



Tangents and normals

IB Mathematics: Analysis and Approaches SL and HL

50 shared questions | 10 additional HL-only challenges

Learning goal	Find and interpret tangent and normal equations; connect algebra and geometry.
Syllabus link	SL 5.4 (shared by AA SL and HL), using SL 5.3 differentiation. The HL-only section deepens the same content through algebra and proof.
Structure	15 short + 15 extended + 20 IB-style + 10 HL-only questions; solution outlines.
Technology	Non-calculator / Paper 1 unless marked GDC / P2 . Required derivatives are supplied where appropriate.

Instructions. Work on separate paper and show your reasoning. Give exact answers unless a question is marked GDC / P2 ; for those questions, give final numerical answers to three significant figures and retain unrounded values in working. Diagrams are illustrative; calculate required coordinates and gradients. SL students complete Questions 1–50. HL students also complete Questions 51–60.

1 | Short response (SL and HL)

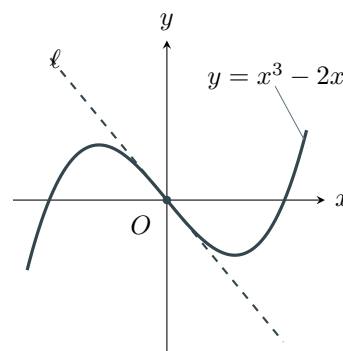
1. A tangent has gradient $-\frac{3}{2}$. State the gradient of the corresponding normal.
2. A normal has gradient -4 . Find the gradient of the corresponding tangent.
3. Given $f(2) = 5$ and $f'(2) = -3$, find the tangent equation at $x = 2$.
4. The tangent at $P(-2, 3)$ has gradient 4. Find the normal equation at P .
5. For $f(x) = x^2 + 2x$, find the point on the curve where $x = 1$ and the tangent gradient there.
6. For $f(x) = 2x^3 - x$, find the tangent equation at $x = -1$.
7. For $y = \frac{6}{x}$, find the normal gradient at $x = 3$.
8. A curve has zero gradient at $P(4, -1)$. Write down its tangent and normal equations at P .
9. A tangent has equation $2x + 5y = 7$. Find the gradient of the corresponding normal.
10. The tangent at $P(2, 1)$ is parallel to $3x - 2y = 8$. Find its equation.
11. The normal at $P(1, -3)$ is parallel to $y = -2x + 7$. Find the tangent equation at P .



12. The tangent to $y = f(x)$ at $x = 1$ is $5x + 2y = 9$. Find $f(1)$ and $f'(1)$.

13. A student says: “If the tangent gradient is -2 , the normal gradient is 2 . Correct the claim.

14. The diagram shows $y = x^3 - 2x$ near the origin and a dashed line. Find the tangent equation at the origin. Explain why a tangent can cross a curve at its point of contact.



15. Write the tangent equation at $(a, f(a))$ in terms of $f'(a)$. State when the normal is vertical.

2 | Extended response (SL and HL)

16. Let $f(x) = 3x^2 - 2x + 1$.

(a) Find $f(1)$ and $f'(1)$.

(b) Find the tangent and normal equations at $x = 1$.

17. Let $f(x) = x^3 - 3x + 2$.

(a) Find the point of contact and tangent gradient at $x = -2$.

(b) Find the tangent and normal equations there.

18. Let $f(x) = \frac{12}{x}$, $x \neq 0$.

(a) Find $f'(x)$.

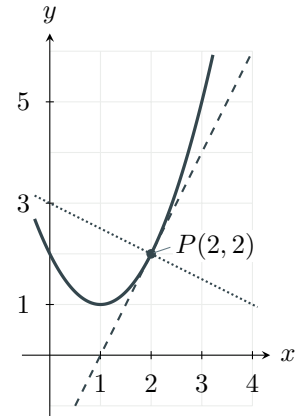
(b) Find the tangent and normal equations at $x = 2$.

19. The curve is $y = x^2 - 6x + 11$.

(a) Find the point where the tangent is horizontal.

(b) Write down the tangent and normal equations there.

20. The diagram shows $y = (x - 1)^2 + 1$ and two lines through $P(2, 2)$.



- (a) Identify the tangent and normal by their line styles. Justify using gradients.
- (b) Find both line equations.

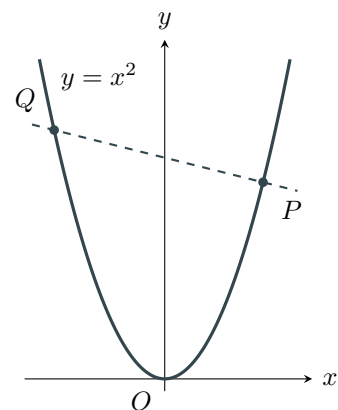
21. For $y = x^2 - 4x + 7$, find the point where the tangent is parallel to $y = 6x + 2$. Hence find the tangent equation.

22. For $y = 2x^2 + x - 1$, find the point where the normal is parallel to $y = -\frac{1}{9}x + 2$. Find the normal equation.

23. The curve $y = ax^2 + bx + 1$ passes through $P(1, 4)$ and has tangent gradient 5 there. Find a , b and the tangent equation.

24. The normal to $y = x^3 + kx + 2$ at $x = 1$ has gradient $-\frac{1}{6}$. Find k and the normal equation.

25. The normal to $y = x^2$ at $P(2, 4)$ meets the curve again at Q .



- (a) Find the normal equation.
- (b) Find the coordinates of Q .

26. Find the tangent to $y = x^2 + 1$ at $x = 2$. Find its intercepts with both coordinate axes and the area of the triangle it forms with the axes.

27. The tangents to $y = x^2$ at $x = -1$ and $x = 3$ intersect at R . Find both tangent equations and the coordinates of R .



28. GDC / P2 Let $f(x) = 1.8^x + 2$.

- (a) Use your GDC to find $f(1.2)$ and the numerical value of $f'(1.2)$.
(b) Find the tangent and normal equations in point-slope form.

29. GDC / P2 Let $f(x) = \ln(x^2 + 2)$.

- (a) Use your GDC to find the point and tangent gradient at $x = 0.8$.
(b) Find the normal equation in point-slope form.

30. The curves are $y = x^2$ and $y = x^3 - x^2 + x$. Show that they have the same tangent at $x = 1$, and find its equation.

3 | Original IB exam-style practice (SL and HL)

31. [Maximum mark: 7]

Consider $f(x) = x^3 - 4x + 1$.

- (a) Find $f'(x)$. [2]
(b) Find the tangent equation at $x = 2$. [3]
(c) Find the normal equation at the same point. [2]

32. [Maximum mark: 8]

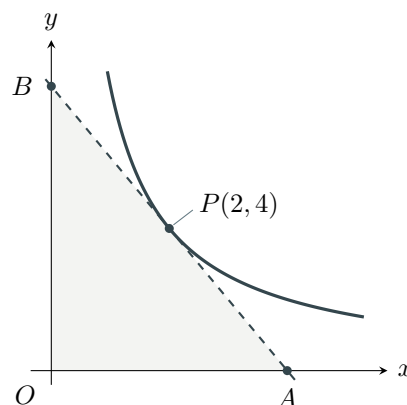
The curve is $y = 2x^2 - 3x + 4$. Its normal at $x = 1$ meets the x -axis at A and the y -axis at B .

- (a) Find the normal equation. [4]
(b) Find the coordinates of A and B . [2]
(c) Find the area of triangle OAB , where O is the origin. [2]

33. [Maximum mark: 8]

The tangent to $y = \frac{8}{x}$, $x > 0$, at $P(2, 4)$ meets the axes at A and B .

- (a) Find the tangent equation. [4]
(b) Find A and B . [2]
(c) Find the shaded area. [2]



**34.** [Maximum mark: 7]

The line $y = 6x - 1$ is tangent to $y = ax^2 + bx + 4$ at $x = 1$.

- (a) Write down two equations in a and b . [3]
- (b) Find a and b . [2]
- (c) Find the normal equation at the point of contact. [2]

35. [Maximum mark: 9]

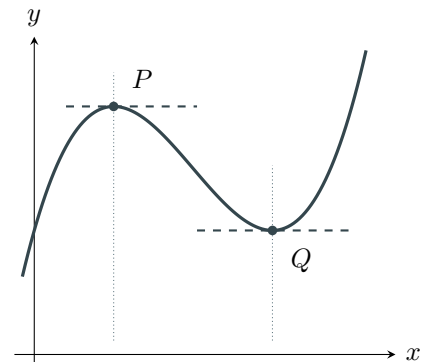
The tangent and normal to $y = x^3 + x$ at $P(1, 2)$ meet the x -axis at A and B , respectively.

- (a) Find the tangent and normal equations. [4]
- (b) Find A and B . [2]
- (c) Find the area of triangle APB . [3]

36. [Maximum mark: 8]

The curve is $y = x^3 - 6x^2 + 9x + 4$.

- (a) Find both points where the tangent is horizontal. Label the left point P and the right point Q . [4]
- (b) Write the tangent and normal equations at each point. [2]
- (c) Find where the normal at P meets the tangent at Q . [2]

**37.** [Maximum mark: 10]

Tangents to $y = x^3 + 2x$ are perpendicular to the line $y = -\frac{1}{5}x + 2$.

- (a) Find all possible points of contact. [4]
- (b) Find the equation of each tangent. [3]
- (c) Find the normal equation at each point of contact. [3]

38. [Maximum mark: 8]

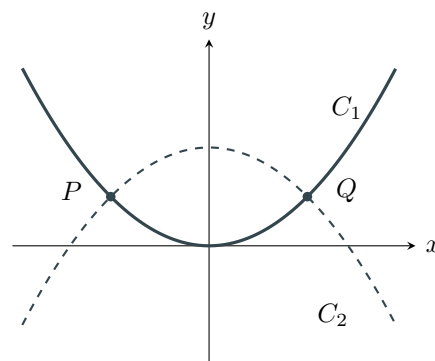
Let $f(x) = \frac{2}{x^2}$, $x \neq 0$.

- (a) Find $f'(x)$. [2]
- (b) Find the tangent equation at $x = 1$. [2]
- (c) Find the normal equation there. [2]
- (d) Find the x -intercept of the normal. [2]

**39.** [Maximum mark: 6]

The curves $C_1 : y = x^2$ and $C_2 : y = \frac{1}{2} - x^2$ intersect at P and Q .

- (a) Find the coordinates of P and Q . [3]
- (b) Show that the tangents to the two curves are perpendicular at each intersection. [3]

**40.** [Maximum mark: 7]

For the curve $y = x^2 - 2x$, let T be the tangent at $x = 3$ and N the normal at $x = 1$.

- (a) Find the equation of T . [3]
- (b) Find the equation of N . [2]
- (c) Find their intersection point. [2]

41. [Maximum mark: 8]

The tangents to $y = x^2 + 1$ at $P(0, 1)$ and $Q(2, 5)$ meet at R .

- (a) Find both tangent equations. [4]
- (b) Find R . [2]
- (c) Find the area of triangle PQR . [2]

42. [Maximum mark: 8]

GDC / P2 The tangent to $f(x) = x^3 - 2x + 1$ at $x = a$, where $a > 0$, has gradient 5.

- (a) Find a . [3]
- (b) Find the normal equation in point-slope form. [3]
- (c) Find where the normal meets the x -axis. [2]

43. [Maximum mark: 8]

GDC / P2 Let $f(x) = e^{0.4x} + 1$.

- (a) Use your GDC to find $f(2)$ and $f'(2)$. [2]
- (b) Find the tangent and normal equations at $x = 2$. [4]
- (c) Find the x -intercept of the tangent. [2]

44. [Maximum mark: 7]

The function $f(x) = 2 \sin x + 1$ has derivative $f'(x) = 2 \cos x$. Angles are in radians.

- (a) Find the point and tangent gradient at $x = \frac{\pi}{3}$. [3]
- (b) Find the tangent equation. [2]
- (c) Find the normal equation. [2]

45. [Maximum mark: 8]

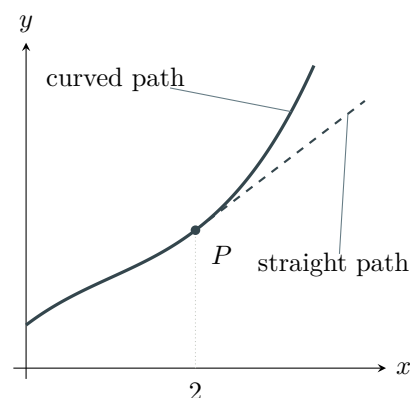
The function $f(x) = \ln x$, $x > 0$, has derivative $f'(x) = \frac{1}{x}$.

- (a) Find the tangent equation at $x = e$. [3]
- (b) Show that this tangent passes through the origin. [1]
- (c) Find the normal equation at $x = e$. [2]
- (d) Find the x -intercept of the normal. [2]

46. [Maximum mark: 9]

 GDC / P2 A curved path is modelled by $y = 0.2x^3 - 0.6x^2 + 1.5x + 1$, with both coordinates in metres. A straight path leaves tangentially at P , where $x = 2$.

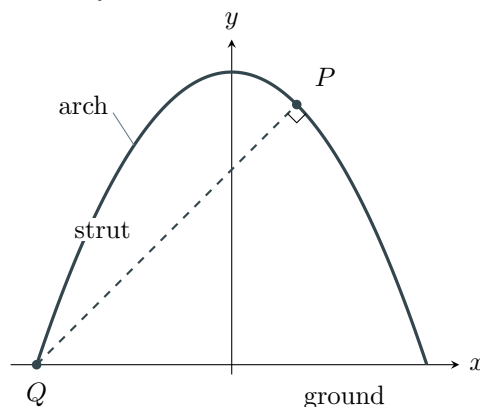
- (a) Find P and the tangent gradient. [3]
- (b) Find the straight-path equation. [2]
- (c) Find y on the straight path at $x = 2.2$. [1]
- (d) Find the vertical separation of the paths there. State which path is higher. [3]



47. [Maximum mark: 9]

The cross-section of an arch is $y = 9 - \frac{1}{4}x^2$, $-6 \leq x \leq 6$, with coordinates in metres. A straight strut is normal to the arch at P , where $x = 2$, and meets the ground $y = 0$ at Q .

- (a) Find P and the normal gradient. [3]
- (b) Find the strut equation and Q . [3]
- (c) Find the exact strut length. [3]



**48.** [Maximum mark: 8]

The tangent to $y = x^3 + 2$ at $x = a$, where $a > 0$, passes through the origin.

- (a) Find the tangent equation in terms of a . [3]
- (b) Find a . [2]
- (c) Find the tangent and normal equations at this point. [3]

49. [Maximum mark: 8]

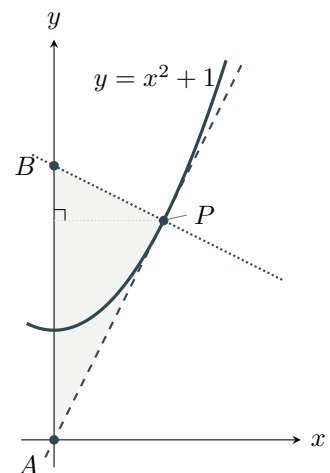
Consider $y = \frac{x^2-4}{x}$, $x \neq 0$. The normal at $x = 2$ meets the curve again at Q .

- (a) Find $\frac{dy}{dx}$. [2]
- (b) Find the normal equation at $x = 2$. [3]
- (c) Find Q . [3]

50. [Maximum mark: 9]

The tangent and normal to $y = x^2 + 1$ at $P(1, 2)$ meet the y -axis at A and B , respectively.

- (a) Find both line equations. [4]
- (b) Find A and B . [2]
- (c) Find the area of triangle APB . [3]





4 | AA HL ONLY: extended challenge

HL students only. Explain your reasoning, including excluded values and special cases.

51. [Maximum mark: 10]

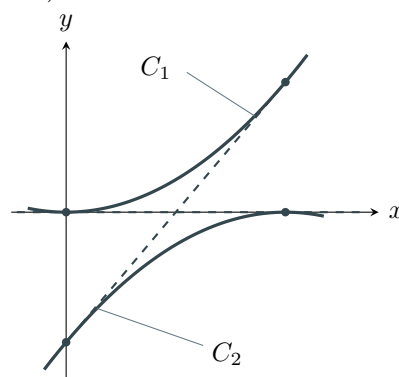
Let $P(a, a^2)$ lie on $y = x^2$. A tangent from a fixed point $Q(h, k)$ touches the curve at P .

- Show that the tangent is $y = 2ax - a^2$. [2]
- Determine conditions on h and k for there to be two, one or no tangents from Q . [5]
- Find the tangents from $Q(1, -3)$. [3]

52. [Maximum mark: 12]

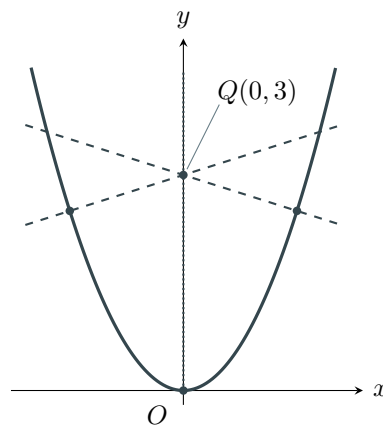
A line $y = mx + c$ is tangent to both $C_1 : y = x^2$ and $C_2 : y = -(x - 2)^2$.

- Show that tangency requires $c = -\frac{m^2}{4}$ and $c = \frac{m^2}{4} - 2m$. [5]
- Find every common tangent. [3]
- Find the two points of contact for each line. [4]



53. [Maximum mark: 9]

Find all normals to $y = x^2$ which pass through $Q(0, 3)$. Account for a possible vertical normal.



54. [Maximum mark: 10]

The normal to $y = x^2$ at $P(a, a^2)$, where $a \neq 0$, meets the curve again at $Q(b, b^2)$. Let $M(u, v)$ be the midpoint of PQ .

- Show that $b = -a - \frac{1}{2a}$. [4]
- Find u and v in terms of a . [3]
- Eliminate a to find an equation for the locus of M . State the restriction on u . [3]

**55.** [Maximum mark: 8]

The tangent to $y = x^3$ at $P(a, a^3)$, where $a \neq 0$, meets the curve again at Q .

- (a) Find the tangent equation in terms of a . [2]
- (b) By factorizing the intersection equation, find Q . [4]
- (c) Explain what changes when $a = 0$. [2]

56. [Maximum mark: 10]

Let $P(a, k/a)$ lie on $y = k/x$, where $a > 0$ and $k > 0$. The tangent at P meets the positive x -axis at A and the positive y -axis at B .

- (a) Find the tangent equation. [3]
- (b) Show that the area of triangle OAB is independent of a . [3]
- (c) Find a in terms of k if the normal at P passes through the origin. [4]

57. [Maximum mark: 12]

The cubic $f(x) = ax^3 + bx^2 + cx + d$ has tangent $y = 5x - 4$ at $x = 1$ and tangent $y = -x + 4$ at $x = -1$.

- (a) Find a , b , c and d . [8]
- (b) Find where the normals at the two given points intersect. [4]

58. [Maximum mark: 10]

 GDC / P2 Normals to $y = x^2$ pass through $Q(2, 3)$. A point of contact has x -coordinate a .

- (a) Show that $2a^3 - 5a - 2 = 0$, accounting for a possible vertical normal. [4]
- (b) Find all possible values of a . [3]
- (c) Find every normal equation in point-slope form, to three significant figures. [3]

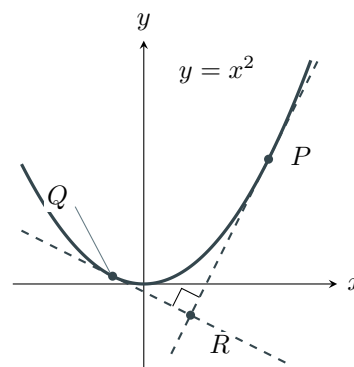
59. [Maximum mark: 7]

For $y = x^3 + kx$, the tangent at $x = 1$ is perpendicular to the tangent at $x = 2$. Find all possible values of k .

60. [Maximum mark: 11]

Tangents to $y = x^2$ at distinct points $P(a, a^2)$ and $Q(b, b^2)$ meet at R .

- (a) Show that $R(\frac{a+b}{2}, ab)$. [4]
- (b) Deduce the locus of R when the tangents are perpendicular. [3]
- (c) Show that every point on this locus is possible. [4]



**Answer key and solution outlines**

Use $m_T = f'(a)$ and $y - f(a) = m(x - a)$. For $m_T \neq 0$, $m_N = -1/m_T$; if $m_T = 0$, the normal is $x = a$. Below, T and N denote tangent and normal.

1 | Short response

1. $m_N = -1/(-3/2) = \frac{2}{3}$.
2. $m_T = -1/(-4) = \frac{1}{4}$.
3. $y - 5 = -3(x - 2)$, so $y = -3x + 11$.
4. $m_N = -\frac{1}{4}$; $y - 3 = -\frac{1}{4}(x + 2)$, so $y = -\frac{1}{4}x + \frac{5}{2}$.
5. $f(1) = 3$ and $f'(x) = 2x + 2$. Thus $P(1, 3)$ and $m_T = 4$.
6. $P(-1, -1)$; $f'(x) = 6x^2 - 1$ gives $m_T = 5$. Hence $y + 1 = 5(x + 1)$, or $y = 5x + 4$.
7. $dy/dx = -6x^{-2}$, so $m_T = -\frac{2}{3}$ and $m_N = \frac{3}{2}$.
8. Tangent $y = -1$; normal $x = 4$.
9. $m_T = -\frac{2}{5}$, so $m_N = \frac{5}{2}$.
10. $m_T = \frac{3}{2}$; $y - 1 = \frac{3}{2}(x - 2)$ gives $y = \frac{3}{2}x - 2$.
11. $m_N = -2$, so $m_T = \frac{1}{2}$. Thus $y + 3 = \frac{1}{2}(x - 1)$, or $y = \frac{1}{2}x - \frac{7}{2}$.
12. Substitute $x = 1$: $f(1) = 2$. Rearranging the line gives gradient $f'(1) = -\frac{5}{2}$.
13. Normal and tangent gradients are negative reciprocals: $m_N = -1/(-2) = \frac{1}{2}$.
14. $f'(x) = 3x^2 - 2$, so the tangent is $y = -2x$. The difference $f(x) - (-2x) = x^3$ changes sign at 0, so the curve crosses its tangent.
15. $y - f(a) = f'(a)(x - a)$. If $f'(a) = 0$, the normal is the vertical line $x = a$.

2 | Extended response

16. (a) $f(1) = 2$ and $f'(x) = 6x - 2$, so $f'(1) = 4$. (b) $T : y - 2 = 4(x - 1)$, giving $y = 4x - 2$; $N : y - 2 = -\frac{1}{4}(x - 1)$, giving $y = -\frac{1}{4}x + \frac{9}{4}$.
17. (a) $f(-2) = 0$ and $f'(-2) = 3(-2)^2 - 3 = 9$. (b) $T : y = 9(x + 2)$; $N : y = -\frac{1}{9}(x + 2)$.
18. (a) $f'(x) = -12x^{-2}$. (b) $P(2, 6)$, $m_T = -3$, $m_N = \frac{1}{3}$. Thus $T : y = -3x + 12$ and $N : y = \frac{1}{3}x + \frac{16}{3}$.
19. (a) $dy/dx = 2x - 6 = 0$ gives $x = 3$, hence $P(3, 2)$. (b) Tangent $y = 2$; normal $x = 3$.
20. (a) $f'(2) = 2$: the rising dashed line is the tangent; the falling dotted line is the normal, with gradient $-\frac{1}{2}$. (b) $T : y = 2x - 2$; $N : y = -\frac{1}{2}x + 3$.
21. $2x - 4 = 6$ gives $x = 5$ and $y = 12$. Thus $y - 12 = 6(x - 5)$, so $T : y = 6x - 18$.
22. $m_N = -\frac{1}{9}$ gives $m_T = 9$. Thus $4x + 1 = 9$, so $P(2, 9)$. The normal is $y - 9 = -\frac{1}{9}(x - 2)$, or $y = -\frac{1}{9}x + \frac{83}{9}$.
23. From the point, $a + b = 3$; from the derivative, $2a + b = 5$. Subtracting gives $a = 2$, then $b = 1$. The tangent is $y - 4 = 5(x - 1)$, or $y = 5x - 1$.
24. $m_T = 6$, so $3 + k = 6$ and $k = 3$. The point is $(1, 6)$; $N : y - 6 = -\frac{1}{6}(x - 1)$, or $y = -\frac{1}{6}x + \frac{37}{6}$.
25. (a) $m_T = 4$, so $N : y - 4 = -\frac{1}{4}(x - 2)$, or $y = -\frac{1}{4}x + \frac{9}{2}$. (b) Substitute $y = x^2$: $4x^2 + x - 18 = (x - 2)(4x + 9) = 0$. Exclude P , giving $Q(-\frac{9}{4}, \frac{81}{16})$.
26. $P(2, 5)$, $m_T = 4$, so $T : y = 4x - 3$. Intercepts are $(\frac{3}{4}, 0)$ and $(0, -3)$. The triangle has area $\frac{1}{2} \times \frac{3}{4} \times 3 = \frac{9}{8}$.
27. Tangents: $y = -2x - 1$ and $y = 6x - 9$. Equating gives $8x = 8$, hence $R(1, -3)$.
28. (a) $f(1.2) \approx 4.024543$, $f'(1.2) \approx 1.189999$. (b) To 3 s.f., $T : y - 4.02 = 1.19(x - 1.2)$ and $N : y - 4.02 = -0.840(x - 1.2)$.

**2 | Extended response (continued)**

- 29.** (a) $f(0.8) \approx 0.970779$, $f'(0.8) \approx 0.606061$. (b) $m_N = -1.65$, so $N : y - 0.971 = -1.65(x - 0.8)$ (3 s.f.).
- 30.** At $x = 1$, both curves have $y = 1$. Their derivatives are $2x$ and $3x^2 - 2x + 1$, both equal to 2 there. Same point and same gradient imply the common tangent $y - 1 = 2(x - 1)$, or $y = 2x - 1$.

3 | IB exam-style: 31–40

- 31.** (a) $f'(x) = 3x^2 - 4$. (b) At $x = 2$, $P(2, 1)$ and $m_T = 8$; $T : y - 1 = 8(x - 2)$, so $y = 8x - 15$. (c) $m_N = -\frac{1}{8}$; $N : y = -\frac{1}{8}x + \frac{5}{4}$.
- 32.** (a) $P(1, 3)$ and $dy/dx = 4x - 3$ gives $m_T = 1$. Hence $m_N = -1$ and $N : y = -x + 4$. (b) Intercepts $(4, 0)$ and $(0, 4)$. (c) Area $= \frac{1}{2}(4)(4) = 8$.
- 33.** (a) $dy/dx = -8x^{-2}$ gives $m_T = -2$. Thus $T : y - 4 = -2(x - 2)$, or $y = -2x + 8$. (b) $A(4, 0)$ and $B(0, 8)$. (c) Area $= \frac{1}{2}(4)(8) = 16$.
- 34.** (a) On the line, $P(1, 5)$; hence $a + b = 1$. Matching gradients gives $2a + b = 6$. (b) Subtract: $a = 5$, $b = -4$. (c) $m_N = -\frac{1}{6}$ and $N : y - 5 = -\frac{1}{6}(x - 1)$, or $y = -\frac{1}{6}x + \frac{31}{6}$.
- 35.** (a) $dy/dx = 3x^2 + 1$ gives $m_T = 4$. Tangent $y = 4x - 2$; normal $y = -\frac{1}{4}x + \frac{9}{4}$. (b) $A(\frac{1}{2}, 0)$ and $B(9, 0)$. (c) $AB = \frac{17}{2}$ and height $= 2$, so area $= \frac{1}{2} \times \frac{17}{2} \times 2 = \frac{17}{2}$.
- 36.** (a) $dy/dx = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$. Zero gradient gives $P(1, 8)$ and $Q(3, 4)$. (b) Tangents $y = 8$, $y = 4$; normals $x = 1$, $x = 3$. (c) The normal at P and tangent at Q meet at $(1, 4)$.
- 37.** (a) Required gradient 5: $3x^2 + 2 = 5$ gives $x = \pm 1$ and points $(1, 3)$, $(-1, -3)$. (b) Tangents $y = 5x - 2$ and $y = 5x + 2$. (c) Normals $y = -\frac{1}{5}x + \frac{16}{5}$ and $y = -\frac{1}{5}x - \frac{16}{5}$, respectively.
- 38.** (a) $f'(x) = -4x^{-3}$. (b) $P(1, 2)$, $m_T = -4$; $T : y = -4x + 6$. (c) $m_N = \frac{1}{4}$; $N : y = \frac{1}{4}x + \frac{7}{4}$. (d) Setting $y = 0$ gives $x = -7$, so the intercept is $(-7, 0)$.
- 39.** (a) $x^2 = \frac{1}{2} - x^2$ gives $x = \pm \frac{1}{2}$ and $y = \frac{1}{4}$. (b) Gradients are $2x$ and $-2x$; at either intersection their product is $-4x^2 = -1$. Hence the tangents are perpendicular.
- 40.** (a) At $x = 3$, $P(3, 3)$ and $m_T = 4$, so $T : y = 4x - 9$. (b) At $x = 1$, $dy/dx = 0$, so the normal is $N : x = 1$. (c) Substitute $x = 1$ into T : the intersection is $(1, -5)$.

3 | IB exam-style: 41–50

- 41.** (a) At P , $m = 0$, giving $y = 1$; at Q , $m = 4$, giving $y = 4x - 3$. (b) $1 = 4x - 3$ gives $R(1, 1)$. (c) Base $PR = 1$ is horizontal and the height from Q is 4, so the area is $\frac{1}{2}(1)(4) = 2$.
- 42.** (a) $f'(a) = 3a^2 - 2 = 5$ gives $a = \sqrt{7/3} \approx 1.527525$. (b) $f(a) \approx 1.509175$; $N : y - 1.51 = -0.200(x - 1.53)$ (3 s.f.). (c) Use unrounded a and $f(a)$: setting $y = 0$ gives $x = a + 5f(a) \approx 9.073401$. Intercept $(9.07, 0)$.
- 43.** (a) $f(2) \approx 3.225541$, $f'(2) \approx 0.890216$. (b) $T : y - 3.23 = 0.890(x - 2)$; $N : y - 3.23 = -1.12(x - 2)$ (3 s.f.). (c) With unrounded values, $x = 2 - 3.225541/0.890216 \approx -1.623322$, so the tangent intercept is $(-1.62, 0)$.
- 44.** (a) $f(\pi/3) = 1 + \sqrt{3}$ and $f'(\pi/3) = 1$. (b) $T : y - (1 + \sqrt{3}) = x - \pi/3$. (c) $m_N = -1$ and $N : y - (1 + \sqrt{3}) = -(x - \pi/3)$.
- 45.** (a) At $x = e$, $P(e, 1)$ and $m_T = 1/e$, so $y - 1 = (x - e)/e$, giving $T : y = x/e$. (b) Substituting $x = 0$ gives $y = 0$. (c) $m_N = -e$; $N : y - 1 = -e(x - e)$, or $y = -ex + e^2 + 1$. (d) The normal has x -intercept $(e + 1/e, 0)$.
- 46.** (a) $P(2, 3.2)$; $dy/dx = 0.6x^2 - 1.2x + 1.5$ gives $m_T = 1.5$. (b) $y - 3.2 = 1.5(x - 2)$, so the straight path is $y = 1.5x + 0.2$. (c) On the straight path, $y = 3.50$ m. (d) The curve gives 3.5256 m, so the straight path lies 0.0256 m below it at $x = 2.2$.
- 47.** (a) $P(2, 8)$; $dy/dx = -x/2$ gives $m_T = -1$, so $m_N = 1$. (b) The strut is $y - 8 = x - 2$, or $y = x + 6$. Thus $Q(-6, 0)$. (c) $PQ = \sqrt{(2 + 6)^2 + (8 - 0)^2} = 8\sqrt{2}$ m.
- 48.** (a) The point is $(a, a^3 + 2)$ and $m_T = 3a^2$, so $y = 3a^2x + 2 - 2a^3$. (b) Passing through $(0, 0)$ gives $2 - 2a^3 = 0$, so $a = 1$. (c) At $(1, 3)$, tangent $y = 3x$ and normal $y = -\frac{1}{3}x + \frac{10}{3}$.
- 49.** (a) Rewrite $y = x - 4x^{-1}$, so $dy/dx = 1 + 4x^{-2}$. (b) $P(2, 0)$, $m_T = 2$ and $N : y = -\frac{1}{2}x + 1$. (c) $x - 4/x = -x/2 + 1$ gives $3x^2 - 2x - 8 = (3x + 4)(x - 2) = 0$. Exclude P , obtaining $Q(-\frac{4}{3}, \frac{5}{3})$.



3 | IB exam-style: 41–50 (continued)

50. (a) $m_T = 2$ and $m_N = -\frac{1}{2}$. Thus $T : y = 2x$ and $N : y = -\frac{1}{2}x + \frac{5}{2}$. (b) $A(0, 0)$, $B(0, \frac{5}{2})$. (c) Base $AB = \frac{5}{2}$ and perpendicular height from P to the y -axis $= 1$. Area $= \frac{1}{2} \times \frac{5}{2} \times 1 = \frac{5}{4}$.

4 | HL ONLY: 51–55

51. (a) At P , $m_T = 2a$, so $T : y = 2ax - a^2$. (b) Passing through Q gives $a^2 - 2ha + k = 0$. Its discriminant is $4(h^2 - k)$. Thus there are two distinct tangents if $k < h^2$, one if $k = h^2$, and none if $k > h^2$. (c) For $Q(1, -3)$, $(a - 3)(a + 1) = 0$. Tangents: $y = 6x - 9$ and $y = -2x - 1$.

52. (a) At tangency with C_1 , $m = 2a$ and $c = -a^2 = -m^2/4$. For C_2 , gradient $m = 4 - 2b$ gives $b = 2 - m/2$. Substitution gives $c = m^2/4 - 2m$. (b) Equate: $-m^2/4 = m^2/4 - 2m$, so $m(m - 4) = 0$. Common tangents are $y = 0$ and $y = 4x - 4$. (c) For $y = 0$: contacts $(0, 0)$ and $(2, 0)$. For $y = 4x - 4$: contacts $(2, 4)$ and $(0, -4)$, respectively.

53. For contact at (a, a^2) with $a \neq 0$, $m_N = -1/(2a)$ and $N : y = -x/(2a) + a^2 + \frac{1}{2}$. Passing through Q gives $a^2 = \frac{5}{2}$, so $a = \pm \frac{\sqrt{10}}{2}$. These normals are $y = 3 \pm x/\sqrt{10}$. At $a = 0$, the tangent is horizontal, giving the additional normal $x = 0$. Therefore there are exactly three normals.

54. (a) The normal is $y = -x/(2a) + a^2 + \frac{1}{2}$. Intersect with $y = x^2$: $(x - a)(2ax + 2a^2 + 1) = 0$. Thus $b = -a - 1/(2a)$. (b) $u = (a + b)/2 = -1/(4a)$; $v = (a^2 + b^2)/2 = a^2 + \frac{1}{2} + 1/(8a^2)$. (c) Since $a = -1/(4u)$,

$$v = 2u^2 + \frac{1}{2} + \frac{1}{16u^2}, \quad u \neq 0.$$

55. (a) $m_T = 3a^2$, so $T : y = 3a^2x - 2a^3$. (b) Intersection gives $x^3 - 3a^2x + 2a^3 = (x - a)^2(x + 2a) = 0$. The other point is $Q(-2a, -8a^3)$. (c) At $a = 0$, the tangent is $y = 0$ and $x^3 = 0$ has only $x = 0$; there is no second distinct intersection.

4 | HL ONLY: 56–60

56. (a) $m_T = -k/a^2$, giving $T : y = -kx/a^2 + 2k/a$. (b) Intercepts are $(2a, 0)$ and $(0, 2k/a)$. Area $OAB = \frac{1}{2}(2a)(2k/a) = \boxed{2k}$, independent of a . (c) $m_N = a^2/k$ gives $N : y = (a^2/k)x + k/a - a^3/k$. It passes through the origin iff $k^2 = a^4$, hence $a = \sqrt{k}$ because $a, k > 0$.

57. (a) The conditions are $f(1) = 1$, $f'(1) = 5$, $f(-1) = 5$, $f'(-1) = -1$. Thus $a + b + c + d = 1$, $3a + 2b + c = 5$, $-a + b - c + d = 5$, $3a - 2b + c = -1$. Subtract the derivative equations: $b = \frac{3}{2}$. The point equations give $a + c = -2$ and $b + d = 3$; the derivative equations give $3a + c = 2$. Hence $a = 2$, $b = \frac{3}{2}$, $c = -4$, $d = \frac{3}{2}$. (b) Normals: $y = -\frac{1}{5}x + \frac{6}{5}$ and $y = x + 6$. Equating gives $x = -4$, $y = 2$.

58. (a) A vertical normal would be $x = 0$, which misses Q . For $a \neq 0$, the normal is $y = -x/(2a) + a^2 + \frac{1}{2}$. Substitute Q : $3 = -1/a + a^2 + \frac{1}{2}$, hence $2a^3 - 5a - 2 = 0$. (b) The three roots are approximately -1.320012 , -0.432320 and 1.752332 . (c) Using $y - 3 = -1/(2a)(x - 2)$, the normals are $y - 3 = 0.379(x - 2)$, $y - 3 = 1.16(x - 2)$ and $y - 3 = -0.285(x - 2)$ (3 s.f.).

59. The gradients are $m_1 = 3 + k$ and $m_2 = 12 + k$. Perpendicularity gives $(k + 3)(k + 12) = -1$, hence $k^2 + 15k + 37 = 0$. Therefore $k = \frac{-15 \pm \sqrt{77}}{2}$. Both values give non-zero gradients and satisfy the required product.

60. (a) Tangents are $y = 2ax - a^2$ and $y = 2bx - b^2$. Since $a \neq b$, equating and dividing by $a - b$ gives $x_R = (a + b)/2$, then $y_R = ab$. (b) Perpendicularity means $(2a)(2b) = -1$, so $y_R = -\frac{1}{4}$. (c) For any real u , choose a, b as the roots of $t^2 - 2ut - \frac{1}{4} = 0$: $a, b = u \pm \sqrt{u^2 + \frac{1}{4}}$. They are distinct real numbers, with sum $2u$ and product $-\frac{1}{4}$. Their tangents meet at $(u, -\frac{1}{4})$ and are perpendicular. Thus every point of this horizontal line occurs.