Trigonometric functions

IB Mathematics AA SL & HL | Section 3.7

60 questions: circular graphs, transformations and periodic models

Learning goal	Interpret, sketch and transform sine, cosine and tangent graphs; build and evaluate periodic models.
Syllabus link	AA SL 3.7 (shared by SL and HL); transformations connect to SL 2.11.
Structure	Q1–15 short response; Q16–30 extended response; Q31–50 Paper 1 style; Q51–60 HL-exclusive challenge within 3.7.
Technology	All questions are designed for non-calculator work. Graphing technology may be used afterwards to check sketches.

Instructions. Use radians unless degrees are explicitly stated. Give exact answers. "Period" means the least positive period. Show working for multi-step questions; label key points and dashed asymptotes on sketches. Use separate paper for longer working.

1 Short response | Questions 1–15

1. Complete the table. Write "not defined" where a function has no amplitude.

Function	Amplitude	Period (radians)	Period (degrees)	Range
$y = \sin x$				
$y = \cos x$				
$y = \tan x$				

2. For $-2\pi \leq x \leq 2\pi$, state all values of x for which $\tan x$ is undefined. Also state the y-intercept of $y = \tan x$.

3. Write down the amplitude, period and range of $f(x) = 3\sin(2x) - 1$.

4. For $g(x) = -2\cos\left(\frac{x}{3}\right) + 4$, state its amplitude, period, maximum and minimum. Explain why the amplitude is not -2.

Section 1 continued | Short response

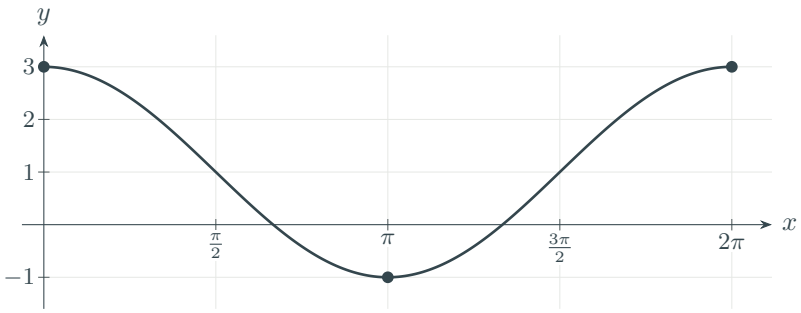
5. For $h(x) = \sin\left(3\left(x - \frac{\pi}{6}\right)\right) + 2$, state its period and midline. Describe the horizontal translation from $y = \sin(3x)$.

6. Which of the following have the same period as $y = \sin x$? Justify your choices.

$$\text{A}: \ \sin\left(x + \frac{\pi}{5}\right) \qquad \text{B}: \ \sin(5x) \qquad \text{C}: \ 5\sin x \qquad \text{D}: \ \sin\left(\frac{x}{5}\right).$$

7. The angle x is measured in degrees. For $y = 4\cos(2x) + 1$, state the period in degrees, maximum value and minimum value.

8. The graph below shows one complete cycle of a cosine function. Read off its amplitude, midline, period and range.



9. Describe a valid ordered sequence of transformations taking $y = \cos x$ to $y = -3\cos(2x) + 5$.

10. The point $P\left(\frac{\pi}{6}, \frac{1}{2}\right)$ lies on $y = \sin x$. Find the corresponding point on $y = 2\sin(3x) - 1$.

Section 1 continued | Short response

11. A function $f(x) = a\cos(bx) + d$, where $a, b > 0$, has maximum 9, minimum 1 and period $\frac{\pi}{2}$. Find a, b and d.

12. Match each function to one row of the table: $f(x) = 2\sin x$, $g(x) = \sin(2x)$ and $h(x) = \sin x + 2$.

Row	Amplitude	Period	Midline
A	1	π	$y = 0$
B	2	2π	$y = 0$
C	1	2π	$y = 2$

13. A student says: "To obtain $y = \sin\left(2x - \frac{\pi}{3}\right)$ from $y = \sin(2x)$, translate the graph $\frac{\pi}{3}$ units to the right." Explain the error and state the correct translation.

14. Sketch $y = -\cos x$ for $0 \le x \le 2\pi$ on the axes below. Label the intercepts and the maximum and minimum points.



15. For $y = 2\tan\left(x - \frac{\pi}{4}\right) + 1$, find the centre of the branch containing $x = \frac{\pi}{4}$ and the equations of its two bounding vertical asymptotes.

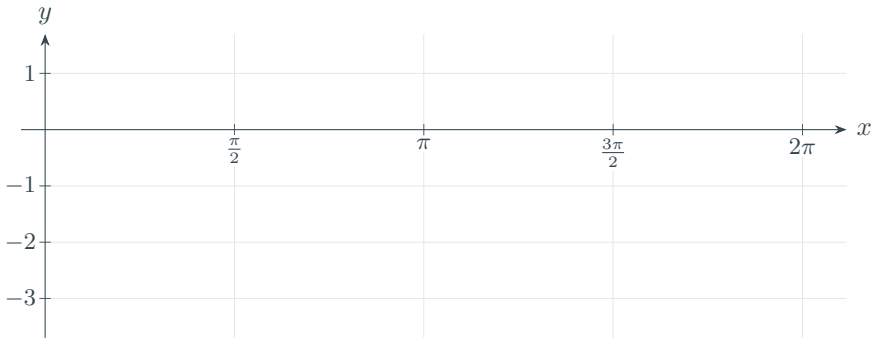
2 Extended response | Questions 16–30

16. Let $f(x) = 2\sin x - 1$, for $0 \leq x \leq 2\pi$.

(a) Complete the table with exact values.

x	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
$f(x)$					

(b) Sketch the graph on the axes below. Label its maximum, minimum and y-intercept, and draw its midline.



17. Let $g(x) = 3\cos\left(2\left(x - \frac{\pi}{4}\right)\right) + 1$.

(a) State its period and range.

(b) Starting at $x = \frac{\pi}{4}$, give the coordinates of five key points, equally spaced in x, that show one complete cycle.

(c) Describe a valid ordered sequence of transformations from $y = \cos x$ to $y = g(x)$.

18. The function $f(x) = \sin x$ is restricted to $0 \leq x \leq \pi$. A new function is defined by $g(x) = 2f\left(\frac{x}{3}\right) - 1$.

(a) Find the domain and range of g.

(b) Find the coordinates of the maximum point of g.

(c) Explain why using the full range $[-1, 1]$ for f would give an incorrect range for g.

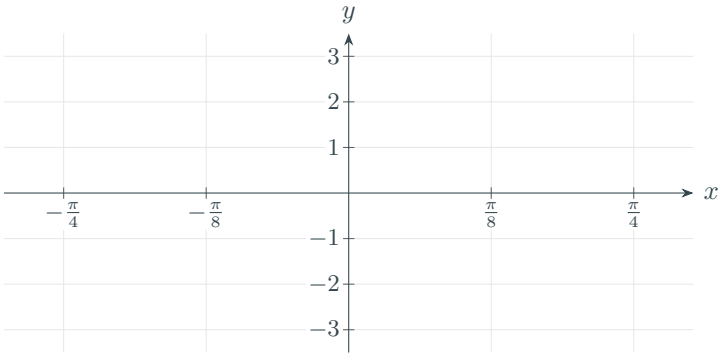
Section 2 continued | Tangent and periodic data

19. Consider $y = \tan(2x)$ on $-\frac{\pi}{4} < x < \frac{\pi}{4}$.

(a) State the equations of the bounding vertical asymptotes.

(b) Find the exact values at $x = -\frac{\pi}{8}$, 0 and $\frac{\pi}{8}$.

(c) Sketch the branch on the axes below, drawing the asymptotes as dashed lines.



20. Let $h(x) = -2\tan\left(x + \frac{\pi}{4}\right) + 3$.

(a) State its period and the centre of the branch containing $x = -\frac{\pi}{4}$.

(b) Find the equations of the two vertical asymptotes bounding this branch.

(c) State whether the branch is increasing or decreasing.

(d) Explain why the function does not have amplitude 2.

21. The temperature in a controlled chamber follows a repeating sinusoidal pattern. The table shows one complete cycle; 26 °C and 14 °C are its maximum and minimum.

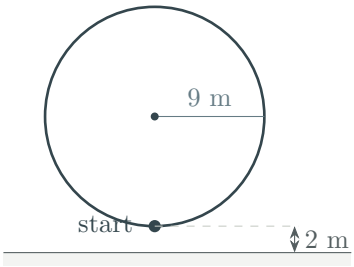
Time t (hours)	0	2	4	6	8
Temperature T (°C)	20	26	20	14	20

(a) Find a, b and d in $T(t) = a\sin(bt) + d$, where $a, b > 0$.

(b) Predict the temperature at $t = 10$ hours.

Section 2 continued | Models in context

22. A Ferris wheel has radius 9 m. Its lowest seat is 2 m above level ground. It turns at constant speed and completes one revolution in 36 seconds. A passenger starts at the lowest point at $t = 0$.



Schematic diagram, not to scale

(a) Find the height of the wheel's centre and the passenger's maximum height.

(b) Find a model $h(t) = d - a\cos(bt)$ for the passenger's height in metres, where $a, b > 0$ and t is in seconds.

(c) Find $h(9)$, $h(18)$ and $h(36)$, and interpret these three positions.

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23. A tide follows a sinusoidal model. Consecutive high tides occur at 03:00 and 15:00, both at 8 m. The low tide between them occurs at 09:00 at 2 m. Let t be the number of hours after midnight.

(a) State the amplitude, midline height and period.

(b) Find a model $h(t) = a\cos\big(b(t-3)\big) + d$, where $a, b > 0$.

(c) Find the predicted depths at midnight and at noon.

(d) State one reason why the actual water depth might differ from the model.

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24. A simplified model for the length of daylight is $L(t) = 12 + 3\sin\left(\frac{\pi t}{6}\right)$, where L is in hours and t is the number of months after a spring equinox, $0 \le t \le 12$.

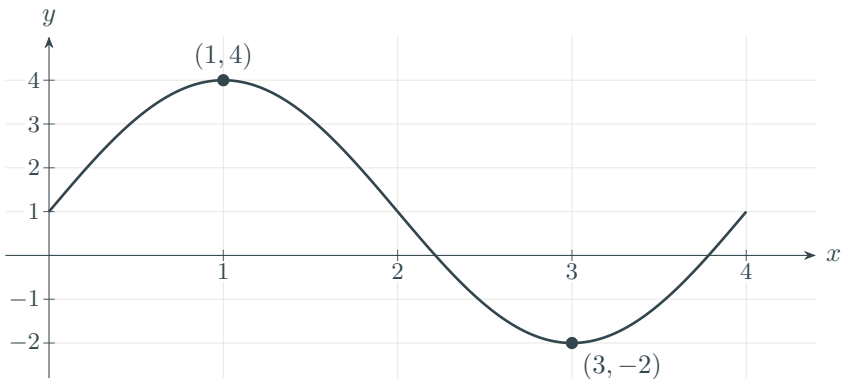
(a) State the minimum and maximum daylight lengths and the period.

(b) Find $L(0)$, $L(3)$, $L(6)$, $L(9)$ and $L(12)$.

(c) State when the model predicts more than 12 hours of daylight.

Section 2 continued | Recovering and comparing functions

25. The graph shows one complete cycle of $f(x) = a\sin(bx) + d$, where $a, b > 0$. Its maximum and minimum are labelled.



(a) Find a, b and d.

(b) State the coordinates of the next maximum point with $x > 4$.

(c) Write an equivalent cosine expression for $f(x)$.

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26. Two periodic signals are modelled by

$$A(t) = 2\cos\left(\frac{\pi t}{4}\right) + 5, \qquad B(t) = 5 - 2\cos\left(\frac{\pi t}{4}\right), \qquad t \geq 0.$$

(a) State the common range and period.

(b) Explain how a maximum of A corresponds to a minimum of B.

(c) Find the positive time delay $k < 8$ for which $B(t) = A(t - k)$, for all $t \geq k$.

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27. Starting with $y = \sin x$, apply these transformations in order: a horizontal stretch of scale factor 2; a translation $\frac{\pi}{3}$ units left; reflection in the x-axis; a vertical stretch of scale factor 3; and a translation 4 units up.

(a) Write the equation of the final graph.

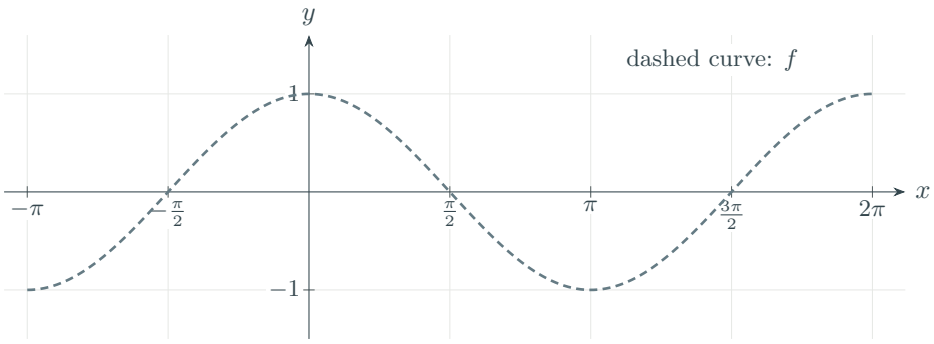
(b) State its amplitude, period and exact y-intercept.

Section 2 continued | Phase shifts and sampling

28. The supplied curve is $f(x) = \cos x$. Let $g(x) = \cos\left(x - \frac{\pi}{3}\right)$.

(a) Sketch g on the same axes for $-\pi \leq x \leq 2\pi$.

(b) State the exact y-intercept of g and the coordinates of every maximum and minimum on this interval.



29. Let $f(x) = 2\sin\left(3x - \frac{\pi}{2}\right) + 4$.

(a) Write f in the form $2\sin(3(x - c)) + 4$.

(b) Complete the key-point table for one cycle, starting at $x = c$.

Position in the cycle	x-coordinate	y-coordinate
Rising through the midline		
Maximum		
Falling through the midline		
Minimum		
Rising through the midline		

30. A temperature cycle has minimum 12 °C, maximum 28 °C and period 24 hours. A maximum occurs at 14:00. Let t be the number of hours after midnight.

(a) Find a, b, c and d in $T(t) = a\cos(b(t - c)) + d$, where $a, b > 0$ and $0 \leq c < 24$.

(b) Predict the temperatures at 08:00 and 20:00.

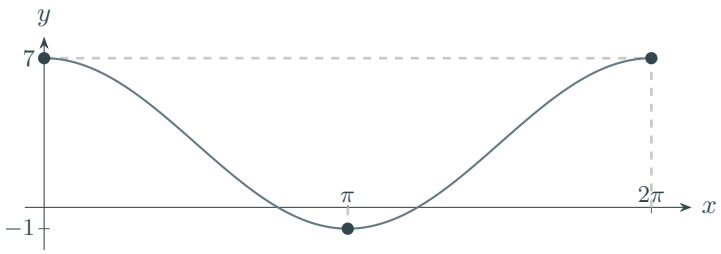
(c) A sensor records only at 08:00 and 20:00 each day. Explain why these readings alone would not reveal the true daily maximum and minimum.

3 IB exam-style / Paper 1 | Questions 31–50

Original non-calculator questions. Marks are shown in square brackets.

31. *[Maximum mark: 6]*

The diagram shows one complete cycle of $f(x) = a\cos(bx) + d$, where $a, b > 0$.



(a) Find a and d. [2]

(b) State the period of f, and hence find b. [2]

(c) Find the range of $g(x) = 2f(x) - 5$. [2]

32. *[Maximum mark: 7]*

Let $f(x) = 3\sin\left(2x - \frac{\pi}{3}\right) - 1$.

(a) Express $f(x)$ in the form $a\sin(b(x + c)) + d$, stating a, b, c and d. [2]

(b) State the amplitude and period of f. [2]

(c) Describe a valid ordered sequence of transformations taking $y = \sin x$ to $y = f(x)$. [3]

33. *[Maximum mark: 6]*

The function $f(x) = p\cos(2x) + q$, where $p < 0$, has maximum value 8 and minimum value -2.

(a) Find p and q. [3]

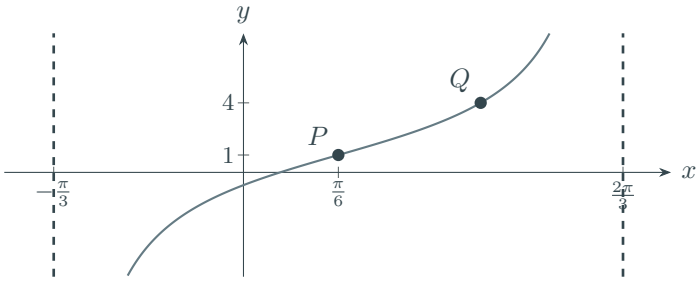
(b) Find the coordinates of all maximum points on $0 \leq x \leq 2\pi$. [2]

(c) State the number of complete cycles on this interval. [1]

Section 3 continued | Original Paper 1-style questions

34. *[Maximum mark: 6]*

The diagram shows a branch of $f(x) = a\tan(b(x-c)) + d$, where $a, b > 0$ and $-\frac{\pi}{3} < c < \frac{2\pi}{3}$. The dashed lines are consecutive vertical asymptotes. The points $P\left(\frac{\pi}{6}, 1\right)$ and $Q\left(\frac{5\pi}{12}, 4\right)$ lie on the curve.



(a) State the period of f, and hence find b. [2]

(b) Find c and d. [2]

(c) Find a. [2]

35. *[Maximum mark: 5]*

The functions $f(x) = 2\cos(3x) + 1$ and $g(x) = 2\sin(3x + c) + 1$ have identical graphs, where $0 \leq c < 2\pi$.

(a) State the period of f. [1]

(b) Find c. [2]

(c) Describe the horizontal translation of smallest positive distance taking $y = 2\sin(3x) + 1$ to $y = f(x)$. State the direction. [2]

36. *[Maximum mark: 8]*

The temperature, T degrees Celsius, is modelled by

$$T(t) = 18 + 6\cos\left(\frac{\pi(t-15)}{12}\right), \qquad 0 \leq t \leq 24,$$

where t is the number of hours after midnight.

(a) State the maximum and minimum temperatures predicted by the model. [2]

(b) Find the period of the model. [2]

(c) State the clock times at which the maximum and minimum occur. [2]

(d) Find $T(9)$, and state whether the temperature is rising or falling at this time. [2]

37. *[Maximum mark: 5]*

Consider $f(x) = 3\cos(2x) - 1$ on the restricted domain $\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$.

(a) Find the values of f at the two endpoints. [2]

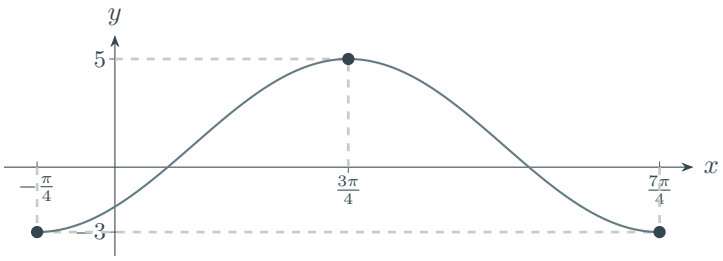
(b) Hence state the range of f, explaining why these endpoint values determine the range. Do not use differentiation. [2]

(c) State the range of $g(x) = f(2x)$ on $\frac{\pi}{12} \leq x \leq \frac{\pi}{6}$. [1]

Section 3 continued | Original Paper 1-style questions

38. *[Maximum mark: 6]*

The diagram shows one complete cycle of $f(x) = a\sin(b(x+c)) + d$, where $a, b > 0$ and $-\pi \leq c \leq \pi$. The marked points are consecutive turning points.



(a) Find a and d. [2]

(b) Find b. [2]

(c) Find c. [2]

39. *[Maximum mark: 6]*

Let $f(x) = -2\sin\left(3\left(x + \frac{\pi}{6}\right)\right) + 4$.

(a) State the amplitude and period of f. [2]

(b) A student claims that $(0, 6)$ is a maximum point of the graph. Explain the error and give the correct turning point at $x = 0$. [2]

(c) The point $P\left(\frac{\pi}{2}, 1\right)$ lies on $y = \sin x$. Using the displayed form of f, find its image under the transformations taking $y = \sin x$ to $y = f(x)$. [2]

40. *[Maximum mark: 8]*

The depth of water in a harbour is modelled by a sinusoidal function $D(t)$, where t is the number of hours after 06:00. The readings include consecutive minimum depths at $t = 0$ and $t = 12$.

t (hours)	0	3	6	9	12
$D(t)$ (metres)	2	5	8	5	2

(a) State the period, amplitude and midline of the model. [3]

(b) Find a model of the form $D(t) = a\cos(bt) + d$, where $a < 0$ and $b > 0$. [3]

(c) State the clock time of the first high tide after 06:00. [1]

(d) State the clock time of the next high tide after 18:00. [1]

Section 3 continued | Original Paper 1-style questions

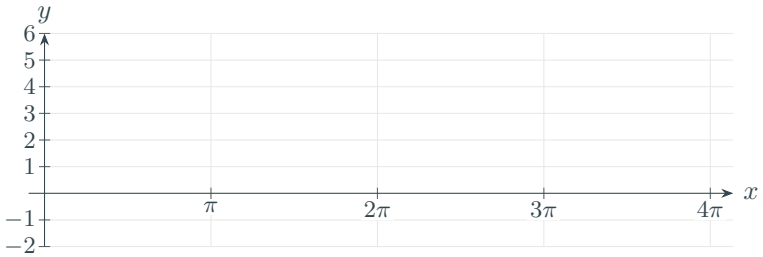
41. *[Maximum mark: 7]*

Let $f(x) = \cos x$ and $g(x) = -3f\left(\frac{x}{2}\right) + 2$.

(a) The point $P(\pi, -1)$ lies on $y = f(x)$. Find its image on $y = g(x)$. [2]

(b) State the period and range of g. [2]

(c) Sketch $y = g(x)$ on the axes below for $0 \leq x \leq 4\pi$. Label the endpoints, maximum point and midline crossings. [3]



42. *[Maximum mark: 8]*

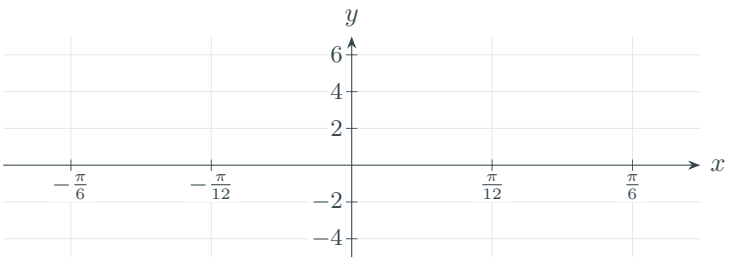
Let $f(x) = 2\tan(3x) + 1$.

(a) State the period of f. [1]

(b) Find the equations of the vertical asymptotes immediately to the left and right of the origin. [2]

(c) Find $f\left(\frac{\pi}{12}\right)$. [2]

(d) Sketch the central branch of $y = f(x)$ on the axes below. Include the asymptotes, the y-intercept and the point found in part (c). [3]



Section 3 continued | Original Paper 1-style questions

43. *[Maximum mark: 8]*

A Ferris wheel has radius 9 m and its centre is 11 m above horizontal ground. It completes one revolution in 36 seconds. At $t = 0$, a passenger is at the same height as the centre and is moving upwards.

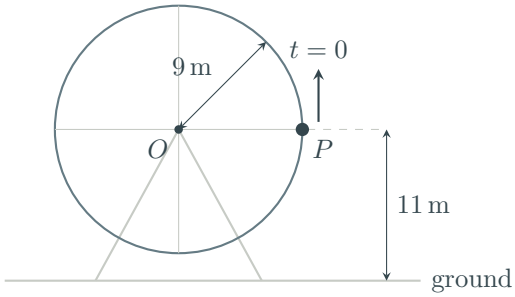


Diagram not drawn to scale

The passenger's height is modelled by $h(t) = a\sin(bt) + d$, where $b > 0$ and h is in metres.

(a) State the range of h. [2]

(b) Find a, b and d. [3]

(c) Find the exact height after 6 seconds. [2]

(d) State the first time at which the passenger reaches the maximum height. [1]

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44. *[Maximum mark: 7]*

In this question, angles are measured in degrees. A sinusoidal function has the form

$$f(x) = a\sin\big(b(x - c)\big) + d, \qquad a, b > 0, \quad 0^\circ \leq c < 120^\circ.$$

The following two points are given.

Point	Coordinates	Feature
P	$(20^\circ, 3)$	midline crossing, graph rising
Q	$(50^\circ, 8)$	first maximum after P

(a) Find a and d. [2]

(b) Find the period, and hence find b and c. [3]

(c) Find the coordinates of the first minimum after Q. [2]

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45. *[Maximum mark: 7]*

Two transformations are applied to $y = \sin x$:

$$A:\ \text{a horizontal stretch of scale factor 2}, \qquad B:\ \text{a translation } \frac{\pi}{3} \text{ to the right.}$$

(a) Find the equation of the graph obtained by applying A followed by B. [2]

(b) Find the equation of the graph obtained by applying B followed by A. [2]

(c) State the period of each resulting graph. [1]

(d) Describe a horizontal translation taking the graph in part (a) to the graph in part (b). [2]

Section 3 continued | Original Paper 1-style questions

46. *[Maximum mark: 7]*

A moving platform has height $h(t) = 4\cos\left(\frac{\pi t}{5}\right) + 7$ metres, where $t \geq 0$ is measured in seconds. A second timer starts when the first timer shows $t = 2$. The second timer displays u seconds.

(a) State the period and range of the height function. [2]
(b) Write the height as a function of u. [2]
(c) Find the first value of $u \geq 0$ at which the platform reaches its maximum height. [2]
(d) Explain why the period is unchanged when the second timer is used. [1]

47. *[Maximum mark: 7]*

Two models are proposed for the height, in metres, of a seat on a rotating wheel:

$$h_1(t) = 12 - 10\cos\left(\frac{\pi t}{20}\right), \qquad h_2(t) = 12 + 10\tan\left(\frac{\pi t}{20}\right),$$

where $t \geq 0$ is in seconds and each function is considered wherever it is defined.

(a) State the period of each function. [2]
(b) State the range of each function. [2]
(c) Give two mathematical reasons why h_2 is unsuitable as a model for continuous wheel motion. [2]
(d) Using h_1, state the first time at which the seat reaches its maximum height. [1]

48. *[Maximum mark: 7]*

The depth of a lake is modelled by $D(t) = 10 - 3\cos\left(\frac{\pi t}{6}\right)$ metres, where t is measured in months.

(a) State the period and range of D. [2]
(b) Find $D(3)$. [1]
(c) A second lake has the same midline and period. Its difference between maximum and minimum depths is twice that of the first lake, and its maximum occurs one month later. Find a model for the second lake. [3]
(d) State the minimum and maximum depths of the second lake. [1]

Section 3 continued | Original Paper 1-style questions

49. *[Maximum mark: 8]*

The graph of $f(x) = a\cos(bx) + d$, where $a, b > 0$, has a maximum at $(0, 7)$ and its next minimum at $\left(\frac{\pi}{3}, -1\right)$.

(a) Find a, b and d. [3]

(b) The function $g(x) = mf(x) + n$ has range $[-4, 2]$ and a minimum at $x = 0$. Find m and n. [4]

(c) Write $g(x)$ in the form $p\cos(qx) + r$. [1]

50. *[Maximum mark: 10]*

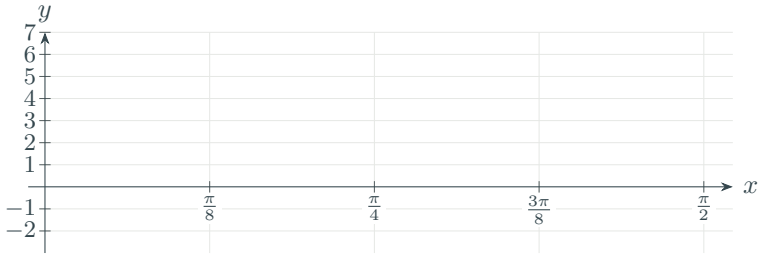
Let $f(x) = a\sin(bx + c) + d$, where $a, b > 0$ and $-\pi < c \leq \pi$. The graph has a minimum at $\left(\frac{\pi}{8}, -2\right)$ and its next maximum at $\left(\frac{3\pi}{8}, 6\right)$. It also passes through $(0, 2)$.

(a) Find a and d. [2]

(b) Find b and c. [3]

(c) Sketch one complete cycle for $0 \leq x \leq \frac{\pi}{2}$ on the axes below. Label the turning points, endpoints and midline crossings. [3]

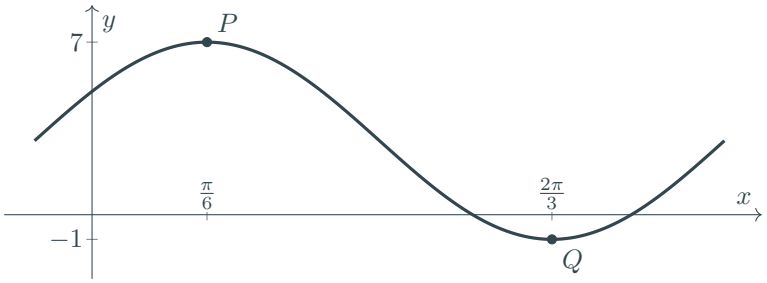
(d) Express $f(x)$ as a cosine function with positive amplitude. [2]



4 AA HL-exclusive challenge | Questions 51–60

This section deepens the shared 3.7 content. SL students may stop after Question 50.

51. The graph of $f(x) = a\sin(b(x+c)) + d$, where $a, b > 0$ and $0 \le c < \pi$, has a maximum at $P\left(\frac{\pi}{6}, 7\right)$. The next minimum is at $Q\left(\frac{2\pi}{3}, -1\right)$.



(a) Determine a, b, c and d. [5]

(b) Let $g(x) = 2f\left(3x - \frac{\pi}{2}\right) - 5$. Find its range, its period, and the coordinates of its first maximum for $x \ge 0$. [4]

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52. The function $f(x) = 3\cos(bx) + 2$, where $b > 0$, gives the following observations.

x	0	1	2	3	4
$f(x)$	5	2	−1	2	5

(a) Find the smallest positive value of b consistent with the table. [2]

(b) Show that $b = \frac{3\pi}{2}$ also fits every observation. State the period in this case. [3]

(c) Evaluate $f\left(\frac{1}{2}\right)$ for each of these two values of b. Explain why the original table alone does not establish that the period is 4. [3]

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53. Let

$$f(x) = 4\sin\left(3x - \frac{\pi}{6}\right) + 1, \qquad g(x) = -\frac{1}{2}f\left(2x + \frac{2\pi}{9}\right) + 3.$$

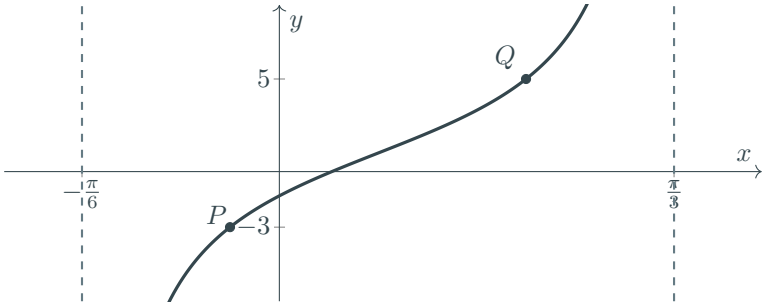
(a) Express $g(x)$ in the form $A\cos(Bx) + D$, where $B > 0$. [4]

(b) State the range and period of g. [2]

(c) The graph of g is translated horizontally to give $y = 2\cos(6x) + \frac{5}{2}$. Find the smallest possible positive translation to the right. [2]

Section 4 continued | AA HL-exclusive challenge

54. A branch of $y = a\tan\big(b(x-c)\big) + d$, with $a, b > 0$, lies between the consecutive vertical asymptotes $x = -\frac{\pi}{6}$ and $x = \frac{\pi}{3}$. Here $-\frac{\pi}{6} < c < \frac{\pi}{3}$. The curve passes through $P\left(-\frac{\pi}{24}, -3\right)$ and $Q\left(\frac{5\pi}{24}, 5\right)$.



(a) Determine a, b, c and d. [6]

(b) Apply a horizontal stretch of scale factor 2, followed by a translation $\frac{\pi}{6}$ units to the left. Find the equation of the resulting curve and the two asymptotes corresponding to those shown. [3]

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55. The depth of water in a tank is modelled by

$$D_k(t) = k + 1 + (k-2)\cos\left(\frac{\pi t}{6}\right), \qquad t \geq 0,$$

where $D_k(t)$ is in metres, t is in hours, and k is a real constant.

(a) Express the minimum and maximum depths in terms of k. [2]

(b) Determine all values of k for which the depth always satisfies $1 \leq D_k(t) \leq 7$. [4]

(c) A student claims that every model in this family has least positive period 12 hours. Identify the exception and explain it. [2]

---

56. Two Ferris wheels rotate at constant speeds. The heights, in metres, of two cabins are modelled by

$$A(t) = 14 - 12\cos\left(\frac{\pi t}{6}\right), \qquad B(t) = 11 + 8\cos\left(\frac{\pi(t-2)}{4}\right),$$

where $t \geq 0$ is measured in minutes.

(a) Find the period and maximum height of each cabin. [4]

(b) Find the first time at which both cabins are at their highest points simultaneously. [3]

(c) Find the smallest positive time by which both wheels have completed a whole number of revolutions since $t = 0$. [2]

Section 4 continued | AA HL-exclusive challenge

57. The function

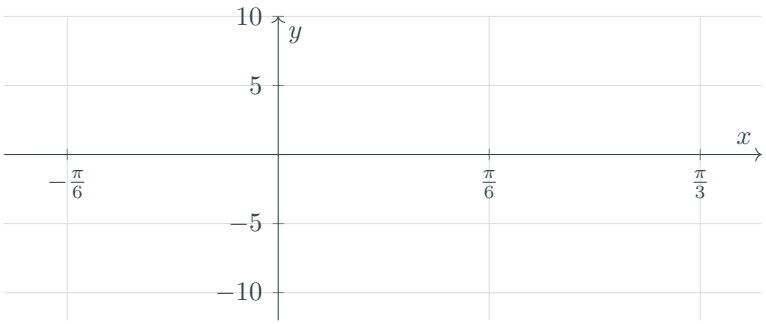
$$f(x) = a\sin\left(2x + \frac{\pi}{6}\right) + d, \qquad a > 0,$$

has range $[2, 8]$ when its domain is $0 \leq x \leq \frac{\pi}{6}$.

(a) Find a and d. [4]

(b) The same formula is now used on $-\frac{\pi}{6} \leq x \leq \frac{\pi}{3}$. Sketch the curve on the supplied axes, labelling its endpoints and maximum. State its range on this larger domain. [4]

(c) Explain why half the length of the original range is not the full graph's amplitude. [1]



58. The function $f(x) = a\sin\big(b(x + c)\big) + d$, where $a, b > 0$, has range $[-2, 8]$ and period $\frac{3\pi}{2}$.

(a) Find a, b and d. [3]

(b) Four values, spaced one quarter of a period apart, are shown below.

x	t	$t + \frac{3\pi}{8}$	$t + \frac{3\pi}{4}$	$t + \frac{9\pi}{8}$
$f(x)$	p	q	r	s

Find $p + q + r + s$. Explain why the result does not depend on t or c. [3]

(c) Hence find $\sum_{j=0}^{19} f\left(t + \frac{3j\pi}{8}\right)$. [1]

Section 4 continued | AA HL-exclusive challenge

59. Consider the complete graph of

$$f(x) = 4\sin\left(3x - \frac{\pi}{4}\right) + 2, \qquad x \in \mathbb{R}.$$

(a) A translation by the vector $\binom{h}{k}$, where $0 < h \leq 2\pi$, leaves the graph unchanged. Find all possible vectors, explaining why the vertical component must be zero. [4]

(b) Find the smallest positive translation to the right that gives the reflection of this graph in the line $y = 2$. [3]

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60. Every point (u, v) on $y = f(x)$, where $f(x) = a\cos(bx) + d$ and $a, b > 0$, is mapped to

$$(3u - \pi,\ 2 - v).$$

The resulting curve $y = g(x)$ has the following consecutive extrema.

	x-coordinate	y-coordinate
Maximum P	$\frac{\pi}{2}$	9
Next minimum Q	2π	-3

(a) Express $g(x)$ in terms of f. [2]

(b) Determine a, b and d. [5]

(c) State a valid ordered sequence of transformations that produces the given point mapping. [3]

Answer key and solution outlines

Equivalent exact expressions and valid alternative transformation sequences are accepted.

Section 1 Short response | Questions 1–15

1.

Function	Amplitude	Period (rad)	Period (deg)	Range
$\sin x$	1	2π	$360°$	$[-1, 1]$
$\cos x$	1	2π	$360°$	$[-1, 1]$
$\tan x$	Not defined	π	$180°$	$\mathbb{R}$

2. Undefined at $x = -\frac{3\pi}{2}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$. The y-intercept is $(0, 0)$.

3. Amplitude 3; period $\frac{2\pi}{2} = \pi$; range $[-4, 2]$.

4. Amplitude $|-2| = 2$; period $\frac{2\pi}{1/3} = 6\pi$; maximum 6; minimum 2. Amplitude is a non-negative distance from the midline; the negative coefficient gives a reflection.

5. Period $\frac{2\pi}{3}$; midline $y = 2$; translation $\frac{\pi}{6}$ units right.

6. A and C. A horizontal translation and a vertical stretch leave the period unchanged. B has period $\frac{2\pi}{5}$; D has period 10π.

7. Period $\frac{360°}{2} = 180°$; maximum 5; minimum -3.

8. Amplitude $\frac{3-(-1)}{2} = 2$; midline $y = 1$; period 2π; range $[-1, 3]$.

9. One valid sequence: horizontal stretch of scale factor $\frac{1}{2}$; vertical stretch of scale factor 3; reflection in the x-axis; translation 5 units up.

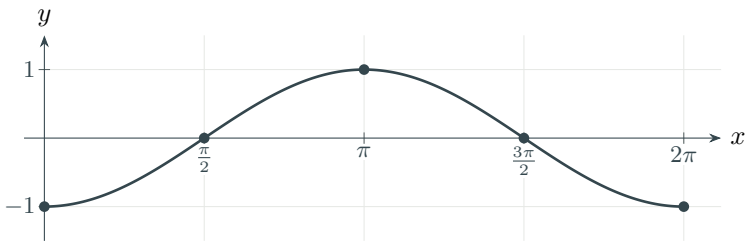
10. Under the transformation, $(u, v) \mapsto \left(\frac{u}{3}, 2v - 1\right)$. Thus $P \mapsto \left(\frac{\pi}{18}, 0\right)$.

11. $a = \frac{9-1}{2} = 4$, $d = \frac{9+1}{2} = 5$ and $\frac{2\pi}{b} = \frac{\pi}{2} \Rightarrow b = 4$.

12. f: B; g: A; h: C.

13. Factor the angle: $2x - \frac{\pi}{3} = 2\left(x - \frac{\pi}{6}\right)$. The translation is $\frac{\pi}{6}$ units right. The shift must be read after factoring out the coefficient of x.

14. Smooth cosine curve through $(0, -1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, 1)$, $\left(\frac{3\pi}{2}, 0\right)$, $(2\pi, -1)$. Maximum $(\pi, 1)$; minima $(0, -1)$ and $(2\pi, -1)$.



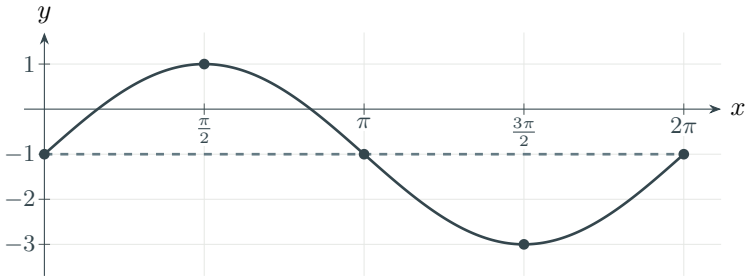
15. Centre $\left(\frac{\pi}{4}, 1\right)$. Set $x - \frac{\pi}{4} = \pm\frac{\pi}{2}$ for the bounding asymptotes: $x = -\frac{\pi}{4}$ and $x = \frac{3\pi}{4}$.

Section 2 Extended response | Questions 16–30

16.

(a) The five values, in order, are $-1,\ 1,\ -1,\ -3,\ -1$.

(b) Maximum $\left(\frac{\pi}{2}, 1\right)$; minimum $\left(\frac{3\pi}{2}, -3\right)$; y-intercept $(0, -1)$; midline $y = -1$.



17.

(a) Period π; range $[-2, 4]$.

(b) A quarter-period is $\frac{\pi}{4}$, so the points are

$$\left(\frac{\pi}{4}, 4\right),\quad \left(\frac{\pi}{2}, 1\right),\quad \left(\frac{3\pi}{4}, -2\right),\quad (\pi, 1),\quad \left(\frac{5\pi}{4}, 4\right).$$

(c) Horizontal stretch of scale factor $\frac{1}{2}$, translation $\frac{\pi}{4}$ right, vertical stretch of scale factor 3, then translation 1 up is one valid sequence.

18.

(a) $0 \le \frac{x}{3} \le \pi \Rightarrow 0 \le x \le 3\pi$. On its restricted domain, $0 \le f \le 1$, so $-1 \le g \le 1$.

(b) The parent maximum occurs at input $\frac{\pi}{2}$. Thus $\frac{x}{3} = \frac{\pi}{2}$, giving $\left(\frac{3\pi}{2}, 1\right)$.

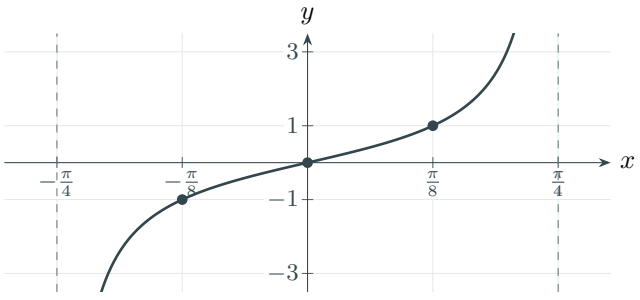
(c) The original domain covers only the non-negative half-wave of sine. Its range is $[0, 1]$, so the negative half of the full sine range is unavailable.

19.

(a) $2x = \pm\frac{\pi}{2} \Rightarrow x = \pm\frac{\pi}{4}$.

(b) $-1, 0, 1$, respectively.

(c) An increasing tangent branch through $\left(-\frac{\pi}{8}, -1\right), (0, 0), \left(\frac{\pi}{8}, 1\right)$, approaching the dashed asymptotes.



20.

(a) Period π; centre $\left(-\frac{\pi}{4}, 3\right)$.

(b) $x + \frac{\pi}{4} = \pm\frac{\pi}{2}$, so $x = -\frac{3\pi}{4}$ and $x = \frac{\pi}{4}$.

(c) Decreasing, because the negative output multiplier reverses the increasing parent branch.

(d) Tangent is unbounded above and below, so it has no maximum, minimum or amplitude. The 2 is a vertical stretch factor.

21.

(a) $a = \frac{26-14}{2} = 6$, $d = \frac{26+14}{2} = 20$, and $b = \frac{2\pi}{8} = \frac{\pi}{4}$. Hence $T(t) = 6\sin\left(\frac{\pi t}{4}\right) + 20$.

(b) $T(10) = 6\sin\left(\frac{5\pi}{2}\right) + 20 = 26\,^{\circ}\text{C}$.

Section 2 Extended response | Questions 16–30 (continued)

22.

(a) Centre: $2+9=11$ m; maximum: $2+18=20$ m.

(b) $a=9$, $d=11$, $b=\frac{2\pi}{36}=\frac{\pi}{18}$. Thus $h(t)=11-9\cos\left(\frac{\pi t}{18}\right)$.

(c) $h(9)=11$ m: level with the centre; $h(18)=20$ m: top; $h(36)=2$ m: back at the bottom.

23.

(a) Amplitude 3 m; midline height 5 m; period 12 hours.

(b) $h(t)=3\cos\left(\frac{\pi}{6}(t-3)\right)+5$.

(c) $h(0)=5$ m and $h(12)=5$ m, since the cosine arguments are $-\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

(d) For example, wind, atmospheric pressure or storm conditions can alter the depth.

24.

(a) Minimum 9 hours; maximum 15 hours; period 12 months.

(b) $12, 15, 12, 9, 12$ hours, respectively.

(c) $0<t<6$: between the spring and autumn equinoxes in this model. Endpoints are excluded because there are exactly 12 hours at each.

25.

(a) $a=\frac{4-(-2)}{2}=3$, $d=\frac{4+(-2)}{2}=1$. The maximum-to-minimum gap is 2, half a period, so $T=4$ and $b=\frac{\pi}{2}$.

(b) The next maximum is one period after $(1,4)$, so $(5,4)$.

(c) One expression is $f(x)=3\cos\left(\frac{\pi}{2}(x-1)\right)+1$.

26.

(a) Range $[3,7]$; period 8.

(b) $A(t)+B(t)=10$. Hence when $A=7$, $B=3$ at the same time. The models are reflections in the midline $y=5$.

(c) A half-cycle delay reverses the cosine displacement: $k=4$.

27.

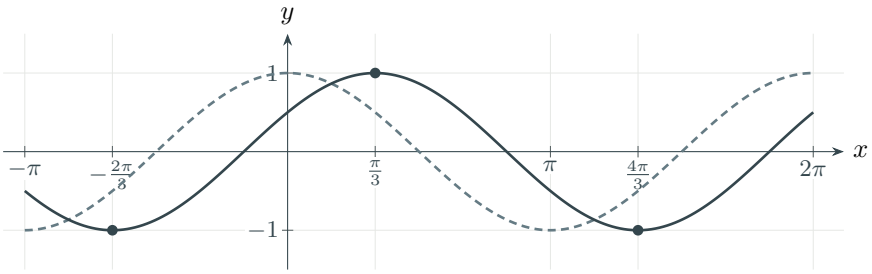
(a) $y=-3\sin\left(\frac{x+\pi/3}{2}\right)+4=-3\sin\left(\frac{x}{2}+\frac{\pi}{6}\right)+4$.

(b) Amplitude 3; period 4π; y-intercept $\left(0,\frac{5}{2}\right)$, since $-3\sin\frac{\pi}{6}+4=\frac{5}{2}$.

28.

(a) Translate the supplied cosine graph $\frac{\pi}{3}$ right. The solid curve below is g; the dashed curve is f.

(b) y-intercept $\left(0,\frac{1}{2}\right)$; maximum $\left(\frac{\pi}{3},1\right)$; minima $\left(-\frac{2\pi}{3},-1\right)$ and $\left(\frac{4\pi}{3},-1\right)$.



29.

(a) $f(x)=2\sin\left(3\left(x-\frac{\pi}{6}\right)\right)+4$, so $c=\frac{\pi}{6}$.

(b) The period is $\frac{2\pi}{3}$ and the quarter-period is $\frac{\pi}{6}$. The coordinate pairs, in table order, are

$$\left(\frac{\pi}{6},4\right),\quad \left(\frac{\pi}{3},6\right),\quad \left(\frac{\pi}{2},4\right),\quad \left(\frac{2\pi}{3},2\right),\quad \left(\frac{5\pi}{6},4\right).$$

30.

(a) $a=8$, $b=\frac{2\pi}{24}=\frac{\pi}{12}$, $c=14$, $d=20$. Hence $T(t)=8\cos\left(\frac{\pi}{12}(t-14)\right)+20$.

(b) Both are 20 °C: the times lie a quarter-cycle before and after the maximum.

(c) The readings sample only the midline, missing all maxima and minima. Many amplitudes would give those same two readings.

Section 3 IB exam-style / Paper 1 | Questions 31–50

31.

(a) $a = (7-(-1))/2 = 4$ and $d = (7+(-1))/2 = 3$.

(b) Consecutive maxima are 2π apart, so $T = 2\pi$ and $b = 2\pi/T = 1$.

(c) Since $-1 \le f(x) \le 7$, $-7 \le 2f(x) - 5 \le 9$. The range is $[-7, 9]$.

32.

(a) $f(x) = 3\sin\left(2\left(x - \dfrac{\pi}{6}\right)\right) - 1$; $a = 3$, $b = 2$, $c = -\pi/6$, $d = -1$.

(b) Amplitude 3; period $2\pi/2 = \pi$.

(c) One valid sequence: horizontal stretch of scale factor $1/2$; vertical stretch of scale factor 3; translation $\pi/6$ right; translation 1 down.

33.

(a) $|p| = (8-(-2))/2 = 5$, so $p = -5$. Also $q = (8+(-2))/2 = 3$.

(b) A maximum occurs when $\cos(2x) = -1$. The points are $(\pi/2, 8)$ and $(3\pi/2, 8)$.

(c) $T = \pi$, so there are 2 complete cycles.

34.

(a) Consecutive tangent asymptotes are one period apart: $T = 2\pi/3 - (-\pi/3) = \pi$. Hence $b = \pi/T = 1$.

(b) The branch centre has x-coordinate $c = (-\pi/3 + 2\pi/3)/2 = \pi/6$. Since $f(c) = d$ and $P = (\pi/6, 1)$, $d = 1$.

(c) Substituting Q gives $4 = a\tan(5\pi/12 - \pi/6) + 1 = a\tan(\pi/4) + 1$, so $a = 3$.

35.

(a) $T = 2\pi/3$.

(b) A cosine cycle is a sine cycle shifted left by a quarter-period. Thus $c = \pi/2$, giving $\sin(3x + \pi/2) = \cos(3x)$.

(c) Translation $\pi/6$ to the left, since $3(x + \pi/6) = 3x + \pi/2$.

36.

(a) Maximum $18 + 6 = 24°$C; minimum $18 - 6 = 12°$C.

(b) $T = 2\pi/(\pi/12) = 24$ hours.

(c) Maximum at $t = 15$: 15:00. Minimum half a period earlier, at $t = 3$: 03:00.

(d) $T(9) = 18 + 6\cos(-\pi/2) = 18°$C. The temperature is rising from the minimum at 03:00 towards the maximum at 15:00.

37.

(a) $f(\pi/6) = 3\cos(\pi/3) - 1 = 1/2$; $f(\pi/3) = 3\cos(2\pi/3) - 1 = -5/2$.

(b) As $2x$ increases from $\pi/3$ to $2\pi/3$, cosine decreases continuously. Hence the range is $[-5/2, 1/2]$.

(c) The input $2x$ spans $[\pi/6, \pi/3]$, so the range is again $[-5/2, 1/2]$.

38.

(a) $a = (5-(-3))/2 = 4$ and $d = (5+(-3))/2 = 1$.

(b) Consecutive minima are $7\pi/4 - (-\pi/4) = 2\pi$ apart, so $b = 1$.

(c) The rising midline crossing is midway between $x = -\pi/4$ and $x = 3\pi/4$, at $x = \pi/4$. Hence $x + c = 0$ there, so $c = -\pi/4$.

39.

(a) Amplitude 2; period $2\pi/3$.

(b) $f(0) = -2\sin(\pi/2) + 4 = 2$. The negative coefficient makes this a minimum, so the correct turning point is $(0, 2)$.

(c) The coordinate mapping is $(x, y) \mapsto (x/3 - \pi/6, -2y + 4)$. Thus $P \mapsto (0, 2)$.

Section 3 IB exam-style / Paper 1 | Questions 31–50 (continued)

40.

(a) Period 12 hours; amplitude $(8-2)/2 = 3\,\text{m}$; midline $D = 5\,\text{m}$.

(b) $b = 2\pi/12 = \pi/6$. The model starts at a minimum, so $a = -3$ and $d = 5$:

$$D(t) = 5 - 3\cos\left(\frac{\pi t}{6}\right).$$

(c) First high tide: $t = 6$, which is 12:00.

(d) The next high tide after $t = 12$ is at $t = 18$: 00:00 the following day.

41.

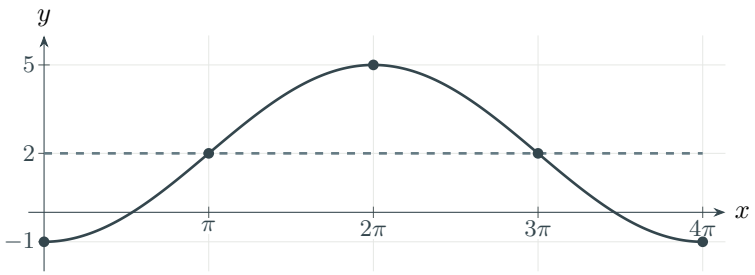
(a) The mapping is $(x, y) \mapsto (2x, -3y + 2)$, so $P \mapsto (2\pi, 5)$.

(b) Period 4π; range $[-1, 5]$.

(c) A smooth inverted cosine cycle through

$$(0, -1), \quad (\pi, 2), \quad (2\pi, 5), \quad (3\pi, 2), \quad (4\pi, -1).$$

The endpoints are minima; $(2\pi, 5)$ is the maximum; the other two points cross the midline $y = 2$.



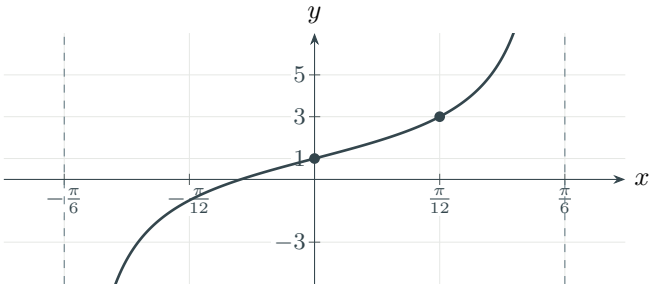
42.

(a) Period $\pi/3$.

(b) $3x = \pm\pi/2$, so the asymptotes are $x = -\pi/6$ and $x = \pi/6$.

(c) $f(\pi/12) = 2\tan(\pi/4) + 1 = 3$.

(d) Draw an increasing tangent branch between the two dashed asymptotes, through $(0, 1)$ and $(\pi/12, 3)$. It approaches $-\infty$ on the right of $x = -\pi/6$ and $+\infty$ on the left of $x = \pi/6$.



Section 3 IB exam-style / Paper 1 | Questions 31–50 (continued)

43.

(a) Minimum $11 - 9 = 2\,\text{m}$; maximum $11 + 9 = 20\,\text{m}$. Range $[2, 20]$ metres.

(b) $b = 2\pi/36 = \pi/18$, $d = 11$. The passenger starts at the midline and moves upwards, so $a = 9$:

$$h(t) = 9\sin\left(\frac{\pi t}{18}\right) + 11.$$

(c) $h(6) = 9\sin(\pi/3) + 11 = 11 + \dfrac{9\sqrt{3}}{2}\,\text{m}$.

(d) One quarter of a revolution: $t = 36/4 = 9$ seconds.

44.

(a) $d = 3$ and $a = 8 - 3 = 5$.

(b) A rising midline crossing to the next maximum is a quarter-period. Thus $T = 4(50° - 20°) = 120°$, $b = 360°/120° = 3$, and $c = 20°$.

(c) The next minimum is half a period after the maximum: $(50° + 60°, 3 - 5) = (110°, -2)$.

45.

(a) A gives $y = \sin(x/2)$; then B gives

$$y = \sin\left(\frac{x - \pi/3}{2}\right) = \sin\left(\frac{x}{2} - \frac{\pi}{6}\right).$$

(b) B gives $y = \sin(x - \pi/3)$; then A gives

$$y = \sin\left(\frac{x}{2} - \frac{\pi}{3}\right) = \sin\left(\frac{x - 2\pi/3}{2}\right).$$

(c) Both have period $2\pi/(1/2) = 4\pi$.

(d) Translation $\pi/3$ to the right. The horizontal shifts in parts (a) and (b) are $\pi/3$ and $2\pi/3$, respectively.

46.

(a) Period $2\pi/(\pi/5) = 10$ seconds; range $[3, 11]$ metres.

(b) Since $t = u + 2$, the height is $H(u) = 4\cos\left(\dfrac{\pi(u+2)}{5}\right) + 7$, $u \geq 0$.

(c) The next maximum on the first timer is at $t = 10$. Hence $u = 10 - 2 = 8$ seconds.

(d) Changing the zero of time translates the graph horizontally; the time between successive cycles remains 10 seconds.

47.

(a) h_1: period $2\pi/(\pi/20) = 40$ seconds. h_2: period $\pi/(\pi/20) = 20$ seconds.

(b) h_1: range $[2, 22]$ metres. h_2: range $\mathbb{R}$.

(c) For example: h_2 is unbounded, whereas a wheel has a finite radius; it is undefined at vertical asymptotes (first at $t = 10$), whereas the seat has a height at every time. Negative predicted heights below the ground are another valid objection.

(d) The cosine coefficient is negative, so the first maximum occurs after half a cycle: $t = 20$ seconds.

Section 3 IB exam-style / Paper 1 | Questions 31–50 (continued)

48.

(a) Period 12 months; range $[7, 13]$ metres.

(b) $D(3) = 10 - 3\cos(\pi/2) = 10\,\text{m}$.

(c) The new amplitude is 6, the midline remains 10, and the one-month delay is a translation to the right:

$$E(t) = 10 - 6\cos\left(\frac{\pi(t-1)}{6}\right).$$

(d) Minimum 4 m; maximum 16 m.

49.

(a) $a = (7-(-1))/2 = 4$, $d = 3$. A maximum to the next minimum is half a period, so $T = 2\pi/3$ and $b = 3$.

(b) The maximum of f becomes the minimum of g, so $m < 0$. Using the extrema gives $7m + n = -4$ and $-m + n = 2$. Subtraction gives $8m = -6$, hence $m = -3/4$ and $n = 5/4$.

(c) $g(x) = -\frac{3}{4}(4\cos(3x) + 3) + \frac{5}{4} = -3\cos(3x) - 1$.

50.

(a) $a = (6-(-2))/2 = 4$ and $d = (6+(-2))/2 = 2$.

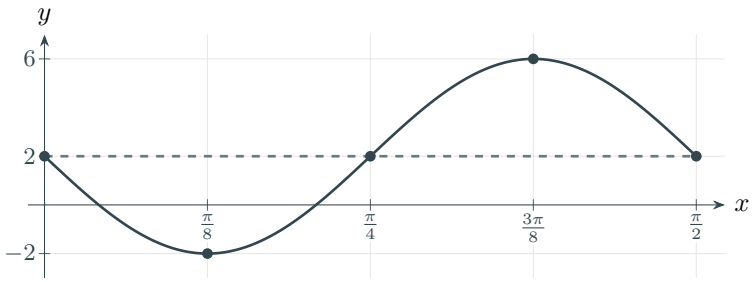
(b) The minimum to the next maximum spans $\pi/4$, so $T = \pi/2$ and $b = 4$. Also $f(0) = 2$ gives $\sin c = 0$. Since $x = \pi/8$ is a minimum, $\sin(\pi/2 + c) = -1$, giving $c = \pi$ in the stated interval.

(c) $f(x) = -4\sin(4x) + 2$. Draw a smooth cycle through

$$(0, 2), \quad (\pi/8, -2), \quad (\pi/4, 2), \quad (3\pi/8, 6), \quad (\pi/2, 2).$$

It initially falls; the two labelled turning points are the minimum and maximum.

(d) One suitable form is $f(x) = 4\cos\left(4\left(x - \frac{3\pi}{8}\right)\right) + 2$.



Section 4 AA HL-exclusive challenge | Questions 51–60

51.

(a) $a = \frac{7-(-1)}{2} = 4$, $d = \frac{7+(-1)}{2} = 3$. The maximum and next minimum are $\frac{\pi}{2}$ apart, so the period is π and $b = 2$. At $x = \frac{\pi}{6}$, the sine is 1; the restriction on c gives $2(\frac{\pi}{6} + c) = \frac{\pi}{2}$, hence $c = \frac{\pi}{12}$. Thus $f(x) = 4\sin(2x + \frac{\pi}{6}) + 3$.

(b) The range is $[2(-1) - 5, 2(7) - 5] = [-7, 9]$, and the period is $\frac{\pi}{3}$. Original maxima occur at $x = \frac{\pi}{6} + n\pi$. Their new coordinates satisfy $3x - \frac{\pi}{2} = \frac{\pi}{6} + n\pi$, giving $x = \frac{2\pi}{9} + \frac{n\pi}{3}$. The first maximum for $x \geq 0$ is $\boxed{(\frac{2\pi}{9}, 9)}$.

52.

(a) $f(1) = 2$ requires $\cos b = 0$. The smallest positive possibility is $\boxed{b = \frac{\pi}{2}}$, which gives all five table entries.

(b) For $b = \frac{3\pi}{2}$, the cosine values at $x = 0, 1, 2, 3, 4$ are $1, 0, -1, 0, 1$, so the same observations result. The period is $\frac{2\pi}{3\pi/2} = \boxed{\frac{4}{3}}$.

(c) For $b = \frac{\pi}{2}$, $f(\frac{1}{2}) = \boxed{2 + \frac{3\sqrt{2}}{2}}$. For $b = \frac{3\pi}{2}$, $f(\frac{1}{2}) = \boxed{2 - \frac{3\sqrt{2}}{2}}$. Both curves agree at the sampled integers, but one completes extra oscillations between them. The given observations therefore do not determine the period uniquely.

53.

(a)
$$g(x) = -\frac{1}{2}\left[4\sin\left(6x + \frac{\pi}{2}\right) + 1\right] + 3 = \boxed{-2\cos(6x) + \frac{5}{2}}.$$

(b) Range $\boxed{[\frac{1}{2}, \frac{9}{2}]}$; period $\boxed{\frac{\pi}{3}}$.

(c) A half-period translation reverses the oscillation about the midline. The smallest positive translation to the right is $\boxed{\frac{\pi}{6}}$. Indeed, $g(x - \frac{\pi}{6}) = 2\cos(6x) + \frac{5}{2}$.

54.

(a) The period is the asymptote spacing, $\frac{\pi}{2}$, so $b = 2$. The centre of the branch is halfway between the asymptotes: $c = \frac{(-\pi/6)+(\pi/3)}{2} = \frac{\pi}{12}$. The arguments at P and Q are $-\frac{\pi}{4}$ and $\frac{\pi}{4}$, respectively. Thus $-a + d = -3$, $a + d = 5$, giving $\boxed{a = 4,\ d = 1}$. The equation is $y = 4\tan(2(x - \frac{\pi}{12})) + 1$.

(b) If the original function is F, the transformed function is
$$F\left(\frac{x + \pi/6}{2}\right) = \boxed{4\tan x + 1}.$$
Each old horizontal coordinate u becomes $2u - \frac{\pi}{6}$. The corresponding asymptotes are $\boxed{x = -\frac{\pi}{2},\ x = \frac{\pi}{2}}$.

55.

(a) Minimum $k + 1 - |k - 2|$; maximum $k + 1 + |k - 2|$.

(b) Require $k + 1 - |k - 2| \geq 1$ and $k + 1 + |k - 2| \leq 7$. For $k \leq 2$, the range is $[2k - 1, 3]$, so $1 \leq k \leq 2$. For $k \geq 2$, the range is $[3, 2k - 1]$, so $2 \leq k \leq 4$. Therefore $\boxed{1 \leq k \leq 4}$.

(c) At $\boxed{k = 2}$, $D_k(t) = 3$ is constant. Every positive time shift is a period, so there is no least positive period. For $k \neq 2$, the least positive period is 12 hours.

56.

(a) Cabin A: period $\boxed{12 \text{ min}}$, maximum height $\boxed{26 \text{ m}}$. Cabin B: period $\boxed{8 \text{ min}}$, maximum height $\boxed{19 \text{ m}}$.

(b) Cabin A reaches its highest point at $t = 6, 18, 30, \ldots$; cabin B does so at $t = 2, 10, 18, 26, \ldots$. The first simultaneous maximum occurs at $\boxed{t = 18 \text{ min}}$.

(c) The least common multiple of the two revolution times is $\text{lcm}(12, 8) = \boxed{24 \text{ min}}$.

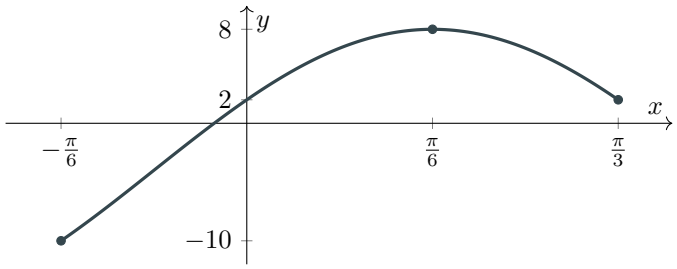
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57.

(a) On the original domain, the angle increases from $\frac{\pi}{6}$ to $\frac{\pi}{2}$. Since $a > 0$, the endpoint values give $\frac{a}{2} + d = 2$ and $a + d = 8$. Subtracting gives $a = 12$, hence $\boxed{a = 12,\ d = -4}$.

(b) The curve increases from $(-\frac{\pi}{6}, -10)$, through $(0, 2)$, to the maximum $(\frac{\pi}{6}, 8)$, then decreases to $(\frac{\pi}{3}, 2)$. The range is $\boxed{[-10, 8]}$.

(c) The original interval contains only part of a cycle and does not contain the full sinusoid's minimum. The full amplitude is 12, not $\frac{8-2}{2} = 3$.



58.

(a) $\boxed{a = 5,\ b = \dfrac{4}{3},\ d = 3}$.

(b) Points half a period apart are equally far above and below the midline $y = 3$. Therefore $p + r = 6$ and $q + s = 6$, so $\boxed{p + q + r + s = 12}$. The phase shift and starting position do not affect these paired sums.

(c) The 20 terms form five successive groups of four quarter-period samples. Each group sums to 12, giving $\boxed{60}$.

59.

(a) The range is $[-2, 6]$. A vertical translation gives $[-2 + k, 6 + k]$, so an unchanged graph requires $k = 0$. The least positive period is $\frac{2\pi}{3}$; the admissible vectors are

$$\boxed{\begin{pmatrix} 2\pi/3 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 4\pi/3 \\ 0 \end{pmatrix},\quad \begin{pmatrix} 2\pi \\ 0 \end{pmatrix}}.$$

(b) Reflection in $y = 2$ gives $y = 4 - f(x)$. A half-period translation interchanges the parts above and below this midline. The smallest positive translation to the right is $\boxed{\dfrac{\pi}{3}}$.

60.

(a) A new horizontal coordinate x comes from $u = \frac{x+\pi}{3}$, so $\boxed{g(x) = 2 - f(\dfrac{x + \pi}{3})}$.

(b) The maximum P of g corresponds to the minimum $(\frac{\pi}{2}, -7)$ of f. The next minimum Q corresponds to the next maximum $(\pi, 5)$ of f. Thus $a = \frac{5-(-7)}{2} = 6$, $d = \frac{5+(-7)}{2} = -1$. The original consecutive extrema are $\frac{\pi}{2}$ apart, so the original period is π, giving $b = 2$. Hence $\boxed{f(x) = 6\cos(2x) - 1}$.

(c) One valid sequence: horizontal stretch of scale factor 3; reflection in the x-axis; translation by the vector $\begin{pmatrix} -\pi \\ 2 \end{pmatrix}$.