $f \circ f$ . [2]

37. Let  $f(x) = x^2 - 4x + 7$ , with domain  $x \geq k$ .

[Maximum mark: 8]

- Write  $f(x)$  in vertex form. [2]
- Find the smallest value of  $k$  for which  $f$  is one-to-one. [2]
- For this value of  $k$ , find  $f^{-1}(x)$ . [3]
- State the domain of the inverse in part (c). [1]

38. Let  $f(x) = 2x - 1$ ,  $g(x) = x + 3$  and  $h(x) = x^2$ , all with domain  $\mathbb{R}$ .

[Maximum mark: 9]

- Find  $(f \circ g \circ h)(x)$ . [2]
- Find  $((f \circ g) \circ h)(x)$ , showing that it agrees with part (a). [2]
- Find  $(h \circ g \circ f)(x)$ . [2]
- Solve  $(f \circ g \circ h)(x) = (h \circ g \circ f)(x)$ . [3]

## Section 3: IB exam-style / Paper 1 (continued)

## Shared SL &amp; HL

39. Let  $f(x) = \frac{x+1}{x-2}$ ,  $x \neq 2$ .

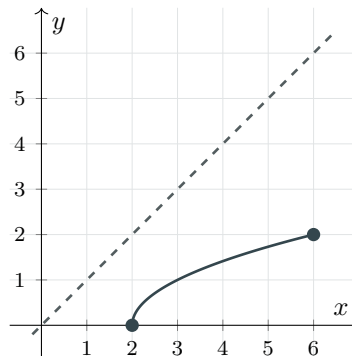
[Maximum mark: 8]

- (a) Find  $f^{-1}(x)$ . [3]  
 (b) State the domain of  $f^{-1}$ . [1]  
 (c) Find  $f^{-1}(4)$ . [1]  
 (d) Solve  $f^{-1}(x) = x$ . [3]

40. The solid graph shows  $f(x) = \sqrt{x-2}$ ,  $2 \leq x \leq 6$ . The dashed line is  $y = x$ .

[Maximum mark: 9]

- (a) Find  $f^{-1}(x)$ . [3]  
 (b) State its domain. [1]  
 (c) Sketch  $f^{-1}$  on the same axes, labelling its endpoints. [3]  
 (d) Show that  $(f^{-1} \circ f)(x) = x$  for  $2 \leq x \leq 6$ . [2]



41. Let  $f(x) = x^2$ ,  $x \geq 0$ , and  $g(x) = x - 4$ ,  $x \in \mathbb{R}$ . Let  $h = f \circ g$ .

[Maximum mark: 7]

- (a) Find an expression for  $h(x)$ . [1]  
 (b) Find the domain of  $h$ . [2]  
 (c) Find  $h^{-1}(x)$ . [3]  
 (d) State the range of  $h^{-1}$ . [1]

42. Let  $f(x) = ax + b$ . It is given that  $(f \circ f)(x) = 9x + 8$  for every real  $x$ .

[Maximum mark: 8]

- (a) Show that  $a^2 = 9$  and  $b(a+1) = 8$ . [2]  
 (b) Find both possible pairs  $(a, b)$ . [4]  
 (c) For the function with negative  $a$ , find  $f^{-1}(x)$ . [2]

## Section 3: IB exam-style / Paper 1 (continued)

## Shared SL &amp; HL

43. The functions  $f$  and  $g$  are bijections of  $\{0, 1, 2, 3, 4\}$  onto itself, as shown.

[Maximum mark: 6]

- (a) Find  $(f \circ g)(1)$ . [1]  
 (b) Find  $f^{-1}(3)$ . [1]  
 (c) Find  $(f \circ g)^{-1}(3)$ . [2]  
 (d) Find  $(g^{-1} \circ f^{-1})(3)$ . [2]

| $x$    | 0 | 1 | 2 | 3 | 4 |
|--------|---|---|---|---|---|
| $f(x)$ | 1 | 4 | 0 | 2 | 3 |
| $g(x)$ | 2 | 0 | 4 | 1 | 3 |

44. A sensor converts a temperature  $x$  in degrees Celsius to degrees Fahrenheit using  $f(x) = \frac{9}{5}x + 32$ . Its display then applies  $g(t) = \frac{t-20}{3}$  to the Fahrenheit reading. Let  $h = g \circ f$ .

[Maximum mark: 6]

- (a) Find and simplify  $h(x)$ . [2]  
 (b) Find  $h^{-1}(x)$ . [2]  
 (c) Find the temperature in degrees Celsius when the display reads 22. [2]

45. Let  $f(x) = \sqrt{9-x}$ ,  $x \leq 9$ , and  $g(x) = x^2$ ,  $x \geq 0$ . Let  $h = f \circ g$ .

[Maximum mark: 10]

- (a) Find an expression for  $h(x)$ . [2]  
 (b) Find the domain of  $h$ . [2]  
 (c) Find the range of  $h$ . [2]  
 (d) Find  $h^{-1}(x)$ . [3]  
 (e) State the domain of  $h^{-1}$ . [1]

46. Let  $f(x) = 2x + 5$ . A function  $g$  satisfies  $(f \circ g)(x) = x^2 + 3x - 1$  for every real  $x$ .

[Maximum mark: 9]

- (a) Find  $g(x)$ . [2]  
 (b) Explain why  $g$  is not one-to-one on  $\mathbb{R}$ . [1]  
 (c) Find the smallest  $k$  such that restricting  $g$  to  $x \geq k$  makes it one-to-one. [2]  
 (d) For this restriction, find  $g^{-1}(x)$ . [3]  
 (e) State the domain of this inverse. [1]

## Section 3: IB exam-style / Paper 1 (continued)

## Shared SL &amp; HL

47. Let  $f(x) = \frac{3x-4}{x-2}$ ,  $x \neq 2$ .

[Maximum mark: 8]

- (a) Write  $f(x)$  in the form  $A + \frac{B}{x-2}$ . [2]
- (b) State the equations of the two asymptotes of  $y = f(x)$ . [2]
- (c) Find  $f^{-1}(x)$ . [3]
- (d) The point  $(3, 5)$  lies on  $y = f(x)$ . State the corresponding point on  $y = f^{-1}(x)$ . [1]

48. Let  $f(x) = x^2 - 1$ ,  $x \geq 0$ , and  $g(x) = \sqrt{x+1}$ ,  $x \geq -1$ .

[Maximum mark: 6]

- (a) Find  $(f \circ g)(x)$ . [1]
- (b) State the domain of  $f \circ g$ . [1]
- (c) Find  $(g \circ f)(x)$ . [1]
- (d) State the domain of  $g \circ f$ . [1]
- (e) Explain how both results can represent an identity function although their domains differ. [2]

49. Let  $f(x) = x^3 - 2$ ,  $g(x) = x + 4$ , and  $h = f \circ g$ , all on  $\mathbb{R}$ .

[Maximum mark: 8]

- (a) Find  $h^{-1}(x)$ . [3]
- (b) Show that  $(g^{-1} \circ f^{-1})(x) = h^{-1}(x)$ . [2]
- (c) Find  $(f^{-1} \circ g^{-1})(x)$ , showing why the order of the inverses matters. [2]
- (d) Find  $h^{-1}(6)$ . [1]

50. Let  $f(x) = (x-2)^2 + 1$ ,  $x \geq 2$ , and  $g(x) = x + 1$ ,  $-1 \leq x \leq 3$ . Let  $h = f \circ g$ .

[Maximum mark: 12]

- (a) Find the domain of  $h$ . [2]
- (b) Find an expression for  $h(x)$ . [1]
- (c) Find the range of  $h$ . [2]
- (d) Sketch  $h$  on the supplied axes, labelling its endpoints. [3]
- (e) Find  $h^{-1}(x)$ . [3]
- (f) State the domain of  $h^{-1}$ . [1]



# Section 4: AA HL exclusive challenge

**HL only**

The shared worksheet ends at Question 50. These questions develop the same syllabus content with more demanding reasoning. Use no calculator.

51. Let  $f(x) = \frac{x}{x-1}$ ,  $x \neq 1$ , and  $g(x) = \frac{1}{x}$ ,  $x \neq 0$ . Let  $h = f \circ g$ .

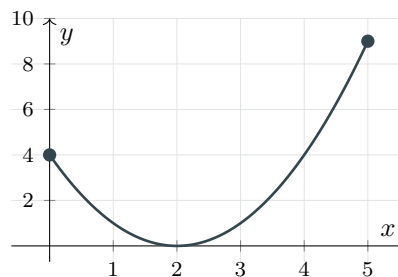
[Maximum mark: 10]

- (a) Find a simplified expression for  $h(x)$ . [2]
- (b) Find the domain of  $h$ , retaining restrictions from the original composition. [2]
- (c) Find the range of  $h$ . [3]
- (d) Find  $h^{-1}(x)$ . [2]
- (e) State the domain of  $h^{-1}$ . [1]

52. Let  $f(x) = \sqrt{9-x}$ ,  $x \leq 9$ , and  $g(x) = (x-2)^2$ ,  $0 \leq x \leq 5$ . Let  $h = f \circ g$ . The graph of  $g$  is shown.

[Maximum mark: 10]

- (a) Find the domain of  $h$ . [2]
- (b) Find the range of  $h$ . [2]
- (c) State the two largest subintervals of this domain, each with an endpoint at 2, on which  $h$  is one-to-one. [2]
- (d) Using the subinterval to the right of 2, find  $h^{-1}(x)$ . [3]
- (e) State the domain of this inverse. [1]



53. For  $a \in \mathbb{R}$ , let  $f_a(x) = \frac{ax+1}{x-a}$ ,  $x \neq a$ .

[Maximum mark: 10]

- (a) Show that  $f_a^{-1} = f_a$ . [4]
- (b) Find the two solutions of  $f_a(x) = x$  in terms of  $a$ . [3]
- (c) One of these solutions is 3. Find  $a$ . [2]
- (d) For this value of  $a$ , find the other solution. [1]

# Section 4: AA HL exclusive challenge (continued)

**HL only**

**54.** Let  $f(x) = px + q$ , where  $p \neq 0$ , and let  $g(x) = 3x + 4$ , both on  $\mathbb{R}$ . A fixed point is an input  $x$  for which the output also equals  $x$ .

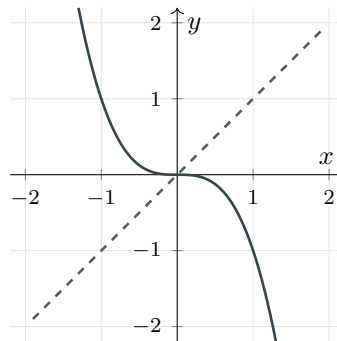
[Maximum mark: 9]

- (a) Find a relation between  $p$  and  $q$  that is necessary and sufficient for  $f \circ g = g \circ f$ . [3]
- (b) Find all such functions  $f$  that are also self-inverse. [4]
- (c) For the non-identity function from part (b), show that  $f$  and  $g$  have the same fixed point. [2]

**55.** Let  $f(x) = -x^3$ ,  $x \in \mathbb{R}$ . Its graph is shown as a solid curve; the dashed line is  $y = x$ .

[Maximum mark: 10]

- (a) Find  $f^{-1}(x)$ . [2]
- (b) Sketch  $f^{-1}$  on the same axes. [2]
- (c) Find all points of intersection of  $y = f(x)$  and  $y = f^{-1}(x)$ . [4]
- (d) A student claims that a function and its inverse can intersect only on  $y = x$ . Use your results to assess this claim. [2]



**56.** Let  $f(x) = x^2$ ,  $-2 \leq x \leq 2$ , and  $g(x) = \sqrt{x}$ ,  $0 \leq x \leq 4$ .

[Maximum mark: 8]

- (a) Show that  $f \circ g$  is the identity on  $[0, 4]$ . [2]
- (b) Find an expression for  $(g \circ f)(x)$  on  $[-2, 2]$ . [2]
- (c) Explain why part (a) alone does not establish that  $g = f^{-1}$ . [3]
- (d) State the restriction of the domain of  $f$  that makes  $g$  its inverse. [1]

# Section 4: AA HL exclusive challenge (continued)

**HL only**

57. Let  $f(x) = \sqrt{\frac{x-1}{x+2}}$ , on its maximal real domain.

[Maximum mark: 12]

- (a) Find the domain of  $f$ . [3]
- (b) Find its range, taking care with the value 1. [3]
- (c) Find  $f^{-1}(x)$ . [3]
- (d) State the domain of  $f^{-1}$ . [1]
- (e) Explain why  $f$  is one-to-one even though its domain consists of two separate intervals. [2]

58. For  $a \in \mathbb{R}$ , define  $f_a(x) = x^2 - 2ax$ ,  $x \geq a$ , and  $g_a(x) = x + a$ ,  $x \in \mathbb{R}$ . Let  $h_a = f_a \circ g_a$ .

[Maximum mark: 8]

- (a) Find an expression for  $h_a(x)$ . [2]
- (b) State the domain of  $h_a$ . [1]
- (c) Find  $h_a^{-1}(x)$ . [2]
- (d) State the domain of  $h_a^{-1}$ . [1]
- (e) It is given that  $h_a^{-1}(7) = 4$ . Find all possible values of  $a$ . [2]

59. Let  $f(x) = \frac{x+2}{x+1}$ ,  $x \neq -1$ .

[Maximum mark: 12]

- (a) Find a simplified expression for  $(f \circ f)(x)$ . [3]
- (b) Find the domain of  $f \circ f$ . [2]
- (c) Find a simplified expression for  $(f \circ f \circ f)(x)$ . [3]
- (d) Find the domain of  $f \circ f \circ f$ . [2]
- (e) Explain why the simplified denominator in part (c) does not identify every excluded input. [2]

60. Let  $f(x) = x^3 + 1$  and  $g(x) = 6 - x$ , both with domain  $\mathbb{R}$ . Define  $h = f^{-1} \circ g \circ f$ .

[Maximum mark: 10]

- (a) Find an expression for  $h(x)$ . [3]
- (b) Show directly that  $(h \circ h)(x) = x$  for every real  $x$ . [3]
- (c) Hence state  $h^{-1}(x)$ . [1]
- (d) Find the fixed point of  $h$ , giving its coordinates on  $y = h(x)$ . [3]

## Answer key and solution outlines

## Check after attempting

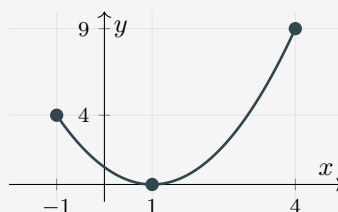
In the answer sketches, solid curves show the given function and purple dashed curves show its inverse. A dotted diagonal is  $y = x$ .

### Section 1: Short response

1. (a)  $g(2) = 6$ , so  $f(6) = 14$ . (b)  $f(2) = 2$ , so  $g(2) = 6$ .
2. (a)  $2(x - 3) + 5 = 2x - 1$ . (b)  $(2x + 5) - 3 = 2x + 2$ .
3. (a)  $f(-1) = 2$ . (b)  $g(-1) = 1$ . (c)  $f(1) = -1$ .
4. (a)  $h = g \circ f$ . (b)  $h(x) = x^2 - 5$ . (c)  $h(-2) = -1$ .
5. (a)  $5 - 2(x^2 - 1) = 7 - 2x^2$ . (b)  $x = \pm\sqrt{7/2} = \pm\frac{\sqrt{14}}{2}$ .
6. (a)  $\sqrt{3x + 3}$ . (b)  $3x + 3 \geq 0$ , so  $[-1, \infty)$ .
7. (a)  $1/(x + 3)$ . (b)  $x = -3$ , because  $g(-3) = 2$ .
8. (a) Require  $-2 \leq x \leq 2$  and  $-3 \leq x + 2 \leq 1$ ; their intersection is  $[-2, -1]$ . (b)  $g(x) \in [0, 1]$ , so the range is  $[0, 1]$ .
9. (a) From  $y = 4x - 7$ ,  $x = (y + 7)/4$ , so  $f^{-1}(x) = (x + 7)/4$ . (b) 4.
10. (a)  $[1, 15]$ . (b)  $[-2, 5]$ . The domain and range interchange.
11. (a)  $f^{-1}(x) = \sqrt[3]{x - 6}$ . (b)  $\sqrt[3]{-8} = -2$ .
12. (a) For example,  $f(-1) = f(1) = -1$ , so a horizontal line meets the graph twice. (b)  $[-2, 0]$  and  $[0, 2]$ .
13. (a)  $x + 2 \geq 0$ , hence  $f^{-1}(x) = -2 + \sqrt{x + 1}$ . (b)  $[-1, \infty)$ .
14. (a)  $I(x) = x$ . (b)  $3x + 2$ . (c)  $3x + 2$ .
15. (a)  $((5x + 6) - 6)/5 = x$ . (b)  $5((x - 6)/5) + 6 = x$ . (c) Both compositions are the identity on  $\mathbb{R}$ , so  $g = f^{-1}$ .

### Section 2: Extended response

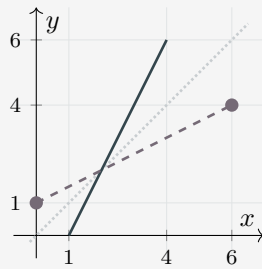
16. (a)  $2x^2 - 3$ . (b)  $(2x + 1)^2 - 2 = 4x^2 + 4x - 1$ . (c)  $2x^2 + 4x + 2 = 0$ , so  $2(x + 1)^2 = 0$  and  $x = -1$ .
17. (a)  $\sqrt{4 - x}$ . (b)  $x \leq 4$ . (c)  $5 - \sqrt{x - 1}$ . (d)  $x \geq 1$ . In each case both the inner input and the outer input must be allowed.
18. (a)  $h(x) = (x - 1)^2$ . (b)  $[-1, 4]$ : the outputs of  $g$  fill  $[-2, 3]$ . (c)  $[0, 9]$ . (d) Upward parabola with vertex  $(1, 0)$  and closed endpoints  $(-1, 4)$ ,  $(4, 9)$ ; see the answer sketch.



19. (a)  $y(x - 3) = 2x + 1$  gives  $x(y - 2) = 3y + 1$ , so  $f^{-1}(x) = (3x + 1)/(x - 2)$ . (b)  $\mathbb{R} \setminus \{2\}$ . (c)  $\mathbb{R} \setminus \{3\}$ . (d)  $x = 2$  and  $y = 3$ .
20. (a)  $f^{-1}(x) = 4 - \sqrt{x - 2}$ . (b) The original input satisfies  $x - 4 \leq 0$ , so  $x - 4 = -\sqrt{y - 2}$ . (c)  $[2, \infty)$ . (d)  $(-\infty, 4]$ .
21. (a) 1. (b) 1. (c)  $f(4) = 1$ . (d)  $g^{-1}(4) = 2$ . (e) The forward journey is  $2 \xrightarrow{g} 4 \xrightarrow{f} 1$ ; reversing it gives  $1 \xrightarrow{f^{-1}} 4 \xrightarrow{g^{-1}} 2$ .

## Section 2: Extended response (continued)

**22. (a)**  $[0, 6]$ . **(b)**  $f^{-1}(x) = (x + 2)/2$ ,  $0 \leq x \leq 6$ . **(c)** A straight segment with closed endpoints  $(0, 1)$  and  $(6, 4)$ , reflected in  $y = x$ ; see the answer sketch.



**23. (a)**  $h(p) = 0.8p + 6$ . **(b)**  $h^{-1}(x) = (x - 6)/0.8$ , for  $x \geq 6$ . **(c)**  $(62 - 6)/0.8 = \$70$ . **(d)** The reversed bill is  $0.8(p + 6) = 0.8p + 4.8$ , which is \$1.20 less.

**24. (a)**  $3ax - 2a + b = 6x + 7$ , so  $a = 2$ ,  $b = 11$ . **(b)**  $3(2x + 11) - 2 = 6x + 31$ . **(c)**  $(x - 11)/2$ .

**25. (a)**  $2x^3 + 2x - 1$ . **(b)** Solve  $2x^3 + 2x - 11 = 0$ :  $x \approx 1.577132$ , hence 1.58. **(c)**  $g(f(x)) = 10$  gives  $f(x) = 6.5$ , so  $f^{-1}(6.5) \approx 1.58$ .

**26. (a)**  $1.4x^2 + 1.5$ . **(b)**  $(1.4x + 0.8)^2 + 0.5$ . **(c)** The common domain is  $x \geq 0$ ; solve  $0.56x^2 + 2.24x - 0.36 = 0$ . The admissible root is  $x \approx 0.154729$ , hence 0.155. **(d)**  $y \approx 1.53352$ , hence 1.53.

**27. (a)**  $[-3/2, \infty)$ . **(b)**  $[-1, \infty)$ . **(c)** From  $y + 1 = \sqrt{2x + 3}$ ,  $f^{-1}(x) = ((x + 1)^2 - 3)/2$ . **(d)**  $[-1, \infty)$ . **(e)**  $-2$  is outside this domain. Squaring does not remove the condition  $y + 1 \geq 0$ .

**28. (a)** Solve  $x^3 + 2x = 7$ :  $x \approx 1.568946$ , hence 1.57. **(b)**  $(7, 1.57)$ . **(c)** Since  $3 < 7 < 12$  and  $f$  is strictly increasing,  $1 < f^{-1}(7) < 2$ .

**29. (a)**  $(\sqrt{x})^2 = x$  for  $x \geq 0$ . **(b)**  $\sqrt{x^2} = |x| = x$  for  $x \geq 0$ . **(c)** For negative  $x$ ,  $\sqrt{x^2} = |x| = -x \neq x$ ; for example  $x = -3$  gives 3, not  $-3$ .

**30. (a)**  $f^{-1}(x) = 5 - x$ . **(b)**  $5 - (5 - x) = x$ . **(c)**  $5 - x = x$  gives  $(5/2, 5/2)$ . **(d)**  $f^{-1} = f$ , but  $f(x)$  does not equal  $x$  for every input; for example  $f(0) = 5$ .

## Section 3: IB exam-style / Paper 1

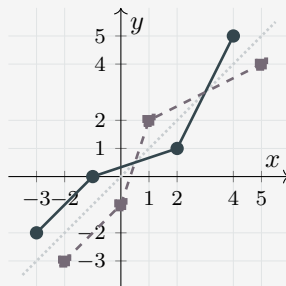
**31. (a)**  $3x^2 - 2$ . **(b)**  $(3x - 2)^2 = 9x^2 - 12x + 4$ . **(c)**  $3x^2 - 2 = 10 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$ .

**32. (a)**  $\sqrt{2x - 3}$ . **(b)**  $2x - 3 \geq 0$ , so  $[3/2, \infty)$ . **(c)**  $2x - 3 = 25$ , so  $x = 14$ , which lies in the domain.

**33. (a)**  $2a + b = 7$  and  $-a + b = 1$ , since  $f(-1) = 1$ . **(b)** Subtracting gives  $3a = 6$ , so  $a = 2$ ,  $b = 3$ . **(c)**  $f^{-1}(x) = (x - 3)/2$ .

**34. (a)**  $[0, \infty)$ . **(b)**  $y = (x - 1)^2$  and  $x - 1 \geq 0$ , so  $f^{-1}(x) = 1 + \sqrt{x}$ . **(c)**  $\sqrt{x} = 3$ , so  $x = 9$ . **(d)** It would not be one-to-one; for example  $f(0) = f(2) = 1$ .

**35. (a)** 2. **(b)**  $[-2, 5]$ . **(c)**  $[-3, 4]$ . **(d)** Join  $(-2, -3)$ ,  $(0, -1)$ ,  $(1, 2)$ ,  $(5, 4)$  in that order with line segments; reflect every original point in  $y = x$ .



**36. (a)**  $y(x + 2) = 3 \Rightarrow x = 3/y - 2$ , so  $f^{-1}(x) = 3/x - 2$ . **(b)**  $\mathbb{R} \setminus \{0\}$ . **(c)**  $3/(3/(x + 2) + 2) = 3(x + 2)/(2x + 7)$ . **(d)** Exclude  $-2$  from the inner function and  $-7/2$ , where  $f(x) = -2$ .

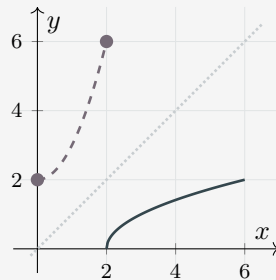
## Section 3: IB exam-style / Paper 1 (continued)

**37.** (a)  $(x-2)^2 + 3$ . (b)  $k = 2$ . Any smaller  $k$  includes inputs on both sides of the vertex with the same output. (c)  $f^{-1}(x) = 2 + \sqrt{x-3}$ . (d)  $[3, \infty)$ .

**38.** (a)  $2(x^2 + 3) - 1 = 2x^2 + 5$ . (b)  $(f \circ g)(u) = 2u + 5$ , hence  $2h(x) + 5 = 2x^2 + 5$ . (c)  $(2x + 2)^2$ . (d)  $2x^2 + 8x - 1 = 0$ , so  $x = -2 \pm 3\sqrt{2}/2$ .

**39.** (a)  $y(x-2) = x+1 \Rightarrow x(y-1) = 2y+1$ , so  $f^{-1}(x) = (2x+1)/(x-1)$ . (b)  $\mathbb{R} \setminus \{1\}$ . (c) 3. (d)  $2x+1 = x(x-1) \Rightarrow x^2 - 3x - 1 = 0$ . Thus  $x = (3 \pm \sqrt{13})/2$ ; both are allowed.

**40.** (a)  $y^2 = x-2 \Rightarrow f^{-1}(x) = x^2 + 2$ . (b)  $[0, 2]$ . (c) The part of  $y = x^2 + 2$  from  $(0, 2)$  to  $(2, 6)$ , with closed endpoints. (d)  $(\sqrt{x-2})^2 + 2 = x$ , valid on the original domain  $[2, 6]$ .



**41.** (a)  $(x-4)^2$ . (b) Require  $g(x) \geq 0$ , giving  $[4, \infty)$ . (c) Since  $x-4 \geq 0$ ,  $h^{-1}(x) = 4 + \sqrt{x}$ . (d)  $[4, \infty)$ .

**42.** (a)  $f(f(x)) = a(ax+b) + b = a^2x + b(a+1)$ ; compare coefficients. (b)  $a = 3$  gives  $b = 2$ ;  $a = -3$  gives  $b = -4$ . (c) For  $f(x) = -3x - 4$ ,  $f^{-1}(x) = -(x+4)/3$ .

**43.** (a)  $f(0) = 1$ . (b) 4. (c)  $f(g(x)) = 3$  requires  $g(x) = 4$ , hence  $x = 2$ . (d)  $g^{-1}(f^{-1}(3)) = g^{-1}(4) = 2$ .

**44.** (a)  $h(x) = ((9x/5 + 32) - 20)/3 = 3x/5 + 4$ . (b)  $h^{-1}(x) = 5(x-4)/3$ . (c)  $h^{-1}(22) = 30$ , so the temperature is  $30^\circ\text{C}$ .

**45.** (a)  $\sqrt{9-x^2}$ . (b)  $x \geq 0$  and  $x^2 \leq 9$ , hence  $[0, 3]$ . (c)  $[0, 3]$ :  $h$  decreases from 3 to 0. (d)  $y^2 = 9 - x^2$  with  $x \geq 0$  gives  $h^{-1}(x) = \sqrt{9-x^2}$ . (e)  $[0, 3]$ .

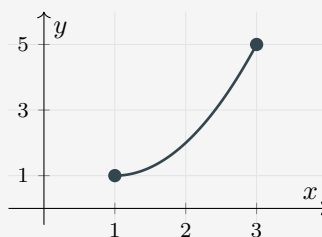
**46.** (a)  $2g(x) + 5 = x^2 + 3x - 1$ , so  $g(x) = (x^2 + 3x - 6)/2$ . (b) Its parabola has equal outputs on opposite sides of its vertex. (c)  $g(x) = \frac{1}{2}(x + \frac{3}{2})^2 - \frac{33}{8}$ , so  $k = -3/2$ . (d)  $g^{-1}(x) = (-3 + \sqrt{8x + 33})/2$ . (e)  $[-33/8, \infty)$ .

**47.** (a)  $f(x) = 3 + 2/(x-2)$ . (b)  $x = 2$  and  $y = 3$ . (c)  $yx - 2y = 3x - 4 \Rightarrow x(y-3) = 2y-4$ , hence  $f^{-1}(x) = (2x-4)/(x-3)$ ,  $x \neq 3$ . (d)  $(5, 3)$ .

**48.** (a)  $(\sqrt{x+1})^2 - 1 = x$ . (b)  $[-1, \infty)$ . (c)  $\sqrt{x^2 - 1 + 1} = \sqrt{x^2} = x$ . (d)  $[0, \infty)$ . (e) Each composition returns every input in its own domain unchanged. They are identity functions on different sets.

**49.** (a)  $h(x) = (x+4)^3 - 2$ , so  $h^{-1}(x) = \sqrt[3]{x+2} - 4$ . (b)  $f^{-1}(x) = \sqrt[3]{x+2}$ ,  $g^{-1}(x) = x-4$ ; composing in this order gives  $\sqrt[3]{x+2} - 4$ . (c)  $\sqrt[3]{x-2}$ , a different function; for instance at  $x = 6$  it gives  $\sqrt[3]{4}$ , not  $-2$ . (d)  $\sqrt[3]{8} - 4 = -2$ .

**50.** (a) Require  $x \in [-1, 3]$  and  $x+1 \geq 2$ , giving  $[1, 3]$ . (b)  $h(x) = (x-1)^2 + 1$ . (c)  $[1, 5]$ . (d) Increasing part of a parabola from the vertex  $(1, 1)$  to  $(3, 5)$ , both endpoints closed. (e)  $h^{-1}(x) = 1 + \sqrt{x-1}$ . (f)  $[1, 5]$ .



## Section 4: AA HL exclusive challenge

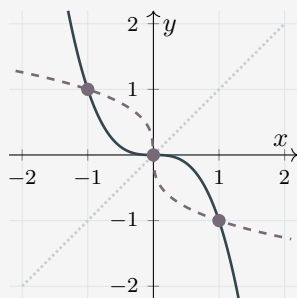
**51. (a)**  $h(x) = (1/x)/(1/x - 1) = 1/(1 - x)$ . **(b)**  $x \neq 0$  and  $1/x \neq 1$ , so  $\mathbb{R} \setminus \{0, 1\}$ . **(c)** The rule never gives 0, and the output 1 would require the excluded input 0. Every other real output is attained, so  $\mathbb{R} \setminus \{0, 1\}$ . **(d)**  $y(1 - x) = 1 \Rightarrow h^{-1}(x) = 1 - 1/x$ . **(e)**  $\mathbb{R} \setminus \{0, 1\}$ .

**52. (a)** Require  $(x - 2)^2 \leq 9$ , hence  $-1 \leq x \leq 5$ ; intersecting with  $[0, 5]$  gives  $[0, 5]$ . **(b)**  $h(x) = \sqrt{9 - (x - 2)^2}$  has maximum 3 at  $x = 2$  and minimum 0 at  $x = 5$ , so  $[0, 3]$ . **(c)**  $[0, 2]$  and  $[2, 5]$ . **(d)** On  $[2, 5]$ ,  $x - 2 \geq 0$ , giving  $h^{-1}(x) = 2 + \sqrt{9 - x^2}$ . **(e)**  $[0, 3]$ .

**53. (a)**  $y(x - a) = ax + 1 \Rightarrow x(y - a) = ay + 1$ , so  $x = (ay + 1)/(y - a)$ . The value  $y = a$  is impossible since it would require  $a^2 + 1 = 0$ . Thus the inverse has the same rule and domain. **(b)**  $x^2 - 2ax - 1 = 0$ , giving  $x = a \pm \sqrt{a^2 + 1}$ , neither equal to  $a$ . **(c)**  $9 - 6a - 1 = 0$ , so  $a = 4/3$ . **(d)** The roots have product  $-1$ , hence the other root is  $-1/3$ .

**54. (a)** Compare  $3px + 4p + q$  with  $3px + 3q + 4$ :  $q = 2p - 2$ . **(b)** Self-inverse requires  $p^2 = 1$  and  $q(p + 1) = 0$ . Together with part (a), this gives  $(p, q) = (1, 0)$  or  $(-1, -4)$ , so  $f(x) = x$  or  $f(x) = -x - 4$ . **(c)**  $-x - 4 = x$  gives  $x = -2$ ; also  $3x + 4 = x$  gives  $x = -2$ . The common fixed point on both graphs is  $(-2, -2)$ .

**55. (a)**  $f^{-1}(x) = -\sqrt[3]{x}$ . **(b)** Reflect the solid curve in  $y = x$ ; see the answer sketch. **(c)**  $-x^3 = -\sqrt[3]{x}$ . Cubing gives  $x^9 = x$ , so  $x(x^8 - 1) = 0$ , giving real solutions  $x = -1, 0, 1$ . The points are  $(-1, 1)$ ,  $(0, 0)$ ,  $(1, -1)$ . **(d)** False: two intersection points lie off  $y = x$ . Reflection pairs those two points with each other.



**56. (a)**  $f(g(x)) = (\sqrt{x})^2 = x$ , for every  $x \in [0, 4]$ . **(b)**  $g(f(x)) = \sqrt{x^2} = |x|$ . **(c)** For negative inputs this is not the identity. The function  $f$  is not one-to-one on  $[-2, 2]$ , so it has no inverse function there. **(d)** Restrict  $f$  to  $[0, 2]$ ; then both compositions are identities on the appropriate domains.

**57. (a)** Require  $(x - 1)/(x + 2) \geq 0$ , with  $x \neq -2$ . Thus  $(-\infty, -2) \cup [1, \infty)$ . **(b)** Write the radicand as  $1 - 3/(x + 2)$ . On  $[1, \infty)$ , outputs fill  $[0, 1)$ ; on  $(-\infty, -2)$ , they fill  $(1, \infty)$ . The range is  $[0, \infty) \setminus \{1\}$ . **(c)**  $y^2(x + 2) = x - 1 \Rightarrow x = (1 + 2y^2)/(1 - y^2)$ . Hence  $f^{-1}(x) = (1 + 2x^2)/(1 - x^2)$ . **(d)**  $x \geq 0$ ,  $x \neq 1$ . **(e)** Each allowed output  $y$  determines exactly one original input by the rearrangement in (c). In particular, the two domain intervals have disjoint output intervals.

**58. (a)**  $(x + a)^2 - 2a(x + a) = x^2 - a^2$ . **(b)** Require  $x + a \geq a$ , so  $x \geq 0$ . **(c)**  $h_a^{-1}(x) = \sqrt{x + a^2}$ , using the positive branch. **(d)**  $[-a^2, \infty)$ . **(e)**  $\sqrt{7 + a^2} = 4 \Rightarrow a^2 = 9$ , so  $a = \pm 3$ .

**59. (a)**  $((x + 2)/(x + 1) + 2)/((x + 2)/(x + 1) + 1) = (3x + 4)/(2x + 3)$ . **(b)** Exclude  $x = -1$  and the solution of  $f(x) = -1$ , namely  $-3/2$ . **(c)** Applying  $f$  again gives  $(7x + 10)/(5x + 7)$ . **(d)** In addition to  $-1, -3/2$ , exclude the solution of  $(f \circ f)(x) = -1$ , namely  $-7/5$ . Thus the domain is  $\mathbb{R} \setminus \{-1, -3/2, -7/5\}$ . **(e)** Algebraic cancellation hides restrictions inherited from the earlier stages of the composition.

**60. (a)**  $f^{-1}(x) = \sqrt[3]{x - 1}$ , so  $h(x) = \sqrt[3]{6 - (x^3 + 1) - 1} = \sqrt[3]{4 - x^3}$ . **(b)**  $h(h(x)) = \sqrt[3]{4 - (\sqrt[3]{4 - x^3})^3} = \sqrt[3]{x^3} = x$ . **(c)**  $h^{-1}(x) = \sqrt[3]{4 - x^3}$ ,  $x \in \mathbb{R}$ . **(d)**  $\sqrt[3]{4 - x^3} = x \Rightarrow 4 - x^3 = x^3$ , so the fixed point is  $(\sqrt[3]{2}, \sqrt[3]{2})$ .