



Integration and substitution

IB Mathematics AA SL/HL

50 shared questions + 10 HL challenges | Standard integrals and the reverse chain rule

Learning goal	Integrate standard functions and composites; choose substitutions and determine constants.
Syllabus link	SL 5.10: rational powers, $1/x$, e^x , $\sin x$, $\cos x$; linear composites and reverse chain rule. Links to SL 5.5 initial conditions and SL 5.8 stationary points.
Structure	15 short + 15 extended + 20 original IB-style questions + 10 HL challenges.
Technology	Non-calculator unless marked  GDC. Unmarked exam questions are Paper 1-style; marked questions are Paper 2-style.

Instructions. Give exact answers unless a decimal is requested. Include $+C$ for an indefinite integral. Show your substitution and differential where used. Work on a connected interval where each integrand is defined; use absolute-value bars in logarithms when needed. Use radians. GDC answers: 3 significant figures. SL students: Q1–50; HL students: also Q51–60. Work on separate paper.

1. Short response

1. Find $\int (3x^{2/3} - 4x^{-3/2} + 2) dx$.

2. Find $\int \frac{5\sqrt{x} - 3}{\sqrt{x}} dx$.

3. Find $\int \frac{8}{x} dx$.

4. Find $\int 5e^x dx$.

5. Find $\int 4 \sin x dx$.

6. Find $\int 3 \cos x dx$.

7. Find $\int (2x - 1)^4 dx$.

8. Find $\int \frac{1}{3x + 2} dx$.

9. Find $\int e^{4-2x} dx$.



10. Find $\int \cos(3x + \pi/6) dx$.

11. Find $\int \sin(5x - 2) dx$.

12. Find $\int 6x(x^2 + 1)^2 dx$.

13. Find $\int \frac{x}{\sqrt{x^2 + 4}} dx$.

14. Find $\int 2xe^{x^2} dx$.

15. Find $\int \frac{\sin x}{\cos x} dx$.

2. Extended response

16. A function satisfies $f'(x) = 3\sqrt{x} - 2x^{-2}$ for $x > 0$, and $f(1) = 5$.

(a) Find the general antiderivative.

(b) Find $f(x)$ and verify the initial condition.

17. Given $f'(x) = 4/(2x - 3)$ for $x > 3/2$ and $f(2) = 7$,

(a) find $f(x)$;

(b) explain why absolute-value bars may be omitted from the logarithm on this domain.

18. A curve satisfies $dy/dx = 6e^{3x} + 2$ and passes through $(0, 4)$. Find its equation and check your result by differentiation.

19. A function satisfies $f'(x) = 5\cos(2x) - 3\sin x$ and $f(0) = 1$. Find $f(x)$ and the exact value of $f(\pi/2)$.

20. A student attempts the integral shown below.

Integrand	$\frac{4(2x + 5)^3}{(x^2 + 3)^2}$
Proposed antiderivative	$(2x + 5)^4 + C$

(a) Differentiate the proposed antiderivative.

(b) Identify the error and give the correct integral.

21. Use the substitution $u = x^2 + 3$ to find $\int \frac{4x}{(x^2 + 3)^2} dx$. Show the integral in terms of u before evaluating it.

22. Find $\int 3x^2\sqrt{x^3 + 2} dx$ using a suitable substitution. State your substitution and differential.

23. Find $\int \frac{2x-1}{x^2-x+5} dx$. Explain why the logarithm can be written without absolute-value bars for all real x .

24. Find $\int \sin x(2+\cos x)^4 dx$. Verify the sign of your answer by differentiation.

25. Find $\int \frac{e^x}{e^x+4} dx$ by substitution. Explain why your answer is defined for all real x .

26. For $x > 0$, find $\int \frac{\ln x}{x} dx$. Explain why directly applying the power rule to $1/x$ would not handle the whole integrand.

27. A function has $g'(x) = 6x/(3x^2 - 1)$ on $x > 1/\sqrt{3}$ and $g(1) = 2$.

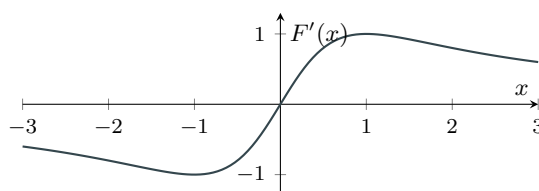
(a) Find $g(x)$.

(b) Find $g(\sqrt{5/3})$ exactly.

28. The graph shows $F'(x) = 2x/(x^2 + 1)$. It is given that $F(0) = 1$.

(a) Find $F(x)$.

(b) Use the derivative to classify its stationary point.



29.  GDC For $x > -2/3$, $F'(x) = \sqrt{3x+2}$ and $F(0) = 1$.

(a) Find an exact expression for $F(x)$.

(b) Use your GDC to calculate $F(2)$ to 3 significant figures.

(c) Check numerically that the derivative of your expression at $x = 2$ equals $\sqrt{8}$.

30.  GDC A function satisfies $F'(x) = 2xe^{-x^2}$ and $F(0) = 5$.

(a) Find $F(x)$ analytically.

(b) Use your GDC to find the positive solution of $F(x) = 5.8$.

(c) Explain why there cannot be two positive solutions.

3. IB exam-style practice

31. For $x > 0$, a curve has $f'(x) = 4x^3 - 3x^{-1/2} + 2/x$ and passes through $(1, 2)$.

- (a) Find $f(x)$. [5 marks]
 (b) Find the equation of the tangent at $(1, 2)$. [3 marks]

32. A function satisfies $f'(x) = 6/(3x - 1)$ for $x > 1/3$, and $f(1) = 4$.

- (a) Find $f(x)$. [4 marks]
 (b) Solve $f(x) = 6$, giving an exact answer. [3 marks]

33. A curve has $f'(x) = 15\sqrt{5x + 2}$ for $x > -2/5$ and $f(0) = 4\sqrt{2} + 1$.

- (a) Find $f(x)$ in the form $A(5x + 2)^{3/2} + B$. [5 marks]
 (b) Find $f(7/5)$ exactly. [2 marks]

34. A curve passes through $(0, 1)$ and has gradient $dy/dx = 8e^{-2x} + 3\cos(3x)$.

- (a) Find the equation of the curve. [5 marks]
 (b) Find the tangent at $x = 0$. [3 marks]

35. A function satisfies $F'(x) = 12x(2x^2 + 1)^2$ and $F(0) = -2$.

- (a) State a suitable substitution and express the integral in the new variable. [2 marks]
 (b) Find $F(x)$. [3 marks]
 (c) Find $F(1)$. [1 marks]

36. Let $F'(x) = 10x/(x^2 + 4)^3$, with $F(0) = 1$.

- (a) Find $F(x)$. [5 marks]
 (b) Find $F(2)$ exactly. [2 marks]
 (c) Explain why F is increasing for $x > 0$. [1 marks]

37. A function satisfies $F'(x) = (3x^2 - 2)e^{x^3 - 2x}$ and $F(0) = 3$.

- (a) Find $F(x)$ by inspection or substitution. [4 marks]
 (b) Find the exact value of $F(2)$. [2 marks]
 (c) Verify your expression for F by differentiation. [2 marks]

38. A function satisfies $F'(x) = \cos x/(4 + \sin x)$ and $F(0) = \ln 2$.

- (a) Find $F(x)$. [4 marks]
 (b) Find $F(\pi/2)$ exactly. [2 marks]
 (c) Explain why your formula is defined for every real x . [1 marks]

39. A function has $F'(x) = 3\sin(2x)/(5 + \cos(2x))^2$ and $F(0) = 0$.

- (a) Using $u = 5 + \cos(2x)$, find $F(x)$. [5 marks]
 (b) Find $F(\pi/2)$ exactly. [2 marks]



40. A function satisfies $F'(x) = e^{2x}/(1 + e^{2x})$ and $F(0) = 2$.

- (a) Find $F(x)$. [5 marks]
(b) Find $F(\ln 2)$, expressing your answer using a single logarithm. [2 marks]
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41. For $x > 0$, a function has $F'(x) = (\ln x)^2/x$ and $F(1) = 4$.

- (a) State a suitable substitution and find $F(x)$. [5 marks]
(b) Find $F(e^3)$. [2 marks]
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42. For $x > 0$, $f'(x) = (x^2 + 2x + 1)/\sqrt{x}$ and $f(1) = 0$.

- (a) Rewrite $f'(x)$ as a sum of powers of x . [2 marks]
(b) Find $f(x)$. [5 marks]
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43. For $x \geq 0$, $F'(x) = 12x/(2x^2 + 3)$ and $F(0) = 0$.

- (a) Find $F(x)$. [4 marks]
(b) Solve $F(x) = 3 \ln 3$ exactly. [3 marks]
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44. Two students find antiderivatives of $\sin x \cos x$. One writes $A(x) = \frac{1}{2} \sin^2 x + C_1$; the other writes $B(x) = -\frac{1}{2} \cos^2 x + C_2$.

- (a) Verify both antiderivatives by differentiation. [4 marks]
(b) Find the condition on C_1, C_2 for A and B to be the same function. [3 marks]
(c) Find the antiderivative satisfying $F(0) = 0$. [2 marks]
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45. For $x > -1$, $F'(x) = a/(x + 1) + 2x$, where a is a constant. The table gives two values of F .

x	0	1
$F(x)$	1	4

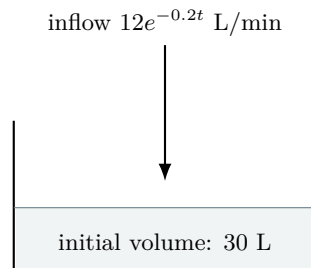
- (a) Find $F(x)$ in terms of a and an integration constant. [3 marks]
(b) Use the table to find both constants. [4 marks]
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46. A function has $F'(x) = kx(x^2 + 1)^2$, $F(0) = 0$ and $F(1) = 7$.

- (a) Find $F(x)$ in terms of k . [4 marks]
(b) Find k and hence an explicit formula for $F(x)$. [3 marks]

47.  *GDC* Water enters a tank at rate $V'(t) = 12e^{-0.2t}$ litres per minute, where $t \geq 0$ is in minutes. Initially the tank contains 30 litres, and no water leaves.

- Find $V(t)$ analytically. [4 marks]
- Find the limiting volume as t increases. [2 marks]
- Use your GDC to find when the tank contains 81 litres. [3 marks]

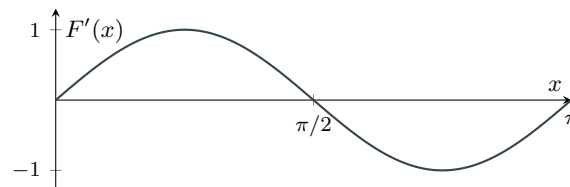


48. A storage device gains charge at a rate $Q'(t) = 8/(t+2)$ coulombs per second for $t \geq 0$. Initially $Q(0) = 10$ coulombs.

- Find $Q(t)$. [4 marks]
- Find, exactly, when the stored charge reaches 20 coulombs. [3 marks]
- State whether the charging rate increases or decreases with time. [1 marks]

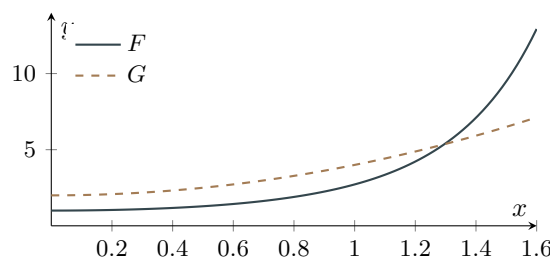
49. The graph shows $F'(x) = \sin(2x)$ on $0 \leq x \leq \pi$. It is given that $F(0) = 2$.

- Find $F(x)$. [4 marks]
- State where F is increasing and decreasing. [2 marks]
- Find the greatest value of F on the given interval. [3 marks]



50.  *GDC* Two functions satisfy $F'(x) = 2xe^{x^2}$, $G'(x) = 4x$, $F(0) = 1$ and $G(0) = 2$. Parts of their graphs are shown.

- Find $F(x)$ and $G(x)$ analytically. [5 marks]
- Use your GDC to find the positive x -coordinate where they intersect. [3 marks]
- Explain why there is also an intersection with a negative x -coordinate. [2 marks]



4. AA HL exclusive challenge

For HL students only. These questions deepen SL 5.10 using nested substitutions, parameters and linked reasoning.

51. A function satisfies $F'(x) = (4x^3 + 6x)/\sqrt{x^4 + 3x^2 + 2}$ and $F(0) = \sqrt{2}$.

- (a) Find $F(x)$. [5 marks]
- (b) Explain why it is defined for all real x . [2 marks]
- (c) Find $F(1)$ exactly. [2 marks]

52. A function satisfies $F'(x) = 2x \ln(1 + x^2)/(1 + x^2)$ and $F(0) = 1$.

- (a) Find $F(x)$ using one or two substitutions. [5 marks]
- (b) Find both real solutions of $F(x) = 3$. [4 marks]

53. A function satisfies $F'(x) = e^x \sqrt{e^x + 1}$ and $F(\ln 3) = 16/3$.

- (a) Find $F(x)$. [5 marks]
- (b) Find $F(0)$ exactly. [2 marks]
- (c) Verify the derivative of your formula. [2 marks]

54. A function satisfies $F'(x) = (x^3 + 2x)/(x^4 + 4x^2 + 7)$ and $F(0) = 0$.

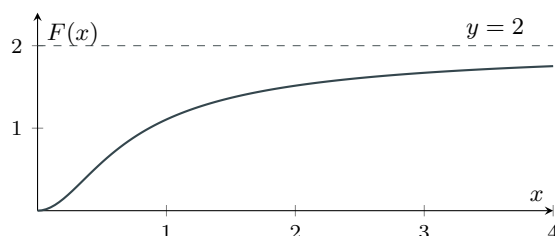
- (a) Find $F(x)$. [5 marks]
- (b) Find the positive solution of $F(x) = \frac{1}{4} \ln(12/7)$. [4 marks]

55. For $0 \leq x \leq \pi/2$, $F'(x) = \sin x \cos^3 x$ and $F(0) = 0$.

- (a) Find $F(x)$. [4 marks]
- (b) Find its greatest value on the interval, justifying your answer. [3 marks]
- (c) Solve $F(x) = 3/16$ exactly in the given interval. [3 marks]

56. For $x \geq 0$, a function satisfies $F'(x) = x/(x^2 + a)^{3/2}$, where $a > 0$, and $F(0) = 0$. Its graph approaches the horizontal asymptote $y = 2$.

- (a) Find $F(x)$ in terms of a . [5 marks]
- (b) Find a . [3 marks]
- (c) Find the exact value of x at which $F(x) = 1$. [3 marks]



57. A function satisfies $F'(x) = (e^x - e^{-x})/(e^x + e^{-x})$ and $F(0) = 0$.

- (a) Find $F(x)$. [5 marks]
- (b) Show that F is even. [2 marks]
- (c) Find and justify its minimum value. [3 marks]

58. For $0 < x < \pi/2$, a function satisfies $F'(x) = \cos x / [\sin x \ln(\sin x)]$ and $F(\pi/6) = 0$.

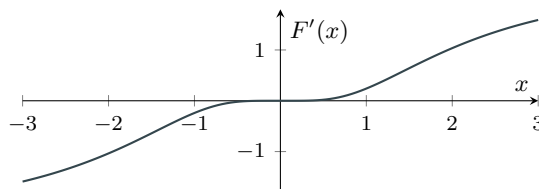
- (a) Find $F(x)$. [6 marks]
- (b) Explain carefully how the domain affects the logarithm in your answer. [3 marks]

59. For constants A, B, k, C , the function $F(x) = k \ln(x^2 + 4x + 7) + C$ satisfies $F'(x) = (Ax + B)/(x^2 + 4x + 7)$, $F(0) = 0$ and $F(1) = 3 \ln(12/7)$.

- (a) Express A and B in terms of k . [3 marks]
- (b) Find all four constants. [5 marks]
- (c) Explain why this logarithmic formula is valid for every real x . [2 marks]

60. The graph shows the derivative $F'(x) = x[\ln(1 + x^2)]^2/(1 + x^2)$, and $F(0) = 0$.

- (a) Find $F(x)$ using a suitable substitution. [5 marks]
- (b) Find and classify its stationary point. [3 marks]
- (c) Find both real solutions of $F(x) = 4/3$. [4 marks]



Answer key and solution outlines

Every indefinite integral represents a family of antiderivatives on its stated interval. Equivalent forms differing only by a constant are accepted. These are original practice questions, not official IB examination items.

1. Short response

1. Use $\int x^n dx = x^{n+1}/(n+1) + C$, $n \neq -1$. The result is $\frac{9}{5}x^{5/3} + 8x^{-1/2} + 2x + C$, for $x > 0$.
2. First write the integrand as $5 - 3x^{-1/2}$. Integrating gives $5x - 6\sqrt{x} + C$, for $x > 0$.
3. $8 \ln |x| + C$, on an interval excluding 0. The power rule with $n \neq -1$ cannot be used when $n = -1$.
4. $5e^x + C$, since the derivative of e^x is e^x .
5. $-4 \cos x + C$, since $(-\cos x)' = \sin x$.
6. $3 \sin x + C$, since $(\sin x)' = \cos x$.
7. With $u = 2x - 1$, $du = 2 dx$: $\frac{1}{2} \int u^4 du = \frac{1}{10}(2x - 1)^5 + C$.
8. $\frac{1}{3} \ln |3x + 2| + C$. The factor $1/3$ compensates for the inner derivative 3.
9. $-\frac{1}{2}e^{4-2x} + C$, because the inner derivative is -2 .
10. $\frac{1}{3} \sin(3x + \pi/6) + C$.
11. $-\frac{1}{5} \cos(5x - 2) + C$.
12. $u = x^2 + 1$, $du = 2x dx$, so $3 \int u^2 du = u^3 + C = (x^2 + 1)^3 + C$.
13. $u = x^2 + 4$, $du = 2x dx$. Thus $\frac{1}{2} \int u^{-1/2} du = \sqrt{x^2 + 4} + C$.
14. $u = x^2$, $du = 2x dx$, so $\int e^u du = e^{x^2} + C$.
15. $u = \cos x$, $du = -\sin x dx$. Hence $-\int du/u = -\ln |\cos x| + C$, where $\cos x \neq 0$.

2. Extended response

16. Integrate powers: $f = 2x^{3/2} + 2x^{-1} + C$. Since $5 = 2 + 2 + C$, $C = 1$. Thus $f(x) = 2x^{3/2} + 2/x + 1$; substitution gives $f(1) = 5$.
17. $f = 2 \ln |2x - 3| + C$. Since $2x - 3 > 0$, this is $2 \ln(2x - 3) + C$. At $x = 2$, $7 = 2 \ln 1 + C$, so $f = 2 \ln(2x - 3) + 7$.
18. $y = 2e^{3x} + 2x + C$. Substituting $(0, 4)$ gives $4 = 2 + C$, so $y = 2e^{3x} + 2x + 2$. Differentiation gives $6e^{3x} + 2$ as required.
19. $f = \frac{5}{2} \sin(2x) + 3 \cos x + C$. The condition gives $1 = 0 + 3 + C$, so $C = -2$. Thus $f(\pi/2) = -2$.
20. The proposed derivative is $8(2x + 5)^3$, twice the required integrand. The chain-rule factor 2 was not compensated for. The correct answer is $\frac{1}{2}(2x + 5)^4 + C$.
21. $du = 2x dx$, so the integral becomes $2 \int u^{-2} du = -2u^{-1} + C = -2/(x^2 + 3) + C$.
22. $u = x^3 + 2$ and $du = 3x^2 dx$. Thus $\int u^{1/2} du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^3 + 2)^{3/2} + C$, where $x^3 + 2 > 0$.
23. The numerator is the derivative of the denominator, so the integral is $\ln |x^2 - x + 5| + C$. Since $x^2 - x + 5 = (x - \frac{1}{2})^2 + \frac{19}{4} > 0$, it is $\ln(x^2 - x + 5) + C$.
24. Let $u = 2 + \cos x$, so $du = -\sin x dx$. The result is $-u^5/5 + C = -\frac{1}{5}(2 + \cos x)^5 + C$. Differentiating multiplies by $5(2 + \cos x)^4(-\sin x)$, giving the required positive integrand.
25. $u = e^x + 4$, $du = e^x dx$, giving $\ln |u| + C = \ln(e^x + 4) + C$. Since $e^x + 4 > 0$ for all real x , the expression is always defined.
26. $u = \ln x$ gives $du = dx/x$, so $\int u du = \frac{1}{2}(\ln x)^2 + C$. The factor $\ln x$ is not constant, and x^{-1} is the exceptional power for the power rule.
27. $u = 3x^2 - 1$, $du = 6x dx$, so $g = \ln(3x^2 - 1) + C$. Since $2 = \ln 2 + C$, $g = \ln[(3x^2 - 1)/2] + 2$. At $x = \sqrt{5/3}$, $g = \ln 2 + 2$.

28. $F = \ln(x^2 + 1) + C$; $F(0) = 1$ gives $C = 1$. Since $x^2 + 1 > 0$, F' changes $- \rightarrow +$ at $x = 0$, so F has a minimum at $(0, 1)$.

x	$-\infty$	0	$+\infty$
$F'(x)$	$-$	0	$+$
$F(x)$			

29. $F = \frac{2}{9}(3x + 2)^{3/2} + C$. The condition gives $C = 1 - 4\sqrt{2}/9$. Thus $F(2) = 1 + 28\sqrt{2}/9 \approx 5.40$. The numerical derivative at 2 is ≈ 2.83 , agreeing with $\sqrt{8}$.

30. With $u = -x^2$, $du = -2x dx$, integration gives $F = -e^{-x^2} + C$. Thus $C = 6$ and $F = 6 - e^{-x^2}$. Solve $e^{-x^2} = 0.2$: $x = \sqrt{\ln 5} \approx 1.27$. For $x > 0$, $F' > 0$, so F is strictly increasing and the positive solution is unique.

3. IB exam-style practice

31. Integrating term by term gives $f = x^4 - 6\sqrt{x} + 2\ln x + C$. From $f(1) = 2$, $2 = 1 - 6 + C$, so $C = 7$. The tangent gradient is $f'(1) = 4 - 3 + 2 = 3$; tangent $y - 2 = 3(x - 1)$, or $y = 3x - 1$.

32. $f = 2\ln(3x - 1) + C$. Since $4 = 2\ln 2 + C$, $f = 2\ln[(3x - 1)/2] + 4$. Setting $f = 6$ gives $\ln[(3x - 1)/2] = 1$, hence $x = (1 + 2e)/3$.

33. $u = 5x + 2$, $du = 5dx$, so $\int 15\sqrt{5x + 2} dx = 3 \int u^{1/2} du = 2u^{3/2} + C$. Thus $A = 2$. At 0, $4\sqrt{2} + 1 = 4\sqrt{2} + B$, so $B = 1$. Since $5(7/5) + 2 = 9$, $f(7/5) = 2(27) + 1 = 55$.

34. $y = -4e^{-2x} + \sin(3x) + C$. The point gives $1 = -4 + C$, so $C = 5$. At 0, the gradient is $8 + 3 = 11$; tangent $y = 11x + 1$.

35. $u = 2x^2 + 1$, $du = 4x dx$, so $\int 12x(2x^2 + 1)^2 dx = 3 \int u^2 du = u^3 + C$. From $F(0) = -2$, $1 + C = -2$, so $F = (2x^2 + 1)^3 - 3$. Therefore $F(1) = 24$.

36. $u = x^2 + 4$, $du = 2x dx$, giving $5 \int u^{-3} du = -\frac{5}{2}u^{-2} + C$. Since $1 = -5/32 + C$, $C = 37/32$. Thus $F = 37/32 - 5/[2(x^2 + 4)^2]$ and $F(2) = 143/128$. For $x > 0$, the numerator and denominator of F' are positive.

37. Let $u = x^3 - 2x$, so $du = (3x^2 - 2)dx$. Then $F = e^{x^3 - 2x} + C$; $3 = 1 + C$ gives $C = 2$. Hence $F(2) = e^4 + 2$. The chain rule gives $F' = e^{x^3 - 2x}(3x^2 - 2)$.

38. $u = 4 + \sin x$, $du = \cos x dx$, so $F = \ln(4 + \sin x) + C$. Since $\ln 2 = \ln 4 + C$, $C = -\ln 2$. Thus $F = \ln[(4 + \sin x)/2]$ and $F(\pi/2) = \ln(5/2)$. The logarithm argument is positive because $4 + \sin x \geq 3$.

39. $du = -2\sin(2x)dx$. Thus $F = -\frac{3}{2} \int u^{-2} du = \frac{3}{2}u^{-1} + C$. Since $0 = \frac{3}{2} \cdot \frac{1}{6} + C$, $C = -1/4$. Hence $F = 3/[2(5 + \cos 2x)] - 1/4$ and $F(\pi/2) = 3/8 - 1/4 = 1/8$.

40. $u = 1 + e^{2x}$, $du = 2e^{2x}dx$, so $F = \frac{1}{2} \ln(1 + e^{2x}) + C$. The condition gives $C = 2 - \frac{1}{2} \ln 2$. At $x = \ln 2$, $e^{2x} = 4$, hence $F(\ln 2) = 2 + \frac{1}{2} \ln(5/2)$.

41. $u = \ln x$, $du = dx/x$, so $F = \int u^2 du = \frac{1}{3}(\ln x)^3 + C$. Since $F(1) = 4$, $C = 4$. Thus $F(e^3) = 27/3 + 4 = 13$.

42. $f' = x^{3/2} + 2x^{1/2} + x^{-1/2}$. Hence $f = \frac{2}{5}x^{5/2} + \frac{4}{3}x^{3/2} + 2\sqrt{x} + C$. At 1, $0 = 2/5 + 4/3 + 2 + C$, so $C = -56/15$.

43. $u = 2x^2 + 3$, $du = 4x dx$, giving $F = 3\ln(2x^2 + 3) + C$. Since $0 = 3\ln 3 + C$, $F = 3\ln[(2x^2 + 3)/3]$. Thus $(2x^2 + 3)/3 = 3$, so $x^2 = 3$. On the given domain, $x = \sqrt{3}$.

44. $A' = \frac{1}{2}(2\sin x \cos x) = \sin x \cos x$; $B' = -\frac{1}{2}(2\cos x)(-\sin x) = \sin x \cos x$. Using $\sin^2 x + \cos^2 x = 1$, $A - B = \frac{1}{2} + C_1 - C_2$. Equality requires $C_2 = C_1 + 1/2$. With $F(0) = 0$, $F = \frac{1}{2} \sin^2 x$, equivalently $(1 - \cos^2 x)/2$.

45. $F = a \ln(x + 1) + x^2 + C$. From $F(0) = 1$, $C = 1$. From $F(1) = 4$, $a \ln 2 + 1 + 1 = 4$, giving $a = 2/\ln 2$. Hence $F = \frac{2}{\ln 2} \ln(x + 1) + x^2 + 1$.

46. $u = x^2 + 1$, $du = 2x dx$, so $F = \frac{k}{6}(x^2 + 1)^3 + C$. The first condition gives $C = -k/6$. The second gives $7 = \frac{k}{6}(8 - 1) = 7k/6$, so $k = 6$ and $F = (x^2 + 1)^3 - 1$.

47. $V = -60e^{-0.2t} + C$. Since $30 = -60 + C$, $V = 90 - 60e^{-0.2t}$ litres. The exponential term tends to zero, so the limiting volume is 90 L. For $V = 81$, $e^{-0.2t} = 0.15$, giving $t = -5 \ln 0.15 \approx 9.49$ minutes.

48. $Q = 8 \ln(t+2) + C$. Since $10 = 8 \ln 2 + C$, $Q = 8 \ln[(t+2)/2] + 10$. Set $Q = 20$: $\ln[(t+2)/2] = 5/4$, hence $t = 2e^{5/4} - 2$ seconds. The rate decreases because the denominator $t+2$ increases.

49. $F = -\frac{1}{2} \cos(2x) + C$. From $2 = -1/2 + C$, $F = \frac{5}{2} - \frac{1}{2} \cos(2x)$. Increasing for $0 < x < \pi/2$; decreasing for $\pi/2 < x < \pi$. The maximum is $F(\pi/2) = 3$; endpoints both give 2.

50. $F = e^{x^2} + C_1$; $F(0) = 1$ gives $C_1 = 0$. $G = 2x^2 + C_2$; $G(0) = 2$ gives $C_2 = 2$. Solving $e^{x^2} = 2x^2 + 2$ gives $x \approx 1.30$. Both functions are even, so reflection in the y -axis gives another intersection at $x \approx -1.30$.

4. AA HL exclusive challenge

51. $u = x^4 + 3x^2 + 2$, $du = (4x^3 + 6x)dx$. Hence $F = 2\sqrt{x^4 + 3x^2 + 2} + C$. At 0, $\sqrt{2} = 2\sqrt{2} + C$, so $C = -\sqrt{2}$. The radicand is $(x^2 + 1)(x^2 + 2) > 0$, so the formula is defined on $\mathbb{R}$. Finally $F(1) = 2\sqrt{6} - \sqrt{2}$.

52. Let $u = \ln(1 + x^2)$, so $du = 2x dx/(1 + x^2)$. Thus $F = \frac{1}{2}[\ln(1 + x^2)]^2 + C$. The condition gives $C = 1$. For $F = 3$, $[\ln(1 + x^2)]^2 = 4$. Since $\ln(1 + x^2) \geq 0$, take $\ln(1 + x^2) = 2$, so $x = \pm\sqrt{e^2 - 1}$.

53. $u = e^x + 1$, $du = e^x dx$ gives $F = \frac{2}{3}(e^x + 1)^{3/2} + C$. At $x = \ln 3$, $16/3 = \frac{2}{3}(4)^{3/2} + C = 16/3 + C$, so $C = 0$. Thus $F(0) = 4\sqrt{2}/3$. Differentiation gives $\frac{2}{3} \cdot \frac{3}{2}(e^x + 1)^{1/2}e^x = e^x\sqrt{e^x + 1}$.

54. The denominator has derivative $4x^3 + 8x = 4(x^3 + 2x)$. Hence $F = \frac{1}{4} \ln(x^4 + 4x^2 + 7) + C = \frac{1}{4} \ln[(x^4 + 4x^2 + 7)/7]$. The equation gives $x^4 + 4x^2 + 7 = 12$. With $u = x^2 \geq 0$, $(u+5)(u-1) = 0$; take $u = 1$ and positive $x = 1$.

55. With $u = \cos x$, $du = -\sin x dx$, $F = -\frac{1}{4} \cos^4 x + C$. From $F(0) = 0$, $C = 1/4$, so $F = (1 - \cos^4 x)/4$. Its derivative is positive in the interior, so the maximum is $F(\pi/2) = 1/4$. For $F = 3/16$, $\cos^4 x = 1/4$. Here $\cos x \geq 0$, hence $\cos x = 1/\sqrt{2}$ and $x = \pi/4$.

56. $u = x^2 + a$, $du = 2x dx$ gives $F = -1/\sqrt{x^2 + a} + C$. From $F(0) = 0$, $C = 1/\sqrt{a}$. As x grows, $F \rightarrow 1/\sqrt{a} = 2$, so $a = 1/4$. Then $F = 2 - 1/\sqrt{x^2 + 1/4}$. Set $F = 1$: $x^2 + 1/4 = 1$, so $x = \sqrt{3}/2$.

57. The numerator is the derivative of the positive denominator. Thus $F = \ln(e^x + e^{-x}) + C$; $0 = \ln 2 + C$, so $F = \ln[(e^x + e^{-x})/2]$. Replacing x by $-x$ interchanges the two exponential terms, leaving F unchanged. The derivative is negative for $x < 0$ and positive for $x > 0$, so the minimum is $F(0) = 0$.

x	$-\infty$	0	$+\infty$
$F'(x)$	-	0	+
$F(x)$			

58. Let $u = \ln(\sin x)$, so $du = (\cos x / \sin x)dx$. Then $F = \int du/u = \ln |\ln(\sin x)| + C$. At $\pi/6$, $\ln(\sin x) = -\ln 2$, so $C = -\ln(\ln 2)$. On this domain, $0 < \sin x < 1$, hence $\ln(\sin x) < 0$ and $F = \ln\left[\frac{-\ln(\sin x)}{\ln 2}\right]$, whose logarithm argument is positive.

59. Differentiation gives $F' = k(2x + 4)/(x^2 + 4x + 7)$, so $A = 2k$, $B = 4k$. From $F(0) = 0$, $C = -k \ln 7$. Then $F(1) = k \ln(12/7) = 3 \ln(12/7)$, so $k = 3$, $A = 6$, $B = 12$, $C = -3 \ln 7$. The logarithm argument is $(x+2)^2 + 3 > 0$ for every real x .

60. $u = \ln(1 + x^2)$ gives $du = 2x dx/(1 + x^2)$. Thus $F = \frac{1}{2} \int u^2 du = \frac{1}{6}[\ln(1 + x^2)]^3 + C$; $F(0) = 0$ gives $C = 0$. The only zero of F' is 0, and its sign changes $- \rightarrow +$, giving a minimum at $(0, 0)$. For $F = 4/3$, $[\ln(1 + x^2)]^3 = 8$, hence $x = \pm\sqrt{e^2 - 1}$.