# Composite and Inverse Functions

IB Mathematics: Analysis and Approaches | SL & HL | SL 2.5

60 questions ▪ exact algebra ▪ domains and ranges ▪ inverse identities

**At a glance**

| | |
|---|---|
| **Learning goal** | Evaluate and construct composite functions; control their domains and ranges; find and verify inverses. |
| **Syllabus link** | IB AA SL and AA HL **SL 2.5**: composite functions, identity function, one-to-one functions, inverse functions. |
| **Structure** | Sections 1–3: 50 shared questions (15 short + 15 extended + 20 IB-style); Section 4: 10 AA HL-exclusive challenges. |
| **Calculator** | Sections 1–4 are designed to be completed **without a calculator** (Paper 1 style). |

**Instructions.** Show algebraic working, use brackets when composing rules, and give exact values unless instructed otherwise. **State domains explicitly** when a function contains a square root, denominator or logarithm. Retain any original exclusions after simplifying. Students studying AA SL may stop after Question 50.

## 1. Short response

Fluency with notation, composition and elementary inverse functions.

**1.** Let $f(x) = 2x - 1$ and $g(x) = x^2 + 2$. Evaluate $f(g(3))$ and $g(f(3))$.

**2.** Let $f(x) = x^2 - 4$ and $g(x) = 3x + 1$. Find and simplify $(f \circ g)(x)$ and $(g \circ f)(x)$.

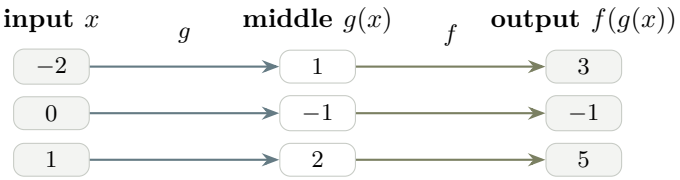
**3.** Let $f(x) = \dfrac{1}{x+2}$ and $g(x) = 2x - 1$. Find each composition and state its domain.

**4.** Let $f(x) = \sqrt{x}$ and $g(x) = x + 5$. Find $(f \circ g)(x)$ and $(g \circ f)(x)$, stating the domain of each.

**5.** The values of two functions are shown below. Find (a) $f(g(-1))$, (b) $g(f(0))$, and (c) $f(f(1))$.

| $x$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|
| $f(x)$ | $2$ | $-1$ | $0$ | $3$ | $1$ |
| $g(x)$ | $0$ | $1$ | $3$ | $2$ | $-1$ |

**6.** A two-stage mapping is represented by the arrows below. (a) Write down $g(-2)$ and $f(2)$. (b) Find $(f \circ g)(-2)$, $(f \circ g)(0)$ and $(f \circ g)(1)$. (c) State the range of $f \circ g$.




**7.** The identity function is $I(x) = x$. If $f(x) = x^2 - 3x$, find (a) $(f \circ I)(x)$, (b) $(I \circ f)(x)$ and (c) $I(-6)$.

**8.** Find the inverse of $f(x) = 5x - 7$, where $x \in \mathbb{R}$.

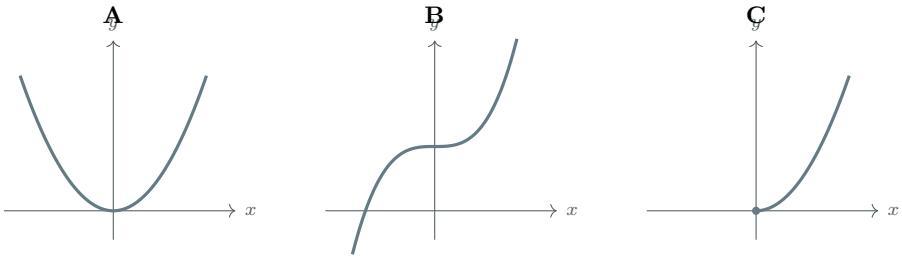
**9.** Find the inverse function of $g(x) = (x + 3)^3$, where $x \in \mathbb{R}$.

**10.** Find $f^{-1}(x)$ for $f(x) = \dfrac{1}{x - 4}$, $x \neq 4$. State the domain of the inverse.

**11.** Given $f(x) = x^2$, $x \geq 0$, find $f^{-1}(x)$ and state its domain.

**12.** Given $f(x) = x^2$, $x \leq 0$, find $f^{-1}(x)$ and state its domain.

**13.** Three graphs are shown. For each labelled graph, state whether it represents a one-to-one function on the displayed domain, giving a reason based on the horizontal line test.




**14.** Let $f(x) = 3 - 2x$ for $-1 \leq x \leq 3$. (a) Find the range of $f$. (b) Find $f^{-1}(x)$, and state its domain and range.

**15.** Let $f(x) = 2x + 4$. Calculate $f^{-1}(4)$ and $\dfrac{1}{f(4)}$. Explain why the values are different.

## 2. Extended response

Domain and range reasoning, restrictions, inverse verification and algebra.

**16.** Let $f(x) = x^2 + 1$, $x \in \mathbb{R}$, and $g(x) = \sqrt{x-2}$, $x \geq 2$.

(a) Find $(f \circ g)(x)$ and state its domain and range.
(b) Find $(g \circ f)(x)$ and state its domain.

**17.** Let $f(x) = \sqrt{x+1}$ for $x \geq -1$ and $g(x) = 2 - x$ for $x \in \mathbb{R}$. Find both compositions, with their domains and ranges.

**18.** Let $f(x) = 1/x$ $(x \neq 0)$, and $g(x) = x/(x-2)$ $(x \neq 2)$.

(a) Simplify $f(g(x))$ and state every excluded input.
(b) Simplify $g(f(x))$ and state every excluded input.

**19.** Let $f(x) = x^2$ for $x \geq 1$, and $g(x) = 3x - 4$ for $2 \leq x \leq 4$.

(a) Find $f \circ g$, giving its domain and range.
(b) Find $g \circ f$, giving its domain and range.

**20.** Let $f(x) = 2x^2 - 3x + 1$ and $g(x) = x + c$. It is given that the coefficient of $x$ in $f(g(x))$ is 5. Find $c$, and then find the simplified expression for $f(g(x))$.

**21.** Let $f(x) = ax + b$ and $g(x) = 2x - 3$, where $a, b \in \mathbb{R}$. If $(f \circ g)(x) = 8x + 5$ for every real $x$, determine $a$ and $b$.

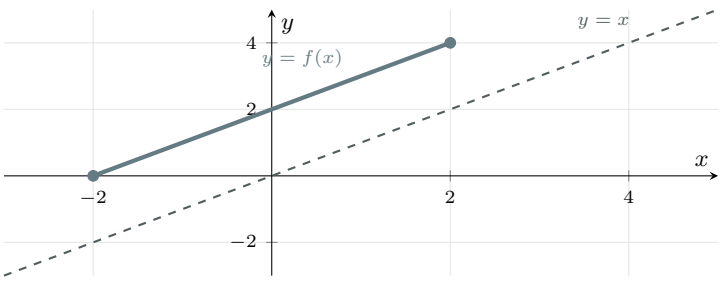
**22.** The function $f(x) = x^2 + 4x + 5$ has domain $x \geq -2$.

(a) Express $f(x)$ in completed-square form and state its range.
(b) Find $f^{-1}(x)$, giving its domain and range.

**23.** The functions $f(x) = 2x + 1$ and $g(x) = (x - 1)/2$ are defined for all real $x$. Show, by calculating both compositions, that $g = f^{-1}$.

**24.** The line segment $y = f(x)$, where $f(x) = x + 2$ and $-2 \leq x \leq 2$, is shown together with $y = x$.

(a) Sketch $y = f^{-1}(x)$ on the same axes, using reflection in $y = x$.
(b) State the domain and range of $f^{-1}$, and the inverse rule.



**25.** Let $f(x) = \dfrac{3x-2}{x+1}$ for $x \neq -1$. Find $f^{-1}(x)$. State its domain and range, explaining the excluded value in its domain.

**26.** Given $f(x) = \sqrt{2x-1}$, $x \geq \frac{1}{2}$, find $f^{-1}(x)$ and its domain and range. Verify that $f(f^{-1}(x)) = x$ for every $x$ in the domain of $f^{-1}$.

**27.** Let $f(x) = e^x + 3$, $x \in \mathbb{R}$. Find $f^{-1}(x)$ and state why its domain is not $\mathbb{R}$.

**28.** Let $g(x) = \ln(x-1)$, $x > 1$. Determine $g^{-1}(x)$, its domain and its range, and verify one inverse identity.

**29.** Let $f(x) = x^2 - 6x + 7$, $x \geq 3$. Find the minimum value of $f$, then determine $f^{-1}(x)$ with the correct square-root branch and domain.

**30.** Let $f(x) = \ln(x-2)$, $x > 2$, and $g(x) = 5 - x$, $x \in \mathbb{R}$. Find and simplify both composites, clearly stating their domains.

## 3. IB exam-style / Paper 1

Original, non-calculator multi-part questions with suggested mark allocations.

**31.** Let $f(x) = 3x - 4$ and $g(x) = x^2 + 1$, $x \in \mathbb{R}$.

(a) Find $(f \circ g)(2)$ and $(g \circ f)(2)$. [2 marks]
(b) Find simplified expressions for $(f \circ g)(x)$ and $(g \circ f)(x)$. [4 marks]

**32.** Consider $f(x) = x^2 + 2x$ and $g(x) = x - 1$, $x \in \mathbb{R}$.

(a) Show that $(f \circ g)(x) = x^2 - 1$. [3 marks]
(b) Find $(g \circ f)(x)$. [2 marks]
(c) Solve $(f \circ g)(x) = 8$. [2 marks]

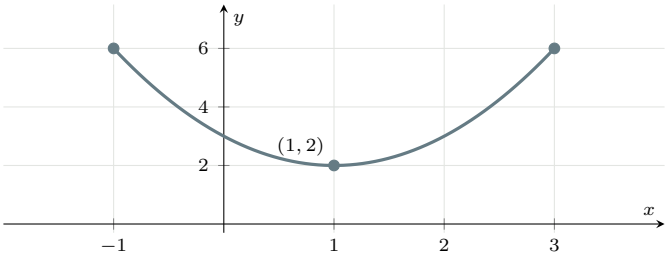
**33.** The functions $f(x) = 5x + 3$ and $g(x) = 2x - 7$ are defined for all real $x$. Let $h = f \circ g$.

(a) Find $h(x)$. [2 marks]
(b) Find $h^{-1}(x)$. [3 marks]
(c) Determine $h^{-1}(8)$. [1 marks]

**34.** Let $f(x) = \dfrac{1}{x-2}$ $(x \neq 2)$ and $g(x) = x + 1$.

(a) Find $f \circ g$ with its domain. [3 marks]
(b) Find $g \circ f$ with its domain. [2 marks]
(c) Explain why $f \circ g \neq g \circ f$. [1 marks]

**35.** The function $f(x) = (x-1)^2 + 2$ is defined for $-1 \leq x \leq 3$. Its graph is shown below.




(a) Explain why $f^{-1}$ is not a function on the original domain. [2 marks]
(b) Restrict $f$ to $1 \leq x \leq 3$. Determine its inverse, including domain and range. [4 marks]

**36.** The function $f(x) = x^3 - 2$ has domain $\mathbb{R}$.

(a) Find $f^{-1}(x)$. [3 marks]
(b) Show algebraically that $(f^{-1} \circ f)(x) = x$. [2 marks]
(c) Hence find $f^{-1}(f(-3))$. [1 marks]

**37.** Let $f(x) = 3 - 2x$, with domain $[-2, 4]$.

(a) Find the range of $f$. [2 marks]
(b) Determine $f^{-1}(x)$ with its domain and range. [3 marks]
(c) State the coordinates on $y = f^{-1}(x)$ corresponding to $(4, -5)$ on $y = f(x)$. [1 marks]

**38.** Let $f(x) = \sqrt{x+4}$, $x \geq -4$, and $g(x) = x - 1$.

(a) Find $(f \circ g)(x)$ with its domain and range. [3 marks]
(b) Find $(g \circ f)(x)$ with its domain and range. [3 marks]

**39.** Let $f(x) = \dfrac{1}{x+1}$ and $g(x) = \dfrac{x}{x-3}$, on their natural real domains.

(a) Express $f(g(x))$ as a single fraction and give all excluded inputs. [4 marks]
(b) Express $g(f(x))$ as a single fraction and give all excluded inputs. [4 marks]

**40.** Two functions are defined by $f(x) = (x-2)^2 + 5$ for $x \geq 2$ and $g(x) = 2 + \sqrt{x-5}$ for $x \geq 5$.

(a) Determine the range of $f$ and the range of $g$. [2 marks]
(b) Show that $f(g(x)) = x$, specifying valid inputs. [2 marks]
(c) Show that $g(f(x)) = x$, specifying valid inputs. [2 marks]

**41.** Let $f(x) = x^2 + 1$ for $x \geq 0$, and let $g(x) = \sqrt{x-1}$ for $x \geq 1$.

(a) Show that $g \circ f$ and $f \circ g$ give the appropriate identity functions. [4 marks]
(b) State why restricting $f$ to $x \geq 0$ is necessary. [2 marks]

**42.** Let $f(x) = 2x + 1$ and $g(x) = ax + b$, where $a, b \in \mathbb{R}$. It is given that $(f \circ g)(x) = 6x - 7$ for every real $x$.

(a) Find $a$ and $b$. [3 marks]
(b) Find $(g \circ f)(x)$ and comment on the order of composition. [3 marks]

**43.** Let $f(x) = x^2 - 2x + 5$, $x \geq 1$.

(a) Find $f^{-1}(x)$, including its domain. [4 marks]
(b) Solve $f(x) = 13$ using the inverse function. [2 marks]

**44.** The function $f(x) = \dfrac{3x}{x+2}$ is defined for $x \neq -2$.

(a) Find $f^{-1}(x)$. [4 marks]
(b) State the domain of $f^{-1}$ and evaluate $f^{-1}(0)$. [2 marks]

**45.** The Fahrenheit temperature $F$ corresponding to $C$ degrees Celsius is modelled by $F = T(C) = \dfrac{9}{5}C + 32$.

(a) Find $T^{-1}(F)$, and explain what the inverse does. [3 marks]
(b) Determine $T^{-1}(212)$, and verify your answer by substitution into $T$. [2 marks]

**46.** Let $f(x) = x^2 - 9$ for $x \geq 0$, and $g(x) = \sqrt{x+9}$ for $x \geq -9$.

(a) Show that $g = f^{-1}$. [4 marks]
(b) Find $f^{-1}(-5)$ and $g(f(4))$. [2 marks]

**47.** The function $f(x) = \dfrac{2x-1}{x+3}$ has domain $x \neq -3$.

(a) Determine $f^{-1}(x)$ and state its domain. [4 marks]
(b) Show that $f^{-1}(f(0)) = 0$. [2 marks]

**48.** The function $f(x) = 2^x$ is restricted to $-2 \leq x \leq 2$. Some values are shown.

| $x$ | $-2$ | $0$ | $2$ |
|---|---|---|---|
| $f(x)$ | $\frac{1}{4}$ | $1$ | $4$ |

(a) Write down three points on the graph of $f^{-1}$ by reflecting the displayed pairs in $y = x$. [3 marks]
(b) Find $f^{-1}(x)$ and state its domain and range. [3 marks]

**49.** Let $f(x) = x^2$ and $g(x) = x - 1$, both defined on $\mathbb{R}$.

(a) Find expressions for $f \circ g$ and $g \circ f$. [3 marks]
(b) Determine every real $x$ for which $f(g(x)) = g(f(x))$. [3 marks]

**50.** A linear function $f(x) = ax + b$, $a \neq 0$, satisfies $(f \circ f)(x) = 4x + 3$ for every real $x$. Determine all possible expressions for $f(x)$. [7 marks]

## 4. AA HL-exclusive challenge

Higher-demand problems based on SL 2.5, with parameter constraints and hidden domain restrictions.

**51.** The function $f(x) = x^2 - 4x + 7$, $x \geq 2$, is proposed to have inverse $g(x) = 2 + \sqrt{x - 3}$. Prove this claim by finding both compositions and stating their correct domains. [7 marks]

**52.** Let $f(x) = \dfrac{x+1}{x-2}$ and $g(x) = \dfrac{2x+1}{x-1}$, each with its largest possible real domain.

(a) Show that $(f \circ g)(x) = x$ and $(g \circ f)(x) = x$. [6 marks]
(b) For each composition, explain why the simplified answer is not a function on all of $\mathbb{R}$. [2 marks]

**53.** A quadratic function $f(x) = x^2 + px + q$ is defined for $x \geq -p/2$. It is known that $f(1) = 2$ and $f^{-1}(6) = 3$.

(a) Determine $p$ and $q$. [4 marks]
(b) Give the domain and range of $f$, and find $f^{-1}(x)$. [4 marks]

**54.** Let $f(x) = \ln(x - 2)$ for $x > 2$, and $g(x) = e^{2x} + 2$ for $x \in \mathbb{R}$.

(a) Simplify $f(g(x))$ and state its domain. [3 marks]
(b) Simplify $g(f(x))$ and state its domain. [3 marks]
(c) Explain why exchanging the order does not preserve the same function. [1 marks]

**55.** Let $f(x) = \sqrt{3 - x}$ for $x \leq 3$, and $g(x) = x^2 - 1$ for $x \geq 0$. Find $f \circ g$ and $g \circ f$, giving the largest allowed domain and exact range of each. [8 marks]

**56.** Two functions $f(x) = ax + b$ and $g(x) = cx + d$ satisfy $(f \circ g)(x) = 6x + 5$ and $(g \circ f)(x) = 6x - 1$ for every real $x$. Given that $a = 2$, determine $b, c, d$. [8 marks]

**57.** Let $f(x) = (x - 1)^3 + 4$, $x \in \mathbb{R}$, and let $h = f^{-1}$.

(a) Derive an expression for $h(x)$. [3 marks]
(b) Solve $h(t) = 3$ and check the answer using $f$. [3 marks]
(c) Evaluate $(h \circ f)(\sqrt{2})$ without expanding the cube. [1 marks]

**58.** A function is defined by $f(x) = \dfrac{x^2 - 4}{x - 2}$ for $x \geq 0$, $x \neq 2$.

(a) Simplify $f(x)$, keeping its domain. [2 marks]
(b) Determine its exact range, taking account of the missing input. [3 marks]
(c) Find $f^{-1}(x)$, and give its domain and range. [3 marks]

**59.** Let $f(x) = \sqrt{x+1}$ $(x \geq -1)$, $g(x) = 2x - 3$ $(x \in \mathbb{R})$ and $h = f \circ g$.

(a) Find $h(x)$ and state its domain and range. [3 marks]
(b) Derive $h^{-1}(x)$ and its domain. [3 marks]
(c) Verify both relevant identity compositions with their restrictions. [3 marks]

**60.** A linear function $f(x) = ax + b$ $(a \neq 0)$ is such that $(f \circ f)(x) = 9x + 8$ for all real $x$, and $f(0) > 0$.

(a) Determine $a$ and $b$, explaining how $f(0) > 0$ is used. [6 marks]
(b) Find $f^{-1}(x)$ and verify $f(f^{-1}(x)) = x$. [3 marks]

# Answer key and solution outlines

IB AA SL/HL | SL 2.5 | Questions 1–60

Exact answers and essential working; match the question numbers above.

## 1. Short response

**1.** $g(3) = 11$, so $f(g(3)) = 21$; $f(3) = 5$, so $g(f(3)) = 27$.

**2.** $f(g(x)) = (3x+1)^2 - 4 = 9x^2 + 6x - 3$; $g(f(x)) = 3(x^2-4) + 1 = 3x^2 - 11$.

**3.** $f(g(x)) = \dfrac{1}{2x+1}$, $x \neq -\frac{1}{2}$; $g(f(x)) = \dfrac{2}{x+2} - 1$, $x \neq -2$.

**4.** $f(g(x)) = \sqrt{x+5}$ for $x \geq -5$; $g(f(x)) = \sqrt{x} + 5$ for $x \geq 0$.

**5.** (a) $g(-1) = 0 \Rightarrow f(0) = -1$. (b) $f(0) = -1 \Rightarrow g(-1) = 0$. (c) $f(1) = 0 \Rightarrow f(0) = -1$.

**6.** (a) $g(-2) = 1$, $f(2) = 5$. (b) $3, -1, 5$, respectively. (c) $\{-1, 3, 5\}$.

**7.** (a), (b) Both equal $x^2 - 3x$. (c) $-6$.

**8.** $y = 5x - 7 \Rightarrow x = (y+7)/5$ after swapping, so $f^{-1}(x) = \dfrac{x+7}{5}$.

**9.** $g^{-1}(x) = \sqrt[3]{x} - 3$; its domain is $\mathbb{R}$.

**10.** $y = 1/(x-4) \Rightarrow x = 4 + 1/y$. Hence $f^{-1}(x) = 4 + \dfrac{1}{x}$, $x \neq 0$.

**11.** $f^{-1}(x) = \sqrt{x}$ for $x \geq 0$.

**12.** $f^{-1}(x) = -\sqrt{x}$ for $x \geq 0$ (the negative branch is required).

**13.** A: not one-to-one (many horizontal lines hit twice). B: one-to-one (a strictly increasing cubic). C: one-to-one on the restricted non-negative domain.

**14.** (a) $f(-1) = 5$, $f(3) = -3$ so range $[-3, 5]$. (b) $f^{-1}(x) = \dfrac{3-x}{2}$, domain $[-3, 5]$, range $[-1, 3]$.

**15.** $f^{-1}(x) = (x-4)/2$, so $f^{-1}(4) = 0$; $1/f(4) = 1/12$. Inverse and reciprocal are different operations.

## 2. Extended response

**16.** (a) $f(g(x)) = x - 1$, $x \geq 2$, range $[1, \infty)$. (b) $g(f(x)) = \sqrt{x^2 - 1}$, which is defined for $x \leq -1$ or $x \geq 1$.

**17.** $f(g(x)) = \sqrt{3-x}$, domain $(-\infty, 3]$, range $[0, \infty)$. $g(f(x)) = 2 - \sqrt{x+1}$, domain $[-1, \infty)$, range $(-\infty, 2]$.

**18.** (a) $f(g(x)) = \dfrac{x-2}{x}$, where $x \neq 0, 2$. (b) $g(f(x)) = \dfrac{1}{1-2x}$, where $x \neq 0, \frac{1}{2}$. Inner-function exclusions remain in force.

**19.** (a) $f(g(x)) = (3x-4)^2$, domain $[2, 4]$, range $[4, 64]$ since $g(x) \in [2, 8]$. (b) $g(f(x)) = 3x^2 - 4$, domain $[\sqrt{2}, 2]$ (as $2 \leq x^2 \leq 4$ and $x \geq 1$), range $[2, 8]$.

**20.** $f(x+c) = 2x^2 + (4c-3)x + (2c^2 - 3c + 1)$. Thus $4c - 3 = 5$, $c = 2$, and $f(g(x)) = 2x^2 + 5x + 3$.

**21.** $f(g(x)) = a(2x-3) + b = 2ax + (b - 3a)$, so $a = 4$ and $b - 12 = 5 \Rightarrow b = 17$.

**22.** (a) $f(x) = (x+2)^2 + 1$, range $[1, \infty)$. (b) $f^{-1}(x) = -2 + \sqrt{x-1}$, domain $[1, \infty)$, range $[-2, \infty)$.

**23.** $f(g(x)) = 2((x-1)/2) + 1 = x$; $g(f(x)) = ((2x+1) - 1)/2 = x$. Hence the functions undo each other on $\mathbb{R}$.

**24.** (a) The reflection is the segment from $(0, -2)$ to $(4, 2)$. (b) $f^{-1}(x) = x - 2$, domain $[0, 4]$, range $[-2, 2]$.

**25.** $y(x+1) = 3x - 2 \Rightarrow x(y-3) = -(y+2)$, so $f^{-1}(x) = \dfrac{x+2}{3-x}$. Domain $\mathbb{R} \setminus \{3\}$ because $f$ never outputs 3; range $\mathbb{R} \setminus \{-1\}$.

**26.** $y = \sqrt{2x-1} \Rightarrow x = (y^2+1)/2$ with $y \geq 0$. Thus $f^{-1}(x) = \frac{x^2+1}{2}$, domain $[0,\infty)$, range $[\frac{1}{2},\infty)$. $f(f^{-1}(x)) = \sqrt{x^2} = x$ for $x \geq 0$.

**27.** $f^{-1}(x) = \ln(x-3)$ for $x > 3$, because the range of $e^x + 3$ is $(3,\infty)$.

**28.** $g^{-1}(x) = e^x + 1$, domain $\mathbb{R}$, range $(1,\infty)$. For $x \in \mathbb{R}$, $g(g^{-1}(x)) = \ln(e^x) = x$.

**29.** $f(x) = (x-3)^2 - 2$, minimum $-2$ at $x = 3$. $f^{-1}(x) = 3 + \sqrt{x+2}$ for $x \geq -2$.

**30.** $f(g(x)) = \ln(3-x)$, $x < 3$. $g(f(x)) = 5 - \ln(x-2)$, $x > 2$.

## 3. IB exam-style / Paper 1

**31.** (a) $g(2) = 5 \Rightarrow f(g(2)) = 11$; $f(2) = 2 \Rightarrow g(f(2)) = 5$. (b) $3(x^2+1) - 4 = 3x^2 - 1$ and $(3x-4)^2 + 1 = 9x^2 - 24x + 17$.

**32.** (a) $(x-1)^2 + 2(x-1) = x^2 - 1$. (b) $g(f(x)) = x^2 + 2x - 1$. (c) $x^2 - 1 = 8 \Rightarrow x = \pm 3$.

**33.** (a) $h(x) = 5(2x-7) + 3 = 10x - 32$. (b) $h^{-1}(x) = (x+32)/10$. (c) $h^{-1}(8) = 4$.

**34.** (a) $f(g(x)) = 1/(x-1)$, $x \neq 1$. (b) $g(f(x)) = 1/(x-2) + 1$, $x \neq 2$. (c) Rules/domains differ; composition is not generally commutative.

**35.** (a) The horizontal line $y = 6$ meets the graph at $x = -1$ and $x = 3$. (b) $f^{-1}(x) = 1 + \sqrt{x-2}$; domain $[2,6]$, range $[1,3]$.

**36.** (a) $f^{-1}(x) = \sqrt[3]{x+2}$. (b) $\sqrt[3]{(x^3-2)+2} = \sqrt[3]{x^3} = x$. (c) $-3$.

**37.** (a) $[-5,7]$. (b) $(3-x)/2$; domain $[-5,7]$, range $[-2,4]$. (c) $(-5,4)$.

**38.** (a) $\sqrt{x+3}$, domain $[-3,\infty)$, range $[0,\infty)$. (b) $\sqrt{x+4} - 1$, domain $[-4,\infty)$, range $[-1,\infty)$.

**39.** (a) $\dfrac{1}{x/(x-3)+1} = \dfrac{x-3}{2x-3}$, $x \neq 3, \frac{3}{2}$. (b) $\dfrac{1/(x+1)}{1/(x+1)-3} = -\dfrac{1}{3x+2}$, $x \neq -1, -\frac{2}{3}$.

**40.** (a) $\operatorname{Ran} f = [5,\infty)$; $\operatorname{Ran} g = [2,\infty)$. (b) $f(g(x)) = (\sqrt{x-5})^2 + 5 = x$ for $x \geq 5$. (c) $g(f(x)) = 2 + \sqrt{(x-2)^2} = 2 + |x-2| = x$ for $x \geq 2$.

**41.** $g(f(x)) = \sqrt{x^2} = x$ for $x \geq 0$, and $f(g(x)) = (\sqrt{x-1})^2 + 1 = x$ for $x \geq 1$. Without the restriction, $f$ is not one-to-one ($f(-1) = f(1)$).

**42.** (a) $2ax + 2b + 1 = 6x - 7 \Rightarrow a = 3$, $b = -4$. (b) $g(f(x)) = 3(2x+1) - 4 = 6x - 1$, different from $6x - 7$.

**43.** $f(x) = (x-1)^2 + 4$. Therefore $f^{-1}(x) = 1 + \sqrt{x-4}$ for $x \geq 4$. Solving $f(x) = 13$ gives $x = f^{-1}(13) = 1 + 3 = 4$.

**44.** $yx + 2y = 3x \Rightarrow x = \dfrac{2y}{3-y}$, so $f^{-1}(x) = \dfrac{2x}{3-x}$. Domain $x \neq 3$ and $f^{-1}(0) = 0$.

**45.** (a) $T^{-1}(F) = \frac{5}{9}(F-32)$, converting Fahrenheit to Celsius. (b) $T^{-1}(212) = 100$; $T(100) = 180 + 32 = 212$.

**46.** $f(g(x)) = (\sqrt{x+9})^2 - 9 = x$ for $x \geq -9$; $g(f(x)) = \sqrt{x^2} = x$ for $x \geq 0$. Thus inverses. Values 2 and 4.

**47.** $y(x+3) = 2x - 1 \Rightarrow x(y-2) = -1 - 3y \Rightarrow f^{-1}(x) = \dfrac{1+3x}{2-x}$, $x \neq 2$. $f(0) = -1/3$ and $f^{-1}(-1/3) = 0$.

**48.** (a) $(\frac{1}{4}, -2)$, $(1,0)$, $(4,2)$. (b) $f^{-1}(x) = \log_2 x$, domain $[\frac{1}{4}, 4]$, range $[-2,2]$.

**49.** $f(g(x)) = (x-1)^2$, $g(f(x)) = x^2 - 1$. Equality requires $x^2 - 2x + 1 = x^2 - 1 \Rightarrow x = 1$.

**50.** $f(f(x)) = a(ax+b) + b = a^2x + b(a+1)$. Thus $a^2 = 4$ and $b(a+1) = 3$. If $a = 2$, $b = 1$, giving $f(x) = 2x + 1$. If $a = -2$, $b = -3$, giving $f(x) = -2x - 3$.

## 4. AA HL-exclusive challenge

**51.** $f(x) = (x-2)^2 + 3$, range $[3, \infty)$. For $x \geq 3$, $f(g(x)) = (\sqrt{x-3})^2 + 3 = x$. For $x \geq 2$, $g(f(x)) = 2 + \sqrt{(x-2)^2} = 2 + |x-2| = x$.

**52.** $f(g(x)) = \dfrac{(2x+1)/(x-1)+1}{(2x+1)/(x-1)-2} = \dfrac{3x}{3} = x$ for $x \neq 1$. $g(f(x)) = \dfrac{2(x+1)/(x-2)+1}{(x+1)/(x-2)-1} = \dfrac{3x}{3} = x$ for $x \neq 2$. The inner functions are undefined at 1 and 2, respectively; cancelling does not restore these inputs.

**53.** $1+p+q=2 \Rightarrow p+q=1$; $f(3)=6 \Rightarrow 9+3p+q=6$, so $3p+q=-3$. Subtract: $2p=-4$, hence $p=-2$, $q=3$. The domain is $x \geq 1$ and $f=(x-1)^2+2$ has range $[2,\infty)$. $f^{-1}(x) = 1+\sqrt{x-2}$ for $x \geq 2$.

**54.** (a) $\ln(e^{2x}) = 2x$ for all $x \in \mathbb{R}$. (b) $e^{2\ln(x-2)} + 2 = (x-2)^2 + 2$ for $x > 2$. (c) Expressions and domains differ.

**55.** $f(g(x)) = \sqrt{4-x^2}$. Inner $g$ needs $x \geq 0$ and $4-x^2 \geq 0$, so domain $[0,2]$, range $[0,2]$. $g(f(x)) = (\sqrt{3-x})^2 - 1 = 2-x$; inner $f$ requires $x \leq 3$ and its output is non-negative, so domain $(-\infty, 3]$, range $[-1, \infty)$.

**56.** Comparing coefficients: $ac=6$ gives $c=3$; $ad+b=5$ gives $2d+b=5$; $cb+d=-1$ gives $3b+d=-1$. Hence $d=-1-3b$, then $-2-6b+b=5$, so $b=-7/5$ and $d=16/5$.

**57.** $h(x) = 1+\sqrt[3]{x-4}$. If $h(t)=3$, then $\sqrt[3]{t-4}=2$, so $t=12$; indeed $f(3)=12$. $(h \circ f)(\sqrt{2}) = \sqrt{2}$ by the inverse identity.

**58.** $f(x) = x+2$ for $x \in [0,\infty) \setminus \{2\}$. Thus range $[2,\infty) \setminus \{4\}$. The inverse rule is $f^{-1}(x) = x-2$, domain $[2,\infty) \setminus \{4\}$, range $[0,\infty) \setminus \{2\}$. The hole at input 2 becomes a missing inverse input 4.

**59.** $h(x) = \sqrt{2x-2}$, domain $[1,\infty)$, range $[0,\infty)$. From $y^2 = 2x-2$ and $y \geq 0$, $h^{-1}(x) = (x^2+2)/2 = 1+x^2/2$ for $x \geq 0$. $h(h^{-1}(x)) = \sqrt{x^2} = x$ for $x \geq 0$; $h^{-1}(h(x)) = 1+(2x-2)/2 = x$ for $x \geq 1$.

**60.** $f(f(x)) = a^2x + b(a+1)$. Hence $a = \pm 3$. For $a=3$, $b=2$; for $a=-3$, $b=-4$. Since $f(0)=b>0$, choose $f(x)=3x+2$. Its inverse is $(x-2)/3$, and $f(f^{-1}(x)) = 3(x-2)/3+2 = x$.