

Discrete Random Variables & Expected Value

IB Mathematics: Analysis and Approaches SL & HL — SL 4.7

60 questions: 50-question core worksheet + 10-question HL-only challenge section

Learning goal

Build, interpret and use discrete probability distributions; calculate expected value; apply expected value to decision-making and fair games.

Syllabus link

IB Mathematics AA SL/HL: **SL 4.7** — discrete random variables and probability distributions; expected value (mean) for discrete data; applications, including fair games.

Structure

Section 1: 15 short response; Section 2: 15 extended response; Section 3: 20 IB Paper 1-style questions; Section 4: 10 HL-only challenge questions.

Calculator / technology

Designed for **non-calculator** practice. Give exact values unless a question explicitly asks otherwise.

Instructions. Show clear working. For every probability distribution, check that all probabilities are between 0 and 1 and that their sum is 1. For expected value, use

$$E(X) = \sum x P(X = x).$$

When a random variable represents a player's *net gain*, a fair game has $E(X) = 0$.

Section 1: Short response

Questions 1–15

1. A discrete random variable X has the distribution

X	-2	0	3
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

List the possible values of X , and find $P(X > 0)$.

2. Determine whether the following table defines a valid probability distribution. Give a reason.

X	0	1	2
$P(X = x)$	0.3	0.4	0.3

3. The distribution of X is

X	1	2	3	4
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	p	$\frac{1}{4}$

Find p .

4. For the distribution

X	0	1	2	3
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$

find $E(X)$.

5. A random variable X takes the values $-1, 2, 5$ with probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, respectively. Find $E(X)$.

6. The probabilities that $X = 0, 2, 4$ are $p, 2p, 3p$, respectively. Find p and hence find $E(X)$.

7. A discrete random variable has probability function

$$P(X = x) = k(x + 1), \quad x \in \{0, 1, 2, 3\}.$$

Find k and $E(X)$.

8. For the distribution

X	-2	-1	1	4
$P(X = x)$	$\frac{1}{5}$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{3}{10}$

find $P(X \geq 1)$.

9. A player rolls a fair six-sided die. The player's net gain X , in euros, is 4 if a 6 is rolled and -1 otherwise. Find $E(X)$.

10. In a game, the net gain X , in euros, is -2 with probability $\frac{1}{2}$ and 3 with probability $\frac{1}{2}$. Determine whether the game is fair.

11. A game costs €2 to play. A player receives a prize of € P with probability $\frac{1}{4}$, and no prize otherwise. Find P so that the game is fair.

12. A fair spinner has four equal sectors labelled 1, 1, 2, 4. Let X be the number on the sector where the spinner stops. Write down the probability distribution of X , and find $E(X)$.

13. A fair die is rolled once and $X = |R - 3|$, where R is the number rolled. Write down the distribution of X and find $E(X)$.

14. A raffle has 100 tickets. One ticket wins €60 and all other tickets win nothing. A ticket costs €1. Find the expected net gain to a person who buys one ticket.

15. Game A has net gain -1 with probability $\frac{3}{4}$ and 3 with probability $\frac{1}{4}$. Game B has net gain -2 with probability $\frac{1}{2}$ and 2 with probability $\frac{1}{2}$. Compare the games using expected value.

Section 2: Extended response

Questions 16–30

16. A bag contains 2 red counters and 3 blue counters. One counter is selected at random. The net gain X is €2 for a red counter and $-\text{€}1$ for a blue counter.

- Write down the probability distribution of X .
- Find $E(X)$ and interpret your answer.

17. A fair die is rolled. A player gains €5 for a 6, gains €1 for a 5, and loses €1 otherwise.

- Construct the probability distribution of the player's net gain X .
- Find $E(X)$.

18. A discrete random variable satisfies

$$P(X = x) = kx, \quad x \in \{1, 2, 3, 4\}.$$

- Find k .
- Find $E(X)$.
- Find $P(X \geq 3)$.

19. A random variable X takes values 0, 1, 2, 3 with probabilities $a, 2a, 3a, 4a$, respectively.

- Find a .
- Find $E(X)$.

20. A random variable X has distribution

X	-1	1	4
$P(X = x)$	p	$\frac{1}{2}$	$\frac{1}{2} - p$

Given that $E(X) = 1$, find p and write the completed distribution.

21. A player rolls a fair die. If a 6 is rolled, the player's net gain is € a . Otherwise the player loses €2. Find a so that the game is fair.

22. The number X of items sold by a small stall during a certain time period has distribution

X	0	1	2	3
$P(X = x)$	0.1	0.3	0.4	0.2

The profit, in euros, is -8, 4, 16, 28 when $X = 0, 1, 2, 3$, respectively.

- Find the expected number of items sold.
- Find the expected profit.

23. A raffle has 200 tickets, each costing €1. There is one prize of €80, one prize of €20, and one prize of €10.

- Construct the distribution of the net gain X for a person who buys one ticket.
- Find $E(X)$.
- State the expected profit to the organizer if all 200 tickets are sold.

24. Two fair coins are tossed. A player's net gain is €3 if two heads occur, €1 if exactly one head occurs, and -€2 if no heads occur.

- Construct the distribution of the net gain X .
- Find $E(X)$.

25. A spinner pays a gross prize X , in euros, with distribution

X	0	4	10
$P(X = x)$	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{1}{8}$

Find the entry fee that would make the game fair.

26. A random variable X takes values 1, 2, 3, 4 with probabilities proportional to 1, 2, 2, 1, respectively.

- Write down the probability distribution of X .
- Find $E(X)$.

27. A random variable has probability function

$$P(X = x) = k(5 - x), \quad x \in \{1, 2, 3, 4\}.$$

Find k , $E(X)$, and $P(X < 3)$.

28. The random variable X has distribution

X	-2	1	a
$P(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

Given that $E(X) = 1$, find a .

29. Game A has net gains -2, 1, 4 with probabilities $\frac{1}{4}, \frac{1}{2}, \frac{1}{4}$, respectively. Game B has net gains 0, 1, 2 with the same respective probabilities.

- Find the expected gain for each game.
- What can be concluded if the games are compared only using expected value?

30. A biased coin lands heads with probability p . A player gains €3 for heads and loses €1 for tails. Find p if the game is fair.

Section 3: IB exam-style / Paper 1

Questions 31–50 Original questions

31. A discrete random variable X has distribution

X	0	1	2	3
$P(X = x)$	k	$2k$	$3k$	$4k$

- Find k . [2 marks]
- Find $P(X \geq 2)$. [1 marks]
- Find $E(X)$. [2 marks]

32. A charity sells 50 tickets for a draw. Each ticket costs €2. One ticket wins €30, two tickets win €10 each, and the remaining tickets win nothing.

- Write down the probability distribution of the net gain X for one ticket. [2 marks]
- Find $E(X)$. [2 marks]
- Hence state whether the game is fair. [1 marks]
- Find the ticket price that would make the game fair, with the prizes unchanged. [2 marks]

33. The probability function of X is

$$P(X = x) = k(2x + 1), \quad x \in \{0, 1, 2, 3\}.$$

- (a) Find k . [2 marks]
 (b) Find $E(X)$. [3 marks]
 (c) Find $P(X > E(X))$. [2 marks]

34. A spinner takes the values 1, 2, 3 with probabilities $\frac{1}{2}, \frac{1}{3}, \frac{1}{6}$, respectively. A player receives a gross prize of $\text{€}(2X)$, where X is the spinner value.

- (a) Find $E(X)$. [2 marks]
 (b) Hence find the entry fee that makes the game fair. [3 marks]

35. A discrete random variable has distribution

X	-2	0	3
$P(X = x)$	p	$\frac{1}{3}$	$\frac{2}{3} - p$

Given that $E(X) = 1$:

- (a) find p ; [3 marks]
 (b) find $P(X > 0)$. [2 marks]

36. A bag contains six cards labelled 1, 1, 2, 3, 3, 3. One card is chosen at random and its number is denoted by X .

- (a) Write down the probability distribution of X . [2 marks]
 (b) Find $E(X)$. [2 marks]
 (c) A game pays a gross prize of $\text{€}(2X)$ and costs $\text{€}5$ to play. Find the expected net gain. [2 marks]

37. A random variable X takes the values 0, 1, 2, with

$$P(X = x) = ax + b.$$

It is given that $P(X = 0) = \frac{1}{6}$.

- (a) Find b . [1 marks]
 (b) Using the fact that the probabilities sum to 1, find a . [3 marks]
 (c) Find $E(X)$. [2 marks]

38. A game is played in two stages. First, a fair spinner selects A or B. If A is selected, the net gain is $\text{€}2$. If B is selected, a fair die is rolled: the net gain is $\text{€}5$ for a 6 and $-\text{€}1$ otherwise.

- (a) Construct the probability distribution of the overall net gain X . [3 marks]
 (b) Find $E(X)$. [2 marks]
 (c) An additional fixed fee c euros is charged to play. Find c so that the complete game is fair. [1 marks]

39. The number X of premium packages sold by an employee in one hour has distribution

X	0	1	2	3
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

The employee earns a fixed $\text{€}15$ plus $\text{€}8$ for each premium package sold.

- (a) Find $E(X)$. [2 marks]
- (b) Write down the possible hourly earnings and their probabilities. [2 marks]
- (c) Find the expected hourly earnings. [2 marks]

40. A board-game score X has distribution

X	-4	-1	2	5
$P(X = x)$	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

- (a) Find $E(X)$. [3 marks]
- (b) A fixed bonus b is added to every possible score. Find b so that the expected score becomes 3. [2 marks]

41. The net gain X , in euros, from a game is given by

X	-3	1	a
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

Find a if the game is fair. [4 marks]

42. A player pays €4 to play a game. The player receives €10 with probability $\frac{1}{5}$, €5 with probability $\frac{1}{5}$, and no prize otherwise.

- (a) Write down the distribution of the player's net gain X . [2 marks]
- (b) Find $E(X)$. [2 marks]
- (c) Hence find the organizer's expected profit from 1000 plays. [1 marks]

43. A random variable X has distribution

X	0	2	5
$P(X = x)$	p	q	$\frac{1}{4}$

Given that $E(X) = 2$, find p and q . [5 marks]

44. A random variable X has distribution

X	1	2	4
$P(X = x)$	k	$2k$	$1 - 3k$

Given that $E(X) = 3$:

- (a) find k ; [3 marks]
- (b) find $P(X \text{ is even})$. [2 marks]

45. A lottery ticket costs €3. The gross prize is €100 with probability $\frac{1}{100}$, €10 with probability $\frac{4}{100}$, and €0 otherwise.

- (a) Find the expected gross prize. [2 marks]
- (b) Find the expected net gain to the player. [2 marks]
- (c) The four €10 prizes remain unchanged. Find the value of the top prize that would make the game fair at a ticket price of €3. [2 marks]

46. A discrete random variable has probability function

$$P(X = x) = k(4 - x), \quad x \in \{0, 1, 2, 3\}.$$

- (a) Find k . [2 marks]
 (b) Find $E(X)$. [2 marks]
 (c) Find $P(X \leq 1)$. [1 marks]

47. Two different cards are chosen at random from three cards labelled 1, 2, 3. Let X be the sum of the two selected labels.

- (a) Write down the possible values of X and their probabilities. [3 marks]
 (b) Find $E(X)$. [2 marks]
 (c) A player receives € X and pays € c to play. Find c for a fair game. [1 marks]

48. A delivery company issues a daily refund X , in euros, with distribution

X	0	5	15
$P(X = x)$	0.7	0.2	0.1

- (a) Find the expected daily refund. [2 marks]
 (b) The company replaces the €15 refund by € r , leaving the probabilities unchanged. Find the greatest value of r for which the expected daily refund is at most €2. [3 marks]

49. For $x \in \{1, 2, 3\}$, the probability function of X is

$$P(X = x) = \frac{c}{2^x}.$$

- (a) Find c . [3 marks]
 (b) Find $E(X)$. [2 marks]

50. A game has net gain X with distribution

X	-2	1	4
$P(X = x)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

- (a) Find $E(X)$. [3 marks]
 (b) The outcome 4 is replaced by a , with the probabilities unchanged. Find a so that the game becomes fair. [3 marks]

Section 4: HL-only challenge

Questions 51–60 SL 4.7 content at greater depth

HL note. These questions deliberately stay inside the content of SL 4.7. Their HL character comes from greater algebraic complexity, parameterisation, modelling, generalisation and multi-stage reasoning — not from introducing AHL 4.14 variance or continuous random variables.

51. For $x \in \{1, 2, 3, 4\}$, a random variable has probability function

$$P(X = x) = kx(5 - x).$$

- (a) Find k . [3 marks]
 (b) Find $E(X)$. [3 marks]
 (c) Explain briefly why the value of $E(X)$ is consistent with the symmetry of the distribution. [2 marks]

52. A family of discrete random variables has distribution

$$\begin{array}{c|ccc} X & -2 & 1 & 4 \\ \hline P(X=x) & p & 1-2p & p \end{array}, \quad 0 \leq p \leq \frac{1}{2}.$$

- (a) Show that this is a valid probability distribution for the stated range of p . [2 marks]
 (b) Show that $E(X) = 1$ for every allowed value of p . [3 marks]
 (c) Interpret what part (b) tells you about comparing members of this family using expected value alone. [2 marks]

53. A random variable X takes values $0, 1, 2, 3$. Its probabilities form an arithmetic sequence:

$$P(X=0) = a, \quad P(X=1) = a+r, \quad P(X=2) = a+2r, \quad P(X=3) = a+3r.$$

Given that $E(X) = 2$, find a and r , and hence write down the distribution. [7 marks]

54. A game costs €3 to play. A player receives a gross prize of € A with probability $\frac{1}{6}$, a gross prize of € B with probability $\frac{1}{3}$, and no prize otherwise. It is given that $A = 2B$.

- (a) Find A and B if the game is fair. [5 marks]
 (b) Check your answer by calculating the expected net gain. [2 marks]

55. A random integer $X \in \{1, 2, 3, 4\}$ is selected with probability proportional to the square of its value; that is,

$$P(X=x) = kx^2.$$

- (a) Find k . [2 marks]
 (b) Find $E(X)$. [3 marks]
 (c) Find $P(X > E(X))$. [2 marks]

56. A game has net gain X , in euros, with

$$\begin{array}{c|ccc} X & -4 & 1 & 10 \\ \hline P(X=x) & \frac{2}{3}-p & \frac{1}{3} & p \end{array}, \quad 0 \leq p \leq \frac{2}{3}.$$

- (a) Find $E(X)$ in terms of p . [3 marks]
 (b) Find the value of p for which the game is fair. [2 marks]
 (c) Determine the range of p for which the game is favourable to the player in the sense that $E(X) > 0$. [2 marks]

57. Let n be a positive integer. A random variable X takes values $1, 2, \dots, n$ with

$$P(X=x) = \frac{2x}{n(n+1)}.$$

You may use

$$\sum_{x=1}^n x = \frac{n(n+1)}{2}, \quad \sum_{x=1}^n x^2 = \frac{n(n+1)(2n+1)}{6}.$$

- (a) Show that the probabilities sum to 1. [3 marks]
 (b) Show that

$$E(X) = \frac{2n+1}{3}.$$

[4 marks]

58. For $r > 0$, a random variable X takes values 0, 1, 2, 3 with probabilities proportional to

$$1, r, r^2, r^3,$$

respectively. It is given that $P(X \geq 2) = \frac{3}{4}$.

- (a) Show that $r = \sqrt{3}$. [5 marks]
(b) Hence find $E(X)$, giving your answer in exact form. [4 marks]
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59. A game has possible net gains $-2, 1, 5$. The probabilities are $p, \frac{1}{2}, \frac{1}{2} - p$, respectively.

- (a) Find p if the game is fair. [4 marks]
(b) For this value of p , find the probability that a player makes a positive gain. [2 marks]
(c) Explain why a probability of winning greater than $\frac{1}{2}$ does not necessarily imply a positive expected gain. [2 marks]
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60. A game is played by first choosing one of three boxes. Box A is chosen with probability $\frac{1}{2}$, Box B with probability $\frac{1}{3}$, and Box C with probability $\frac{1}{6}$.

- If Box A is chosen, the net gain is -2 or 4 , each with probability $\frac{1}{2}$.
 - If Box B is chosen, the net gain is always 1 .
 - If Box C is chosen, the net gain is -5 or 7 , each with probability $\frac{1}{2}$.
- (a) Construct the overall probability distribution of the net gain X . [5 marks]
(b) Find $E(X)$. [3 marks]
(c) Find the additional fixed fee that would make the complete game fair. [1 marks]

Answer key and solution outlines

Questions 1–60

Section 1: Short response

1. Values: $-2, 0, 3$; $P(X > 0) = \frac{1}{4}$. 2. Yes; all probabilities are in $[0, 1]$ and sum to 1. 3. $p = \frac{1}{4}$.
4. $E(X) = \frac{0+1+2+3}{4} = \frac{3}{2}$. 5. $E(X) = -\frac{1}{2} + \frac{2}{3} + \frac{5}{6} = 1$. 6. $6p = 1 \Rightarrow p = \frac{1}{6}$; $E(X) = \frac{8}{3}$.
7. $10k = 1 \Rightarrow k = \frac{1}{10}$; $E(X) = 2$. 8. $\frac{2}{5} + \frac{3}{10} = \frac{7}{10}$. 9. $E(X) = \frac{1}{6}(4) + \frac{5}{6}(-1) = -\frac{1}{6}$.
10. $E(X) = \frac{1}{2}(-2) + \frac{1}{2}(3) = \frac{1}{2} \neq 0$; not fair. 11. $\frac{1}{4}(P - 2) + \frac{3}{4}(-2) = 0 \Rightarrow P = 8$.
12. $P(1) = \frac{1}{2}, P(2) = \frac{1}{4}, P(4) = \frac{1}{4}$; $E(X) = 2$.
13. $P(0) = \frac{1}{6}, P(1) = \frac{1}{3}, P(2) = \frac{1}{3}, P(3) = \frac{1}{6}$; $E(X) = \frac{3}{2}$.
14. Net gains 59 and -1 : $E(X) = \frac{1}{100}(59) + \frac{99}{100}(-1) = -0.40$ euros.
15. Both have $E(X) = 0$, so both are fair and are indistinguishable by expected value alone.

Section 2: Extended response

16. $P(X = 2) = \frac{2}{5}, P(X = -1) = \frac{3}{5}$. $E(X) = \frac{1}{5}$ euro: an average gain of €0.20 per play over many plays.
17. $P(-1) = \frac{4}{6}, P(1) = \frac{1}{6}, P(5) = \frac{1}{6}$. $E(X) = \frac{1}{3}$.
18. $10k = 1 \Rightarrow k = \frac{1}{10}$; $E(X) = 3$; $P(X \geq 3) = \frac{7}{10}$.
19. $10a = 1 \Rightarrow a = \frac{1}{10}$; $E(X) = 2$.
20. $-p + \frac{1}{2} + 4(\frac{1}{2} - p) = 1 \Rightarrow p = \frac{3}{10}$. Thus probabilities are $\frac{3}{10}, \frac{1}{2}, \frac{1}{5}$.
21. $\frac{a}{6} + \frac{5}{6}(-2) = 0 \Rightarrow a = 10$.
22. $E(X) = 0(0.1) + 1(0.3) + 2(0.4) + 3(0.2) = \frac{17}{10}$. Expected profit = $-8(0.1) + 4(0.3) + 16(0.4) + 28(0.2) = \frac{62}{5} = 12.4$ euros.
23. Net gains: 79 with $\frac{1}{200}$, 19 with $\frac{1}{200}$, 9 with $\frac{1}{200}$, -1 with $\frac{197}{200}$. $E(X) = -\frac{9}{20} = -0.45$ euro. Organizer expected profit for 200 tickets: €90.
24. $P(3) = \frac{1}{4}, P(1) = \frac{1}{2}, P(-2) = \frac{1}{4}$; $E(X) = \frac{3}{4}$ euro.
25. Expected gross prize = $4 \cdot \frac{3}{8} + 10 \cdot \frac{1}{8} = \frac{11}{4}$. Fair entry fee = € $\frac{11}{4}$.
26. Distribution $\frac{1}{6}, \frac{1}{3}, \frac{1}{3}, \frac{1}{6}$; $E(X) = \frac{5}{2}$.
27. Weights 4, 3, 2, 1, so $k = \frac{1}{10}$; $E(X) = 2$; $P(X < 3) = \frac{7}{10}$.
28. $-\frac{1}{2} + \frac{1}{2} + \frac{a}{4} = 1 \Rightarrow a = 4$.
29. $E(A) = 1, E(B) = 1$. By expected value alone the games are equally favourable.
30. $3p - (1 - p) = 0 \Rightarrow p = \frac{1}{4}$.

Section 3: IB exam-style / Paper 1

31. (a) $k = \frac{1}{10}$. (b) $\frac{7}{10}$. (c) $E(X) = 2$.
32. (a) $X = 28$ with $\frac{1}{50}$, $X = 8$ with $\frac{2}{50}$, $X = -2$ with $\frac{47}{50}$. (b) $E(X) = -1$ euro. (c) Not fair. (d) Fair ticket price $= \frac{30+10+10}{50} = \text{€}1$.
33. (a) $16k = 1 \Rightarrow k = \frac{1}{16}$. (b) $E(X) = \frac{17}{8}$. (c) Only $X = 3$ exceeds $\frac{17}{8}$, so $P = \frac{7}{16}$.
34. (a) $E(X) = \frac{5}{3}$. (b) Expected gross prize $= 2E(X) = \frac{10}{3}$, so fair fee is $\text{€}\frac{10}{3}$.
35. $-2p + 3(\frac{2}{3} - p) = 1 \Rightarrow p = \frac{1}{5}$. Then $P(X > 0) = \frac{2}{3} - \frac{1}{5} = \frac{7}{15}$.
36. (a) $P(1) = \frac{1}{3}$, $P(2) = \frac{1}{6}$, $P(3) = \frac{1}{2}$. (b) $E(X) = \frac{13}{6}$. (c) Expected gross prize $= \frac{13}{3}$; expected net gain $= \frac{13}{3} - 5 = -\frac{2}{3}$ euro.
37. $b = \frac{1}{6}$. Sum: $b + (a + b) + (2a + b) = 1 \Rightarrow 3a + 3b = 1$, so $a = \frac{1}{6}$. Distribution probabilities: $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}$. $E(X) = \frac{4}{3}$.
38. Overall: $P(X = 2) = \frac{1}{2}$, $P(X = 5) = \frac{1}{12}$, $P(X = -1) = \frac{5}{12}$. $E(X) = 1$ euro. Additional fair fee: $c = 1$ euro.
39. (a) $E(X) = \frac{3}{2}$. (b) Earnings 15, 23, 31, 39 with probabilities $\frac{1}{8}, \frac{3}{8}, \frac{3}{8}, \frac{1}{8}$. (c) Expected earnings $= \text{€}27$.
40. (a) $E(X) = \frac{1}{2}$. (b) $\frac{1}{2} + b = 3 \Rightarrow b = \frac{5}{2}$.
41. $-3(\frac{1}{2}) + 1(\frac{1}{3}) + a(\frac{1}{6}) = 0 \Rightarrow a = 7$.
42. (a) Net gains: 6 with $\frac{1}{5}$, 1 with $\frac{1}{5}$, -4 with $\frac{3}{5}$. (b) $E(X) = -1$ euro. (c) Organizer's expected profit: $\text{€}1000$.
43. $p + q = \frac{3}{4}$ and $2q + \frac{5}{4} = 2$, giving $q = \frac{3}{8}$, $p = \frac{3}{8}$.
44. $k + 4k + 4(1 - 3k) = 3 \Rightarrow k = \frac{1}{7}$. Then $P(\text{even}) = 2k + (1 - 3k) = \frac{6}{7}$.
45. Expected gross prize $= \frac{100+40}{100} = \frac{7}{5} = 1.4$ euros. Expected net gain $= \frac{7}{5} - 3 = -\frac{8}{5}$ euros. For fairness, $(P + 40)/100 = 3 \Rightarrow P = 260$ euros.
46. Weights 4, 3, 2, 1, so $k = \frac{1}{10}$. $E(X) = 1$. $P(X \leq 1) = \frac{7}{10}$.
47. Possible pairs $\{1, 2\}, \{1, 3\}, \{2, 3\}$ are equally likely, so $X = 3, 4, 5$, each with probability $\frac{1}{3}$. $E(X) = 4$. Fair fee $c = 4$ euros.
48. Expected refund $= 5(0.2) + 15(0.1) = \frac{5}{2} = 2.5$ euros. New mean $= 1 + 0.1r \leq 2 \Rightarrow r \leq 10$; greatest $r = 10$.
49. $c(\frac{1}{2} + \frac{1}{4} + \frac{1}{8}) = 1 \Rightarrow c = \frac{8}{7}$. Thus probabilities are $\frac{4}{7}, \frac{2}{7}, \frac{1}{7}$, and $E(X) = \frac{11}{7}$.
50. (a) $E(X) = -\frac{2}{3} + \frac{1}{2} + \frac{4}{6} = \frac{1}{2}$. (b) $-\frac{2}{3} + \frac{1}{2} + \frac{a}{6} = 0 \Rightarrow a = 1$.

Section 4: HL-only challenge

51. Weights are 4, 6, 6, 4, sum 20, so $k = \frac{1}{20}$. Then

$$E(X) = \frac{1(4) + 2(6) + 3(6) + 4(4)}{20} = \frac{50}{20} = \frac{5}{2}.$$

The probabilities are symmetric about $x = 2.5$, so the mean $\frac{5}{2}$ is consistent with that symmetry.

52. Probabilities sum to $p + (1 - 2p) + p = 1$; the range ensures each is non-negative. Also

$$E(X) = -2p + (1 - 2p) + 4p = 1.$$

Hence every member has the same expected value; expected value alone cannot distinguish their differing probability patterns.

53. Sum: $4a + 6r = 1$. Mean: $6a + 14r = 2$. Solving gives $r = \frac{1}{10}$, $a = \frac{1}{10}$. Distribution: $\frac{1}{10}, \frac{2}{10}, \frac{3}{10}, \frac{4}{10}$.

54. Fairness requires expected gross prize = 3: $\frac{A}{6} + \frac{B}{3} = 3$, with $A = 2B$. Thus $\frac{2B}{6} + \frac{B}{3} = 3 \Rightarrow \frac{2B}{3} = 3$, so $B = \frac{9}{2}$, $A = 9$. Expected net gain = $3 - 3 = 0$.

55. $k(1 + 4 + 9 + 16) = 1 \Rightarrow k = \frac{1}{30}$. Then

$$E(X) = \frac{1 + 8 + 27 + 64}{30} = \frac{10}{3}.$$

Only $X = 4$ exceeds $\frac{10}{3}$, so $P(X > E(X)) = \frac{16}{30} = \frac{8}{15}$.

56.

$$E(X) = -4\left(\frac{2}{3} - p\right) + \frac{1}{3} + 10p = -\frac{7}{3} + 14p.$$

Fair when $-\frac{7}{3} + 14p = 0 \Rightarrow p = \frac{1}{6}$. Favourable when $p > \frac{1}{6}$ (within the stated domain).

57.

$$\sum_{x=1}^n P(X = x) = \frac{2}{n(n+1)} \sum_{x=1}^n x = 1.$$

Also

$$E(X) = \sum_{x=1}^n x \frac{2x}{n(n+1)} = \frac{2}{n(n+1)} \sum_{x=1}^n x^2 = \frac{2n+1}{3}.$$

58. With normalizing denominator $1 + r + r^2 + r^3$,

$$\frac{r^2 + r^3}{1 + r + r^2 + r^3} = \frac{3}{4}.$$

Hence $4r^2(1 + r) = 3(1 + r + r^2 + r^3)$, so

$$r^3 + r^2 - 3r - 3 = (r + 1)(r^2 - 3) = 0.$$

Since $r > 0$, $r = \sqrt{3}$. Then

$$E(X) = \frac{r + 2r^2 + 3r^3}{1 + r + r^2 + r^3} = 3 - \frac{\sqrt{3}}{2}.$$

59. Fairness:

$$-2p + \frac{1}{2} + 5\left(\frac{1}{2} - p\right) = 0 \Rightarrow 3 - 7p = 0 \Rightarrow p = \frac{3}{7}.$$

Then $P(X > 0) = 1 - p = \frac{4}{7} > \frac{1}{2}$. A game can win frequently but still have zero or negative expected gain if the losses are sufficiently large relative to the gains.

60. Overall distribution:

$$P(-5) = \frac{1}{12}, \quad P(-2) = \frac{1}{4}, \quad P(1) = \frac{1}{3}, \quad P(4) = \frac{1}{4}, \quad P(7) = \frac{1}{12}.$$

Then

$$E(X) = -\frac{5}{12} - \frac{1}{2} + \frac{1}{3} + 1 + \frac{7}{12} = 1.$$

An additional fee of €1 makes the expected net gain zero.