Trigonometric equations

IB Mathematics: Analysis and Approaches SL & HL

Section 3.8 ▪ 50 shared questions + 10 HL questions

Learning goal	Find every solution of a trigonometric equation in a given interval, using algebra and graphs.
Syllabus link	AA **SL 3.8**, shared by SL and HL; identities and graphs from SL 3.6–3.7.
Structure	Q1–15 short response; Q16–30 extended response; Q31–50 Paper 1 practice; Q51–56 HL exam practice; Q57–60 further challenge.
Technology	Use a graphical calculator for **Q23, Q26 and Q29**. All other questions are non-calculator.

Instructions

Work on separate paper and show your reasoning. Give exact answers unless stated otherwise. Angles are in **radians** unless degrees are shown. Find **all** solutions in each stated interval and check whether its endpoints are included. For numerical answers, round only at the final step.

1 Short response

Build fluency with exact values, intervals and transformed angles.

1. Solve $\sin x = \dfrac{1}{2}$ for $0° \le x < 360°$.

2. Solve $\cos x = -\dfrac{\sqrt{2}}{2}$ for $0 \le x < 2\pi$.

3. Solve $\tan x = -\sqrt{3}$ for $-180° < x \le 180°$.

4. Solve $\sin x = -\dfrac{\sqrt{3}}{2}$ for $-\pi \le x \le \pi$.

5. Solve $\cos x = 1$ for $0 \le x \le 4\pi$.

6. Solve $2\sin x + \sqrt{2} = 0$ for $0° \le x \le 360°$.

7. Solve $2\cos x - 1 = 0$ for $-\pi \le x \le \pi$.

8. Solve $\tan x = 0$ for $-\pi \le x < 2\pi$.

9. Solve $\sin(2x) = 1$ for $0 \le x \le 2\pi$.

10. Solve $\cos(3x) = 0$ for $0 \leq x \leq \pi$.

11. Solve $\sin\left(x - \frac{\pi}{6}\right) = 0$ for $0 \leq x < 2\pi$.

12. Solve $\tan\left(x + \frac{\pi}{4}\right) = 1$ for $0 \leq x < 2\pi$.

13. Solve $\cos^2 x = \frac{1}{4}$ for $0 \leq x < 2\pi$.

14. Explain why $\sin x = \frac{3}{2}$ has no real solutions.

15. A student solves $\cos x(2\sin x - 1) = 0$ by dividing by $\cos x$. State the two solutions lost on $0 \leq x < 2\pi$.

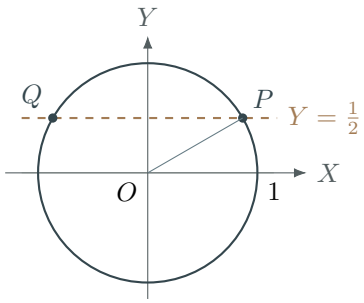
2 Extended response

Connect algebra, graphs and context. Technology: Q23, Q26 and Q29 only.

16. The line $Y = \frac{1}{2}$ meets the unit circle at P and Q. Angles are measured anticlockwise from the positive X-axis.

(a) Use the diagram to solve $\sin x = \frac{1}{2}$ for $-\pi \leq x \leq \pi$.

(b) Hence solve $\sin(2x) = \frac{1}{2}$ for $-\pi \leq x \leq \pi$.



17. Consider $\cos\left(2x - \frac{\pi}{3}\right) = -\frac{1}{2}$, where $0 \leq x \leq 2\pi$.

(a) Let $u = 2x - \frac{\pi}{3}$. State the corresponding interval for u.

(b) Find all values of x that satisfy the equation.

18. Solve $2\sin^2 x + \sin x - 1 = 0$ for $-\pi \leq x \leq \pi$.

(a) Factorize the quadratic in $\sin x$.

(b) Find all solutions in the stated interval.

19. Consider $2\cos^2 x + \cos x - 1 = 0$ on $0 \leq x < 2\pi$.

(a) Write the left-hand side as a product of two factors.

(b) Hence solve the equation.

20. Solve $\tan^2 x + (1 - \sqrt{3})\tan x - \sqrt{3} = 0$ for $0 < x < \pi$.

21. Consider $\cos(2x) = 3\sin x - 1$, where $0 \leq x < 2\pi$.

(a) Rewrite the equation as a quadratic in $\sin x$ and factorize it.

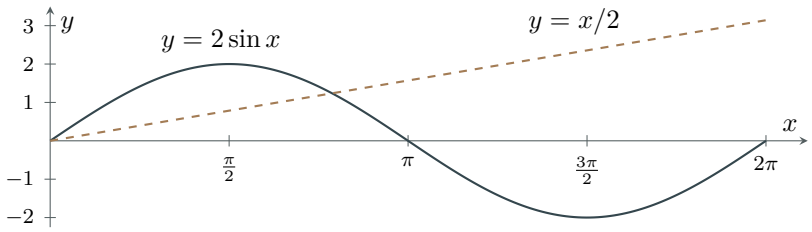
(b) Solve the original equation, explaining why one algebraic value must be rejected.

22. Solve $\sin(2x) = \sqrt{2}\cos x$ for $-\pi \leq x \leq \pi$. Do not divide by a trigonometric expression.

23. Technology permitted. The graphs of $y = 2\sin x$ and $y = x/2$ are shown for $0 \leq x \leq 2\pi$. Use radian mode.

(a) State the number of solutions of $2\sin x = x/2$ in the given interval.

(b) Use a graphing calculator to find all solutions. Give non-zero answers to three significant figures.



24. Consider $\tan x(\tan x - \sqrt{3}) = 0$ for $0 \leq x < 2\pi$.

(a) State the values of x in this interval at which the left-hand side is undefined.

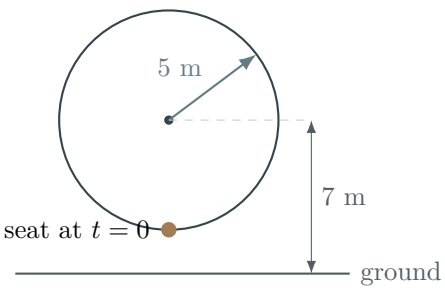
(b) Find all solutions of the equation.

25. A seat on a Ferris wheel has height $h(t) = 7 - 5\cos\left(\dfrac{\pi t}{15}\right)$ metres above the ground, where t is measured in seconds and $0 \leq t \leq 30$.

(a) Find the initial height and the maximum height of the seat.

(b) Find the two times when the seat is $\dfrac{19}{2}$ m above the ground.

(c) Find all times in the stated interval when the seat is 2 m above the ground.

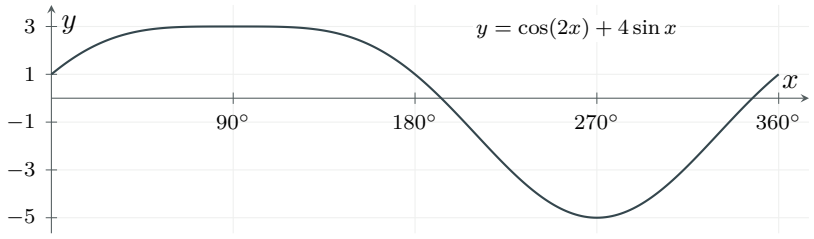


26. **Technology permitted.** Consider $\cos(2x) + 4\sin x = 2$ for $0° \leq x \leq 360°$. The curve $y = \cos(2x) + 4\sin x$ is shown. Use degree mode.

(a) Write the equation as a quadratic in $\sin x$, and find the admissible value of $\sin x$.

(b) Find all solutions of the original equation, giving angles to three significant figures.

(c) Draw the line $y = 2$ on the diagram and explain how the graph confirms the number of solutions.

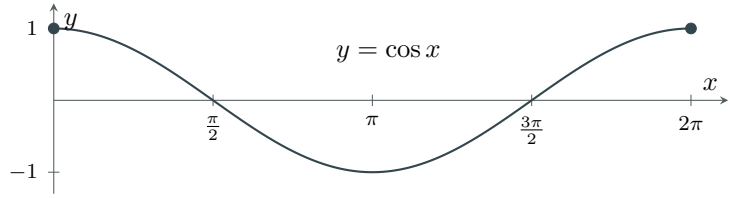


27. The graph of $y = \cos x$ is shown on the closed interval $0 \leq x \leq 2\pi$.

(a) Complete the table for the number of solutions of $\cos x = k$.

(b) State the value of k for which there is exactly one solution.

(c) Explain why $k = 1$ does not give exactly one solution.



k	$k < -1$	$k = -1$	$-1 < k < 1$	$k = 1$	$k > 1$
Number of solutions					

28. A student solves $\sin(2x) = \sqrt{3}\sin x$, for $0 \leq x \leq 2\pi$, as follows: $2\sin x\cos x = \sqrt{3}\sin x$, so $2\cos x = \sqrt{3}$.

(a) Explain why this method may lose solutions.

(b) Find all solutions of the original equation in the stated interval.

29. **Technology permitted.** The water levels in two tanks, in centimetres, are modelled by

$$A(t) = 12 + t + 2\sin\left(\frac{\pi t}{3}\right), \qquad B(t) = 14 + \frac{t}{2},$$

where t is the number of hours after observations begin and $0 \leq t \leq 12$. Use radian mode.

(a) Plot both models on a graphing calculator and find all times when the water levels are equal. Give your answers to three significant figures.

(b) Find the intervals during which the water level in tank A is greater than that in tank B. Give decimal boundaries to three significant figures.

30. Consider the equations $\sin x = \cos x$ and $\sin^2 x = \cos^2 x$ on $0 \leq x < 2\pi$.

(a) Solve $\sin^2 x = \cos^2 x$.

(b) Identify which of these values also satisfy $\sin x = \cos x$.

(c) Explain why squaring an equation can introduce extra solutions.

3 IB exam-style / Paper 1

Original non-calculator questions. Marks are shown in square brackets.

31. **[Maximum mark: 6]**

Solve the equation $2\sin^2 x - 3\sin x + 1 = 0$ for $0 \leq x \leq 2\pi$.

(a) Factorize the left-hand side as a product of two expressions in $\sin x$. [2]

(b) Hence find all values of x. [3]

(c) Explain why the equation has three distinct solutions, although there are two possible values of $\sin x$. [1]

32. **[Maximum mark: 6]**

Consider $\cos(2x) + \cos x = 0$ on $-\pi \leq x \leq \pi$.

(a) Show that the equation can be written as $2\cos^2 x + \cos x - 1 = 0$. [1]

(b) Hence solve the equation, including any solutions at the endpoints. [5]

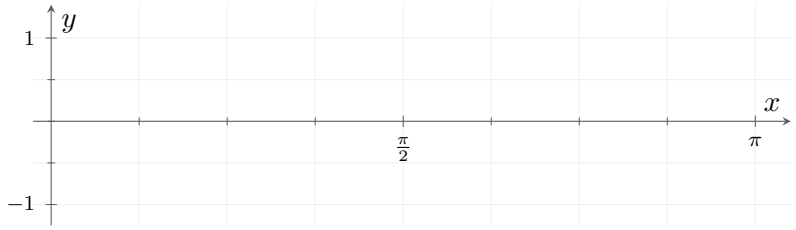
33. **[Maximum mark: 10]**

Let $f(x) = \sin(2x)$ and $g(x) = \sin x$, where $0 \leq x \leq \pi$.

(a) Find the exact coordinates of all points of intersection of the graphs of f and g. [6]

(b) On the supplied axes, sketch both graphs and label the coordinates of their intersections. [3]

(c) State the interval on which $f(x) > g(x)$. [1]



34. Solve $3\tan^2 x - 1 = 0$ for $-\pi < x < \pi$. [5]

35. **[Maximum mark: 5]**

The equation $\sin\left(\frac{x}{2}\right) = \frac{\sqrt{3}}{2}$ is to be solved on $-2\pi \leq x \leq 5\pi$.

(a) Let $u = \frac{x}{2}$. State the corresponding interval for u. [1]

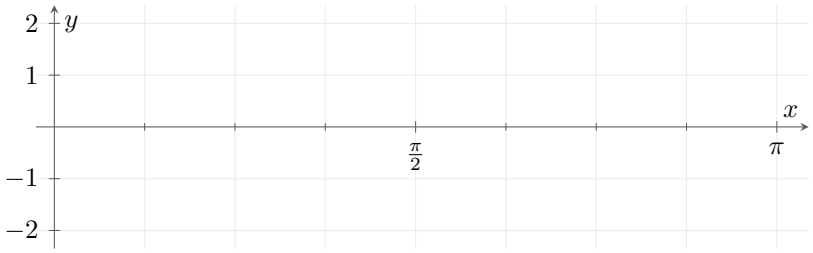
(b) Find every admissible value of u, and hence every value of x. [4]

36. **[Maximum mark: 7]**

Let $f(x) = 2\cos\left(2x - \frac{\pi}{3}\right)$, where $0 \leq x \leq \pi$.

(a) Solve $f(x) = 1$. [4]

(b) On the supplied axes, sketch $y = f(x)$ and $y = 1$. Label the x-coordinates of their intersections. [3]



37. **[Maximum mark: 6]**

The acute angle α satisfies $\tan\alpha = 2$. Consider $\tan^2 x - \tan x - 2 = 0$ for $0 \leq x < 2\pi$.

(a) Factorize the equation as a quadratic in $\tan x$. [2]

(b) Express all its solutions exactly, using α where necessary. [4]

38. Solve $\tan^2(2x) = 1$ for $0 \leq x \leq \pi$. [5]

39. **[Maximum mark: 7]**

Let $u \in \mathbb{R}$, with $u \neq -1$.

(a) Show that $2u + 1 - \dfrac{3}{u+1} = \dfrac{(2u-1)(u+2)}{u+1}$. [2]

(b) Hence, or otherwise, solve $2\sin(2x) + 1 - \dfrac{3}{1+\sin(2x)} = 0$ for $0 \leq x \leq \pi$. [5]

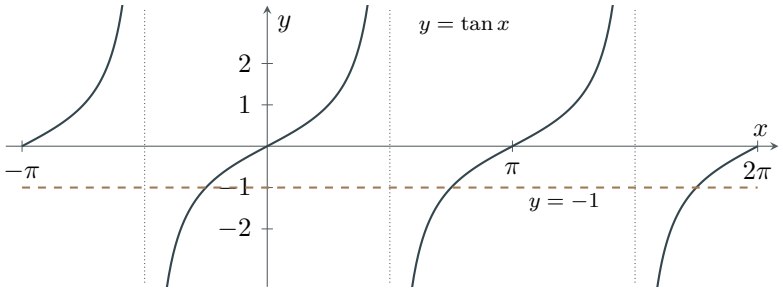
40. **[Maximum mark: 6]**

The graph shows $y = \tan x$ (solid) and $y = -1$ (dashed) on $-\pi \leq x \leq 2\pi$. Dotted vertical lines indicate asymptotes.

(a) State the number of intersections of the solid and dashed graphs. [1]

(b) Find their exact x-coordinates. [3]

(c) The dashed line is moved to $y = 0$. Find all intersection x-coordinates on the same closed interval. [2]



41. **[Maximum mark: 7]**

The functions f and g are defined for all real x by $f(x) = \sin(2x) - \cos x$ and $g(x) = 2x$.

(a) Find $(f \circ g)(x)$. [2]

(b) Solve $(f \circ g)(x) = 0$ for $0 \leq x \leq \pi$. [5]

42. Find the least positive value of x satisfying $\cos\left(\frac{x}{3} + \frac{\pi}{4}\right) = \frac{\sqrt{3}}{2}$, where $0 < x \leq 6\pi$. [5]

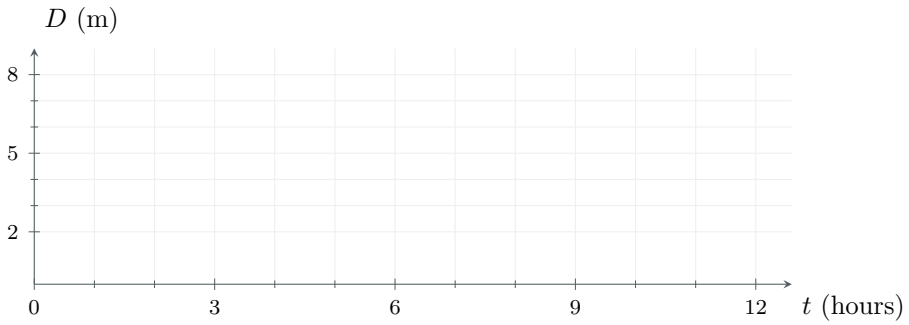
43. **[Maximum mark: 12]**

The depth of water in a harbour varies between 2 m and 8 m, completing one full cycle every 12 hours. At $t = 0$ the depth is at its minimum. The depth is modelled by $D(t) = a\cos(bt) + d$ metres, where $0 \leq t \leq 12$, $a < 0$ and $b > 0$.

(a) Find the values of a, b and d. [4]

(b) On the supplied axes, sketch $y = D(t)$. Label the coordinates of both endpoints and the maximum point. [3]

(c) Find the total time during this cycle for which the depth is at least 6.5 m. [5]



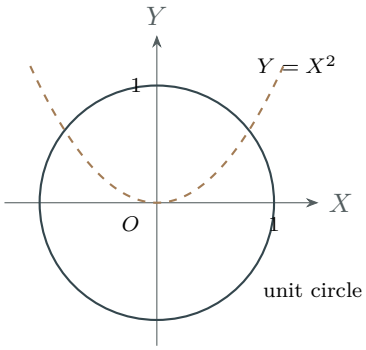
44. **[Maximum mark: 7]**

A point $P = (\cos\theta, \sin\theta)$ lies on both the unit circle and the parabola $Y = X^2$, where $0 \leq \theta < 2\pi$.

(a) Show that $\sin^2\theta + \sin\theta - 1 = 0$. [2]

(b) Find the admissible value of $\sin\theta$, giving a reason for rejecting the other root. [3]

(c) Let α be the acute angle satisfying $\sin\alpha = \frac{\sqrt{5}-1}{2}$. Express all possible values of θ in terms of α. [2]



45. **[Maximum mark: 7]**

Let $f(x) = 2\cos^2 x + \cos x$ for $0 \le x \le 2\pi$.

(a) Solve $f(x) = 0$. [3]

(b) Solve $f(x) = 3$, explaining why one algebraic root must be rejected. [4]

46. **[Maximum mark: 6]**

The equation $\cos(3x) = -\frac{1}{2}$ is to be solved for $-\pi \le x \le \pi$.

(a) Find all solutions of the equation. [5]

(b) Hence state the sum of the solutions, explaining the symmetry used. [1]

47. Solve $2\cos^2 x - 5\cos x + 2 = 0$ for $-2\pi \le x \le 2\pi$. [5]

48. **[Maximum mark: 7]**

Consider $\sin^2 x - (k+1)\sin x + k = 0$ on $0 \le x < 2\pi$, where k is real.

(a) Factorize the left-hand side. [2]

(b) Find the set of values of k for which the equation has exactly three distinct solutions. [3]

(c) Find these three solutions when $k = \frac{1}{2}$. [2]

49. **[Maximum mark: 8]**

Consider the expression $\sin(2x) - \cos(2x) - 1$.

(a) Show that $\sin(2x) - \cos(2x) - 1 = 2\cos x(\sin x - \cos x)$. [2]

(b) Hence, or otherwise, solve $\sin(2x) - \cos(2x) - 1 + \sqrt{2}(\sin x - \cos x) = 0$ for $0 \le x < 2\pi$. [6]

50. **[Maximum mark: 7]**

The equation $\dfrac{\sin x}{1 + \cos x} = 1$ is to be solved for $0 \le x < 2\pi$.

(a) State the value of x that must be excluded because the denominator is zero. [1]

(b) Solve the equation. If you square an equation, check every candidate in the original equation. [6]

4 AA HL-exclusive practice and challenge

Q51–56: independent exam practice. Q57–60: further challenge using the same section 3.8 skills.

51. Solve $\cos(4x) = \sin(2x)$ for $-\pi \le x \le \pi$. [7]

52. Solve $\tan(3x + 15°) = \sqrt{3}$ for $-60° \le x \le 120°$. [5]

53. Solve $\dfrac{\sin(2x)}{1 - \cos x} = 2\sqrt{3}\cos x$ for $0 \le x \le 2\pi$. [8]

54. Solve $\cos^2 x = \cos^2(2x)$ on $0 \le x \le 2\pi$. [7]

55. Solve $\sin(2x) = \sqrt{3}(1 + \cos(2x))$ for $-\pi \leq x \leq \pi$. [6]

56. **[Maximum mark: 7]**

Consider $\sin(4x) = \sin(2x)$ on $-\frac{\pi}{4} \leq x \leq \frac{5\pi}{4}$.

(a) Find all solutions of the equation. [6]

(b) Find the sum of the solutions. [1]

> **Further challenge: Questions 57–60**
> These questions extend the topic through solution counts and parameter reasoning.

57. **[Maximum mark: 7]**

The positive integer n is such that $\sin(nx) = \frac{1}{2}$ has exactly ten distinct solutions on $0 \leq x \leq 3\pi$.

(a) Find n, justifying your count of solutions. [5]

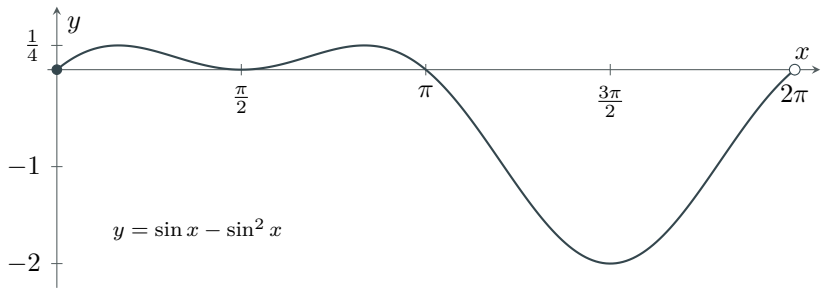
(b) Find the smallest and the largest solution. [2]

58. **[Maximum mark: 8]**

The graph shows $y = \sin x - \sin^2 x$ for $0 \leq x < 2\pi$; the open endpoint is excluded. Let $N(k)$ be the number of distinct solutions of $\sin x - \sin^2 x = k$ on this interval.

(a) Use $u = \sin x$ to show that solutions can exist only when $-2 \leq k \leq \frac{1}{4}$. [3]

(b) Determine $N(k)$ throughout this range, including the special cases $k = -2$, $k = 0$ and $k = \frac{1}{4}$. [5]



59. **[Maximum mark: 9]**

Consider $\cos^2 x - 2a\cos x + a = 0$ on $0 \leq x < 2\pi$, where a is real.

(a) Show that a necessary condition for any solution is $a \leq 0$ or $a \geq 1$. [2]

(b) Find all values of a for which there are exactly four distinct solutions. [5]

(c) Find all values of a for which there are exactly three distinct solutions. [2]

60. **[Maximum mark: 9]**

Consider $2\sin^2 x - (1 + 2a)|\sin x| + a = 0$ on $0 \leq x < 2\pi$, where a is real.

(a) Using $u = |\sin x|$, factorize the equation and state the allowed range of u. [2]

(b) Find all values of a for which the original equation has exactly eight distinct solutions. [3]

(c) Find all values of a for which it has exactly six solutions, and all values for which it has exactly four solutions. [4]

Answer key and solution outlines
Check every branch and interval endpoint, as well as the final values.

Parts are labelled to match the questions. All angles are in radians unless a degree symbol is shown.

Section 1: Short response

1. The reference angle is $30°$ and sine is positive in quadrants I and II: $x = 30°, 150°$.

2. The reference angle is $\pi/4$ and cosine is negative in quadrants II and III: $x = \dfrac{3\pi}{4}, \dfrac{5\pi}{4}$.

3. The reference angle is $60°$. Tangent repeats every $180°$, giving $x = -60°, 120°$.

4. Both solutions lie below the horizontal axis on the unit circle: $x = -\dfrac{2\pi}{3}, -\dfrac{\pi}{3}$.

5. Cosine equals 1 at complete revolutions: $x = 0, 2\pi, 4\pi$. Both stated endpoints are included.

6. $\sin x = -\sqrt{2}/2$, so $x = 225°, 315°$.

7. $\cos x = 1/2$, so $x = -\dfrac{\pi}{3}, \dfrac{\pi}{3}$.

8. Tangent is zero at integer multiples of π: $x = -\pi, 0, \pi$. The endpoint 2π is excluded.

9. $0 \le 2x \le 4\pi$, so $2x = \pi/2, 5\pi/2$. Hence $x = \dfrac{\pi}{4}, \dfrac{5\pi}{4}$.

10. $0 \le 3x \le 3\pi$, so $3x = \pi/2, 3\pi/2, 5\pi/2$. Hence $x = \dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}$.

11. $-\pi/6 \le x - \pi/6 < 11\pi/6$. The inner angle is 0 or π, giving $x = \dfrac{\pi}{6}, \dfrac{7\pi}{6}$.

12. $\pi/4 \le x + \pi/4 < 9\pi/4$. The inner angle is $\pi/4$ or $5\pi/4$, giving $x = 0, \pi$.

13. $\cos x = \pm 1/2$, so $x = \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}$.

14. For every real x, $-1 \le \sin x \le 1$. The value $3/2$ is outside this range.

15. The division removes the case $\cos x = 0$, losing $x = \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

Section 2: Extended response

16. (a) $x = \dfrac{\pi}{6}, \dfrac{5\pi}{6}$.

(b) Let $u = 2x$, so $-2\pi \le u \le 2\pi$. Then $u = -\dfrac{11\pi}{6}, -\dfrac{7\pi}{6}, \dfrac{\pi}{6}, \dfrac{5\pi}{6}$, giving

$$x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, \frac{\pi}{12}, \frac{5\pi}{12}.$$

17. (a) $-\dfrac{\pi}{3} \le u \le \dfrac{11\pi}{3}$.

(b) In this interval, $u = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}, \dfrac{10\pi}{3}$. Since $x = (u + \pi/3)/2$,

$$x = \frac{\pi}{2}, \frac{5\pi}{6}, \frac{3\pi}{2}, \frac{11\pi}{6}.$$

Section 2: Extended response (continued)

18. **(a)** Let $s = \sin x$. Then $(2s-1)(s+1) = 0$.
(b) $s = 1/2$ or $s = -1$. Therefore

$$x = -\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

19. **(a)** $(2\cos x - 1)(\cos x + 1) = 0$.
(b) $\cos x = 1/2$ or -1, giving $x = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$.

20. $(\tan x - \sqrt{3})(\tan x + 1) = 0$. Thus $\tan x = \sqrt{3}$ or -1, giving

$$x = \frac{\pi}{3}, \frac{3\pi}{4}.$$

Neither solution is a point where tangent is undefined.

21. **(a)** Using $\cos(2x) = 1 - 2\sin^2 x$ gives $2\sin^2 x + 3\sin x - 2 = 0$, or $(2\sin x - 1)(\sin x + 2) = 0$.
(b) $\sin x = 1/2$ or -2. Reject -2 because sine lies in $[-1, 1]$. Hence $x = \frac{\pi}{6}, \frac{5\pi}{6}$.

22. $2\sin x \cos x = \sqrt{2}\cos x$, so $\cos x(2\sin x - \sqrt{2}) = 0$. Thus $\cos x = 0$ or $\sin x = \sqrt{2}/2$, giving

$$x = -\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}.$$

23. **(a)** There are two intersections, one at the origin and one with $2 < x < 3$.
(b) Solving $2\sin x - x/2 = 0$ graphically gives $x = 0$ and $x = 2.474576\ldots$. Hence $x = 0, 2.47$ (non-zero solution to 3 significant figures). Check the endpoint $x = 0$ explicitly.

24. **(a)** Tangent is undefined at $x = \frac{\pi}{2}, \frac{3\pi}{2}$.
(b) $\tan x = 0$ or $\sqrt{3}$, giving $x = 0, \frac{\pi}{3}, \pi, \frac{4\pi}{3}$. None is excluded by part (a).

25. **(a)** $h(0) = 7 - 5 = 2$ m. The maximum height is $7 + 5 = 12$ m.
(b) $7 - 5\cos(\pi t/15) = 19/2$ gives $\cos(\pi t/15) = -1/2$. On $0 \le \pi t/15 \le 2\pi$, the angles are $2\pi/3, 4\pi/3$, so $t = 10, 20$ s.
(c) $h(t) = 2$ gives $\cos(\pi t/15) = 1$. Hence $t = 0, 30$ s.

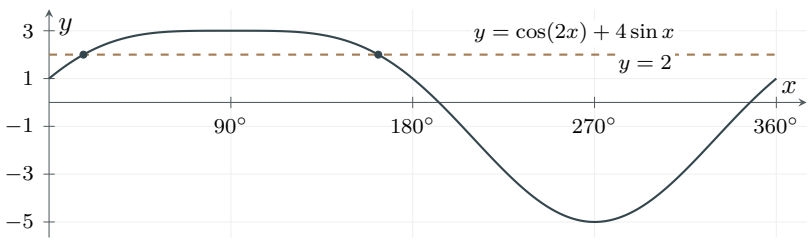
26. **(a)** Using $\cos(2x) = 1 - 2\sin^2 x$ gives

$$2\sin^2 x - 4\sin x + 1 = 0, \qquad \sin x = 1 \pm \frac{\sqrt{2}}{2}.$$

Reject $1 + \sqrt{2}/2 > 1$. The admissible value is $1 - \sqrt{2}/2$.
(b) In degree mode, $\alpha = \sin^{-1}(1 - \sqrt{2}/2) = 17.031248\ldots°$. Sine is positive in quadrants I and II, so the angles are α and $180° - \alpha$. Thus

$$x = 17.0°,\ 163° \quad \text{(3 significant figures)}.$$

(c) The horizontal line $y = 2$ intersects the curve twice in the stated interval, confirming both solutions.



Section 2: Extended response (continued)

27. **(a)** Reading the five columns from left to right: $0, 1, 2, 2, 0$.
(b) Exactly one solution occurs when $k = -1$, at $x = \pi$.
(c) When $k = 1$, the distinct values $x = 0$ and $x = 2\pi$ both satisfy the equation and both endpoints are included.

28. **(a)** Dividing by $\sin x$ discards any solutions for which $\sin x = 0$.
(b) Factor instead: $\sin x(2\cos x - \sqrt{3}) = 0$. Thus $\sin x = 0$ gives $0, \pi, 2\pi$, while $\cos x = \sqrt{3}/2$ gives $\pi/6, 11\pi/6$. The complete set is

$$x = 0, \frac{\pi}{6}, \pi, \frac{11\pi}{6}, 2\pi.$$

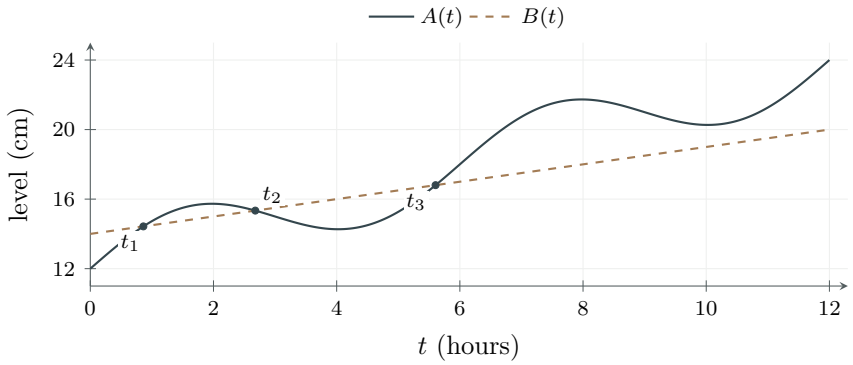
29. **(a)** Solving $A(t) - B(t) = 0$, or $\frac{t}{2} + 2\sin(\pi t/3) - 2 = 0$, gives three intersections:

$$t_1 = 0.8614519\ldots, \quad t_2 = 2.6784657\ldots, \quad t_3 = 5.6055771\ldots.$$

Hence the times are 0.861, 2.68 and 5.61 hours (3 significant figures).
(b) Using the unrounded intersection times, $A(t) > B(t)$ for

$$t_1 < t < t_2 \quad \text{or} \quad t_3 < t \leq 12.$$

Thus the intervals are approximately $(0.861, 2.68)$ and $(5.61, 12]$ hours. The rounded boundaries approximate the true intersection times. The crossings are excluded, while $t = 12$ is included because $A(12) = 24 > B(12) = 20$.



30. **(a)** $\sin^2 x = \cos^2 x$ and $\sin^2 x + \cos^2 x = 1$ imply $\cos^2 x = 1/2$. Thus

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}.$$

(b) The original equation requires sine and cosine to have the same sign. Only $x = \pi/4, 5\pi/4$ satisfy it.
(c) Squaring also admits $\sin x = -\cos x$, so $3\pi/4$ and $7\pi/4$ are extraneous for the original equation.

Section 3: IB exam-style / Paper 1

31. **(a)** $(2\sin x - 1)(\sin x - 1) = 0$.
(b) $\sin x = \frac{1}{2}$ or 1, so $x = \frac{\pi}{6}, \frac{\pi}{2}, \frac{5\pi}{6}$.
(c) $\sin x = 1$ occurs only once on the given interval, at the maximum of the sine curve.

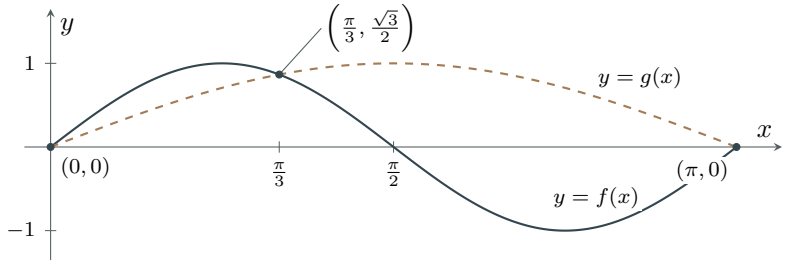
32. **(a)** Use $\cos(2x) = 2\cos^2 x - 1$.
(b) $(2\cos x - 1)(\cos x + 1) = 0$, so $\cos x = \frac{1}{2}$ or -1. Therefore $x = -\pi, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$. Both endpoints belong to the interval.

Section 3: IB exam-style / Paper 1 (continued)

33. **(a)** $\sin(2x) = \sin x$ gives $\sin x(2\cos x - 1) = 0$. Thus $x = 0, \pi$ or $\pi/3$ on the given interval. Substituting into $g(x) = \sin x$ gives the intersection coordinates

$$(0,0), \quad \left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right), \quad (\pi, 0).$$

(b) The solid curve is $y = \sin(2x)$; the dashed curve is $y = \sin x$. Their common endpoints and interior intersection are labelled below.
(c) $f(x) > g(x)$ for $0 < x < \pi/3$. On $(0, \pi)$, $\sin x > 0$, so the sign of $f(x) - g(x)$ is the sign of $2\cos x - 1$.



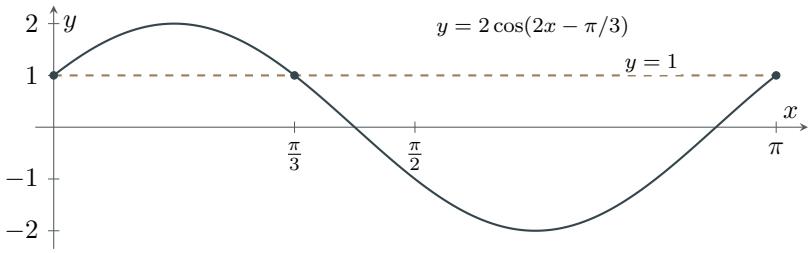
34. $\tan x = \pm 1/\sqrt{3}$. Taking all four angles in $-\pi < x < \pi$ gives

$$x = -\frac{5\pi}{6}, -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}.$$

Tangent is undefined at $\pm\pi/2$; none of the four solutions is excluded.

35. **(a)** $-\pi \le u \le \frac{5\pi}{2}$.
(b) In this interval, $u = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{7\pi}{3}$. Multiplying by 2 gives $x = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{14\pi}{3}$.

36. **(a)** Put $u = 2x - \frac{\pi}{3}$, so $-\frac{\pi}{3} \le u \le \frac{5\pi}{3}$. From $\cos u = \frac{1}{2}$, obtain $u = -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}$. Thus $x = 0, \frac{\pi}{3}, \pi$.
(b) The cosine curve has amplitude 2, period π, maximum $(\frac{\pi}{6}, 2)$ and minimum $(\frac{2\pi}{3}, -2)$; it meets $y = 1$ at the three values above.



37. **(a)** $(\tan x - 2)(\tan x + 1) = 0$.
(b) $\tan x = 2$ gives $x = \alpha, \alpha + \pi$; $\tan x = -1$ gives $x = \frac{3\pi}{4}, \frac{7\pi}{4}$. Therefore $x = \alpha, \frac{3\pi}{4}, \alpha + \pi, \frac{7\pi}{4}$.

38. $\tan(2x) = \pm 1$, with $0 \le 2x \le 2\pi$. Therefore $2x = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4$, giving

$$x = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}.$$

The excluded values $x = \pi/4, 3\pi/4$ are not solutions.

39. **(a)** The numerator over the common denominator is $(2u+1)(u+1) - 3 = 2u^2 + 3u - 2 = (2u-1)(u+2)$.
(b) The denominator excludes $\sin(2x) = -1$, which occurs at $x = 3\pi/4$. For the remaining domain, the equation gives $\sin(2x) = 1/2$ or -2. Reject -2 as outside the sine range. Since $0 \le 2x \le 2\pi$, $2x = \pi/6$ or $5\pi/6$. Therefore

$$x = \frac{\pi}{12}, \frac{5\pi}{12}.$$

Both are in the domain of the original equation.

Section 3: IB exam-style / Paper 1 (continued)

40. **(a)** Three.
(b) $\tan x = -1$ gives $x = -\frac{\pi}{4},\ \frac{3\pi}{4},\ \frac{7\pi}{4}$.
(c) $\tan x = 0$ gives $x = -\pi,\ 0,\ \pi,\ 2\pi$. There are four because both endpoints are included.

41. **(a)** $(f \circ g)(x) = f(2x) = \sin(4x) - \cos(2x)$.
(b) $\sin(4x) - \cos(2x) = \cos(2x)[2\sin(2x) - 1] = 0$. For $0 \le 2x \le 2\pi$, $\cos(2x) = 0$ gives $x = \pi/4, 3\pi/4$; $\sin(2x) = 1/2$ gives $x = \pi/12, 5\pi/12$. Hence

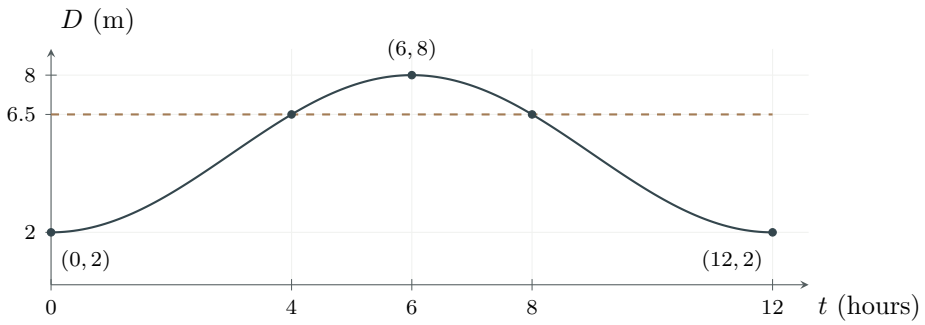
$$x = \frac{\pi}{12},\ \frac{\pi}{4},\ \frac{5\pi}{12},\ \frac{3\pi}{4}.$$

42. Let $u = x/3 + \pi/4$, so $\pi/4 < u \le 9\pi/4$. In this interval, $\cos u = \sqrt{3}/2$ gives $u = 11\pi/6$ or $13\pi/6$. Therefore $x = 3(u - \pi/4)$ gives $19\pi/4$ or $23\pi/4$. The least positive value is $\boxed{19\pi/4}$. The principal angle $u = \pi/6$ would give a negative x and is outside the transformed interval.

43. **(a)** The midline is $d = (8 + 2)/2 = 5$ and the amplitude is $(8 - 2)/2 = 3$. Since the depth starts at its minimum, $a = -3$. The period is 12, so $b = 2\pi/12 = \pi/6$.
(b) The graph starts at $(0, 2)$, reaches its maximum at $(6, 8)$ and ends at $(12, 2)$.
(c) $5 - 3\cos(\pi t/6) \ge 6.5$ gives $\cos(\pi t/6) \le -1/2$. Over one cycle,

$$\frac{2\pi}{3} \le \frac{\pi t}{6} \le \frac{4\pi}{3} \quad \Longrightarrow \quad 4 \le t \le 8.$$

The total time is $8 - 4 = \boxed{4 \text{ hours}}$.



44. **(a)** $\sin\theta = \cos^2\theta = 1 - \sin^2\theta$.
(b) $\sin\theta = \frac{-1 \pm \sqrt{5}}{2}$. The negative root is less than -1 and is impossible; hence $\sin\theta = \frac{\sqrt{5}-1}{2}$.
(c) $\theta = \alpha$ or $\pi - \alpha$, corresponding to the two upper-half-plane intersections.

45. **(a)** $\cos x(2\cos x + 1) = 0$, so $x = \frac{\pi}{2},\ \frac{2\pi}{3},\ \frac{4\pi}{3},\ \frac{3\pi}{2}$.
(b) $2\cos^2 x + \cos x - 3 = (2\cos x + 3)(\cos x - 1) = 0$. Reject $\cos x = -\frac{3}{2}$; $\cos x = 1$ gives $x = 0, 2\pi$.

46. **(a)** Since $-3\pi \le 3x \le 3\pi$, the admissible values of $3x$ are $\pm\frac{2\pi}{3},\ \pm\frac{4\pi}{3},\ \pm\frac{8\pi}{3}$. Thus $x = -\frac{8\pi}{9},\ -\frac{4\pi}{9},\ -\frac{2\pi}{9},\ \frac{2\pi}{9},\ \frac{4\pi}{9},\ \frac{8\pi}{9}$.
(b) The sum is 0: each positive root is paired with its negative because cosine is even and the interval is symmetric.

47. $(2\cos x - 1)(\cos x - 2) = 0$, so $\cos x = 1/2$ or 2. Reject 2 because cosine lies in $[-1, 1]$. The full solution set is

$$x = -\frac{5\pi}{3}, -\frac{\pi}{3}, \frac{\pi}{3}, \frac{5\pi}{3}.$$

48. **(a)** $(\sin x - 1)(\sin x - k) = 0$.
(b) $\sin x = 1$ always gives $x = \frac{\pi}{2}$. A second, distinct sine value gives two further roots precisely when $-1 < k < 1$. At $k = \pm 1$ a branch reaches an extremum or coincides; outside $[-1, 1]$ it has no roots.
(c) $x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6}$.

Section 3: IB exam-style / Paper 1 (continued)

49. **(a)** Using the double-angle identities,

$$\sin(2x) - \cos(2x) - 1 = 2\sin x\cos x - (2\cos^2 x - 1) - 1 = 2\cos x(\sin x - \cos x).$$

(b) The equation becomes $(2\cos x + \sqrt{2})(\sin x - \cos x) = 0$. The first factor gives $x = 3\pi/4, 5\pi/4$; the second gives $x = \pi/4, 5\pi/4$. Count the shared value $5\pi/4$ once. Thus

$$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}.$$

50. **(a)** Exclude $x = \pi$.
(b) Multiplying by the nonzero denominator gives $\sin x = 1 + \cos x$. Squaring and using $\sin^2 x = 1 - \cos^2 x$ gives $2\cos x(\cos x + 1) = 0$. The candidates are $\frac{\pi}{2}, \pi, \frac{3\pi}{2}$. Reject π (undefined) and $\frac{3\pi}{2}$ (left side of the original equation is -1). The only solution is $x = \frac{\pi}{2}$.

Section 4: AA HL-exclusive practice and challenge

51. Put $u = \sin(2x)$. Then $1 - 2u^2 = u$, so $(2u - 1)(u + 1) = 0$. On $-2\pi \le 2x \le 2\pi$, $u = \frac{1}{2}$ gives $2x = -\frac{11\pi}{6}, -\frac{7\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}$; $u = -1$ gives $2x = -\frac{\pi}{2}, \frac{3\pi}{2}$. Hence $x = -\frac{11\pi}{12}, -\frac{7\pi}{12}, -\frac{\pi}{4}, \frac{\pi}{12}, \frac{5\pi}{12}, \frac{3\pi}{4}$.

52. Let $u = 3x + 15°$, so $-165° \le u \le 375°$. The values with $\tan u = \sqrt{3}$ are $u = -120°, 60°, 240°$. Using $x = (u - 15°)/3$ gives $\boxed{x = -45°, 15°, 75°}$.

53. The denominator excludes $x = 0, 2\pi$. Multiplying by $1 - \cos x$ and using the double-angle identity gives

$$2\cos x[\sin x - \sqrt{3}(1 - \cos x)] = 0.$$

The factor $\cos x = 0$ gives $x = \pi/2, 3\pi/2$. For the other factor, square $\sin x = \sqrt{3}(1 - \cos x)$ and use $\sin^2 x = 1 - \cos^2 x$ to obtain $2(2\cos x - 1)(\cos x - 1) = 0$. The branch $\cos x = 1$ is excluded. The branch $\cos x = 1/2$ gives $\pi/3, 5\pi/3$, but only $\pi/3$ satisfies the unsquared equation. Thus all solutions are

$$x = \frac{\pi}{3}, \frac{\pi}{2}, \frac{3\pi}{2}.$$

54. The equation gives $\cos(2x) = \cos x$ or $\cos(2x) = -\cos x$. Writing $c = \cos x$ gives $(2c + 1)(c - 1) = 0$ or $(2c - 1)(c + 1) = 0$. Thus $c \in \{-1, -\frac{1}{2}, \frac{1}{2}, 1\}$. The seven solutions are $x = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$.

55. Using $\sin(2x) = 2\sin x\cos x$ and $1 + \cos(2x) = 2\cos^2 x$ gives

$$2\cos x(\sin x - \sqrt{3}\cos x) = 0.$$

The first factor gives $x = -\pi/2, \pi/2$. In the second factor $\cos x \neq 0$, so $\tan x = \sqrt{3}$, giving $x = -2\pi/3, \pi/3$. The complete set is

$$x = -\frac{2\pi}{3}, -\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{2}.$$

56. **(a)** $\sin(2x)[2\cos(2x) - 1] = 0$. The first factor gives $x = 0, \frac{\pi}{2}, \pi$. The second gives $x = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}$. Thus the seven values, in order, are $-\frac{\pi}{6}$, 0, $\frac{\pi}{6}$, $\frac{\pi}{2}$, $\frac{5\pi}{6}$, π, $\frac{7\pi}{6}$.
(b) Their sum is $\frac{7\pi}{2}$; alternatively, they form three pairs with sum π, together with the central value $\frac{\pi}{2}$.

Further challenge: Questions 57–60

57. **(a)** Put $u = nx$, so $0 \le u \le 3n\pi$. Each full 2π cycle contributes two roots. If n is even, there are $3n$ roots; if n is odd, the last positive half-cycle contributes two, giving $3n + 1$ roots. Thus $3n + 1 = 10$ gives $n = 3$; the even case cannot give ten.
(b) On $0 \le 3x \le 9\pi$, the first and last values of $3x$ are $\frac{\pi}{6}$ and $\frac{53\pi}{6}$. Hence $x_{\min} = \frac{\pi}{18}$ and $x_{\max} = \frac{53\pi}{18}$.

Section 4: AA HL-exclusive practice and challenge (continued)

58. **(a)** $k = u - u^2 = \frac{1}{4} - (u - \frac{1}{2})^2$ with $-1 \leq u \leq 1$. Its minimum is -2 at $u = -1$ and maximum $\frac{1}{4}$ at $u = \frac{1}{2}$.
(b) $N(-2) = 1$; $N(k) = 2$ for $-2 < k < 0$; $N(0) = 3$; $N(k) = 4$ for $0 < k < \frac{1}{4}$; and $N(\frac{1}{4}) = 2$. For $k < 0$, just one root of the quadratic lies in $[-1, 1]$. For $0 < k < \frac{1}{4}$, both roots lie strictly between 0 and 1. At $k = 0$, $\sin x = 0$ or 1 gives $0, \frac{\pi}{2}, \pi$. At the maximum, the repeated sine value $\frac{1}{2}$ still gives two distinct angles.

59. **(a)** With $u = \cos x$, the discriminant is $4a(a - 1) \geq 0$.
(b) The roots are $u = a \pm \sqrt{a^2 - a}$. For $a < 0$, the larger root lies in $(0, \frac{1}{2})$; the smaller lies in $(-1, 0)$ precisely when $-\frac{1}{3} < a < 0$ (the boundary $u = -1$ gives $1 + 3a = 0$). Both then produce two angles, giving exactly four solutions. For $a > 1$, one root exceeds 1; at $a = 0$ or 1 the roots coincide. Hence $-\frac{1}{3} < a < 0$.
(c) Only $a = -\frac{1}{3}$: the two cosine values are -1 and $\frac{1}{3}$, producing one and two angles respectively. At $a = 0$ there are two solutions, and at $a = 1$ only one.

60. **(a)** $(2u - 1)(u - a) = 0$, with $0 \leq u \leq 1$.
(b) $u = \frac{1}{2}$ always gives four angles. A second distinct value strictly between 0 and 1 gives four more, so exactly eight occur for $0 < a < 1$, $a \neq \frac{1}{2}$.
(c) Exactly six occur at $a = 0$ or $a = 1$, because either adds two angles. Exactly four occur for $a < 0$, $a > 1$, or $a = \frac{1}{2}$: the extra branch is inadmissible or coincides with the first. These cases exhaust all real a.