



Differentiation Rules

IB Mathematics: Analysis and Approaches SL / HL | SL 5.6

50 shared questions + 10 AA HL challenges

Learning goal	Differentiate standard and composite functions; combine chain, product and quotient rules to solve gradient and tangent problems.
Syllabus link	SL 5.6: rational powers, $\sin x$, $\cos x$, e^x , $\ln x$; sums and multiples; chain, product and quotient rules. Tangents/normals revisit SL 5.4.
Structure	1–15 short response; 16–30 extended response; 31–50 IB exam-style. 51–60: AA HL exclusive challenge.
Technology	 marks GDC questions (Paper 2 style). Work without a calculator on all other questions (Paper 1 style).

Instructions. Work on separate paper and show your method. Give exact answers unless a decimal is requested or a numerical method is used; give numerical results to 3 significant figures. Use radians for trigonometric functions. State derivative rules clearly and retain domain restrictions.

The HL section combines the SL 5.6 rules in more demanding problems; no additional AHL differentiation techniques are assumed.

1. Short response

15 questions

1. Differentiate $f(x) = 3\sqrt{x} - 2x^{-3}$, for $x > 0$.

2. Differentiate $y = 4x^{3/2} - 5x^{2/3}$, for $x > 0$.

3. Find $f'(x)$ for $f(x) = \sin x + 2 \cos x$.

4. Differentiate $f(x) = 3e^x - 4 \ln x$, for $x > 0$.

5. Differentiate $y = \sin(3x)$.

6. Differentiate $y = \cos(2x - 1)$.

7. Differentiate $f(x) = e^{x^2}$.

8. Differentiate $f(x) = \ln(5x + 2)$, for $x > -2/5$.

9. Differentiate $y = (2x - 1)^6$.

10. Differentiate $y = (x^2 + 1)^{-2}$.

11. Use the product rule to differentiate $f(x) = x^2 e^x$.

12. Differentiate $y = (x + 1) \sin x$.



13. Differentiate $f(x) = \frac{x+2}{x-1}$, for $x \neq 1$.

14. The table gives values of two differentiable functions.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
2	3	-1	4	5

Find $h'(2)$ if (i) $h(x) = f(x)g(x)$; (ii) $h(x) = f(x)/g(x)$.

15. A student writes $\frac{d}{dx} \ln(x^2 + 1) = \frac{1}{x^2 + 1}$. Explain and correct the error.

2. Extended response

15 questions

16. Let $f(x) = \sqrt{3x^2 + 1}$. Find $f'(x)$ and the exact gradient at $x = 1$.

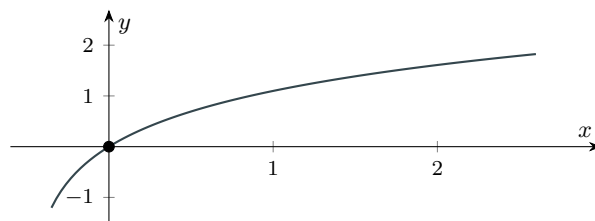
17. Differentiate $f(x) = x^3 \sin(2x)$ and find $f'(\pi/2)$ exactly.

18. For $x > 0$, let $f(x) = \frac{\ln x}{x}$. Show that $f'(x) = \frac{1 - \ln x}{x^2}$ and find where the curve has a horizontal tangent.

19. Let $f(x) = \frac{e^x}{x+1}$, for $x \neq -1$. Find $f'(x)$ in simplified form and the equation of the tangent at $x = 0$.

20. Differentiate $f(x) = (\sin x)^3 + \cos(x^3)$. Explain the different inner derivatives in the two terms.

21. The curve $y = \ln(2x + 1)$ passes through the origin. Find the equation of its tangent there.



22. Find the equation of the normal to $y = e^{-x}$ at its y -intercept.

23. Let $f(x) = (x^2 + 1)e^{-x}$. Show that $f'(x) = -(x - 1)^2 e^{-x}$. Find the equation of the horizontal tangent and explain why f never increases.

24.  For $f(x) = \frac{2x+1}{x^2+1}$, find $f'(x)$. Use your GDC to find both points where the tangent is horizontal, to three significant figures.



25. Use the table to find (i) $\frac{d}{dx}f(g(x))$ at $x = 1$; (ii) $\frac{d}{dx}g(f(x))$ at $x = 1$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	2	2	-3
2	5	4	1	2
3	2	-2	4	-1

26. Let $f(x) = e^{ax} + bx$, where a, b are constants. Given $f'(0) = 5$ and $f(1) = e^2 + 3$, verify that $a = 2$, $b = 3$ satisfy both conditions. Find the tangent at $x = 0$ for these values.

27. For $x > 0$, let $f(x) = x^2 \ln x$. Find the exact coordinates of the point where the tangent is horizontal.

28. Show, by simplifying first and then differentiating, that the derivative of $f(x) = \frac{x^2 - 1}{x - 1}$ is 1 on its domain. Explain why this does not give a derivative at $x = 1$.

29. Find and simplify the derivative of $f(x) = \ln(\sqrt{x^2 + 4})$ in two ways: using the chain rule directly and using a logarithm law first.

30.  A GDC graph suggests a horizontal tangent to $f(x) = xe^{-x} + 0.1x$ near $x = 1.5$. Find $f'(x)$ and use your GDC to locate **all** horizontal tangents for $0 \leq x \leq 6$.

3. IB exam-style practice

20 questions

31.

[Maximum mark: 5]

Let $f(x) = (3x - 2)^5$.

- (a) Find $f'(x)$. [2]
(b) Find the equation of the tangent at $x = 1$. [3]

32.

[Maximum mark: 6]

Let $f(x) = \sqrt{4x + 5}$, for $x \geq -5/4$.

- (a) Find $f'(x)$ for $x > -5/4$. [2]
(b) The tangent at $x = 1$ meets the x -axis at P . Find the coordinates of P . [4]

33.

[Maximum mark: 7]

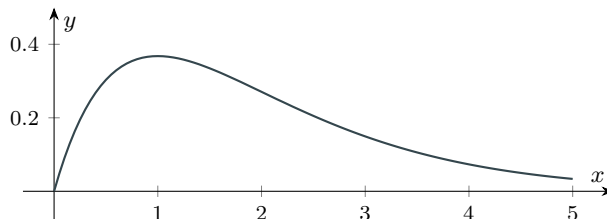
For $x > 0$, let $f(x) = x \ln x$.

- (a) Find $f'(x)$. [2]
(b) Find the exact point at which the tangent is parallel to $y = 2x + 7$. [3]
(c) Find the equation of that tangent. [2]

34.

[Maximum mark: 6]

The graph shows $f(x) = xe^{-x}$ for $x \geq 0$.



- Show that $f'(x) = e^{-x}(1 - x)$. [2]
- Find the exact coordinates of the point where the tangent is horizontal. [2]
- Find the equation of the tangent at the origin. [2]

35.

[Maximum mark: 8]

Let $f(x) = \frac{x^2 + 1}{x + 2}$, where $x \neq -2$.

- Find $f'(x)$ in simplified form. [3]
- Find the exact x -coordinates where the tangent is horizontal. [2]
- Find the equation of the normal at $x = 0$. [3]

36.

[Maximum mark: 6]

Let $f(x) = \sin(2x) + \cos x$.

- Find $f'(x)$. [2]
- Find the exact gradient at $x = \pi/6$. [2]
- Find the tangent equation at $x = \pi/6$. [2]

37.

[Maximum mark: 7]

Let $f(x) = \ln(x^2 + 1)$.

- Find $f'(x)$. [2]
- Find all points at which the tangent is parallel to $y = x$. [3]
- Find the equation of the tangent at each such point. [2]

38.

[Maximum mark: 7]

A function is defined by $f(x) = e^{\sin x}$.

- Find $f'(x)$. [2]
- Find the exact gradients at $x = 0$ and $x = \pi$. [2]
- Show that the two tangents at these points intersect at $(\pi/2, 1 + \pi/2)$. [3]

39.

[Maximum mark: 6]

For $x > 0$, let $f(x) = \frac{\ln(2x)}{x}$.

- Show that $f'(x) = \frac{1 - \ln(2x)}{x^2}$. [3]
- Find the exact coordinates of the point with a horizontal tangent. [3]

40.

[Maximum mark: 6]

Let $f(x) = (x^2 - 1)^3$.

- (a) Find $f'(x)$ in factorized form. [2]
 (b) Find all points at which the tangent is horizontal. [3]
 (c) Explain why solving only $x^2 - 1 = 0$ would miss a solution. [1]

41.

[Maximum mark: 6]

The curve $y = \frac{\sin x}{x}$ is defined for $x > 0$.

- (a) Find $\frac{dy}{dx}$. [3]
 (b) Find the exact equation of the tangent at $x = \pi$. [3]

42.

[Maximum mark: 6]

The differentiable functions f and g have the values shown.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	3	4	-1
2	5	-2	3	2
4	1	6	2	5

Let $h(x) = f(g(x))$ and $p(x) = f(x)g(x)$.

- (a) Find $h(1)$ and $h'(1)$. [3]
 (b) Find the equation of the tangent to $y = p(x)$ at $x = 1$. [3]

43.

[Maximum mark: 7]

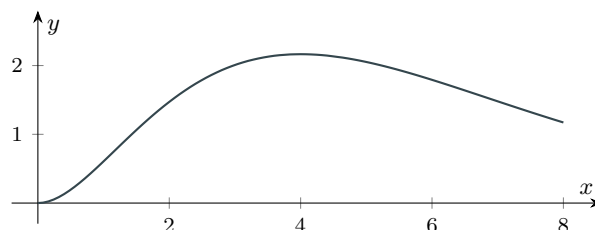
Let $f(x) = (ax + b)e^x$. The tangent at $x = 0$ is $y = 5x + 2$.

- (a) Find b . [1]
 (b) Find $f'(x)$ in terms of a, b and hence find a . [3]
 (c) Find the exact point where the tangent to this curve is horizontal. [3]

44. 

[Maximum mark: 9]

Let $f(x) = x^2e^{-x/2}$, for $0 \leq x \leq 8$.



- (a) Show that $f'(x) = \frac{1}{2}x(4 - x)e^{-x/2}$. [3]
 (b) Find the coordinates of the interior point with a horizontal tangent. [2]
 (c) Use your GDC to find all points on the domain where the gradient is 0.5. [4]

45.

[Maximum mark: 6]

For $x > 0$, let $f(x) = \sqrt{x} \ln x$.

- (a) Show that $f'(x) = \frac{\ln x + 2}{2\sqrt{x}}$. [3]
 (b) Find the exact coordinates of the point with a horizontal tangent. [3]



46.

[Maximum mark: 7]

For $x > 0$, let $f(x) = \frac{e^{2x}}{x}$.

- (a) Find $f'(x)$ in factorized form. [3]
- (b) Find the equation of the horizontal tangent. [3]
- (c) State the interval on which $f'(x) < 0$. [1]

47.

[Maximum mark: 7]

Let $f(x) = \cos(x^2)$.

- (a) Find $f'(x)$. [2]
- (b) Find all x -coordinates in $0 \leq x \leq \sqrt{2\pi}$ where the tangent is horizontal. [3]
- (c) Find the exact gradient at $x = \sqrt{\pi/2}$. [2]

48.

[Maximum mark: 8]

Let $f(x) = \ln(x+1) - \frac{x}{x+1}$, for $x > -1$.

- (a) Show that $f'(x) = \frac{x}{(x+1)^2}$. [3]
- (b) Find all points where the tangent has gradient $\frac{1}{4}$. [3]
- (c) Find the equation of the tangent at the origin. [2]

49. 

[Maximum mark: 9]

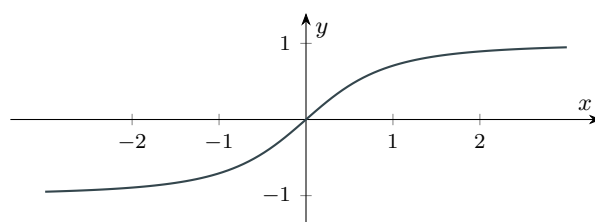
Let $f(x) = e^{-0.2x} \sin x$, for $0 \leq x \leq 2\pi$.

- (a) Find $f'(x)$ in factorized form. [3]
- (b) Use your GDC to find the coordinates of all interior points with horizontal tangents. [4]
- (c) Find the exact gradient at $x = \pi$. [2]

50.

[Maximum mark: 8]

The curve $y = \frac{x}{\sqrt{x^2+1}}$ is shown.



- (a) Use the product and chain rules to show that $\frac{dy}{dx} = (x^2+1)^{-3/2}$. [4]
- (b) Explain why the curve has no horizontal tangent. [1]
- (c) Find the exact points where the gradient is $\frac{1}{2\sqrt{2}}$. [3]



4. AA HL exclusive challenge

10 questions

AA SL students: the shared worksheet ends at Question 50.

51.

[Maximum mark: 7]

Let $f(x) = e^{x^2} \sin x$.

- (a) Find $f'(x)$ in factorized form. [3]
- (b) Find the exact equation of the tangent at $x = \pi/2$. [3]
- (c) State the exact y -intercept of this tangent. [1]

52.

[Maximum mark: 8]

For $x > -1$, let $f(x) = \ln\left(\frac{x^2 + 1}{(x + 1)^2}\right)$.

- (a) Find $f'(x)$ and simplify it to a single fraction. [4]
- (b) Find the exact equation of the horizontal tangent. [3]
- (c) State where f is decreasing on the stated domain. [1]

53.

[Maximum mark: 9]

The curve $y = (x^2 + ax + b)e^{-x}$ has tangent $y = 3x + 2$ at $x = 0$.

- (a) Find a and b . [4]
- (b) Find the exact x -coordinates of all horizontal tangents. [3]
- (c) Find the interval on which the curve is increasing. [2]

54.

[Maximum mark: 8]

Let $f(x) = \frac{x \sin x}{1 + \cos x}$, for $0 < x < \pi$.

- (a) Show that $f'(x) = \frac{x + \sin x}{1 + \cos x}$. [5]
- (b) Explain why this curve has no horizontal tangent on the stated domain. [2]
- (c) Find the exact gradient at $x = \pi/2$. [1]

55.

[Maximum mark: 8]

The table gives values of differentiable functions f and g .

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	2	3	3	2
2	6	1	5	-1
3	4	-2	1	4

Let $h(x) = \frac{f(g(x))}{g(f(x))}$.

- (a) Find $h(1)$. [1]
- (b) Find $h'(1)$, showing how you combine the quotient and chain rules. [5]
- (c) Find the equation of the tangent at $x = 1$. [2]

56.

[Maximum mark: 10]

For $x > 0$, let $f(x) = \frac{(\ln x)^2}{x}$.

- (a) Find and factorize $f'(x)$. [3]
- (b) Find the exact points with horizontal tangents. [3]
- (c) Find every point on the curve whose tangent passes through the origin. [4]

57.

[Maximum mark: 9]

Let $f(x) = \frac{e^x}{x+a}$, where $a \neq 0$. The tangent to the curve at $x = 0$ passes through $(1, 0)$.

- Find $f'(x)$ in terms of a . [3]
- Find a and the tangent equation. [4]
- For this value of a , find the exact point with a horizontal tangent. [2]

58.

[Maximum mark: 8]

Let $f(x) = \ln(1 + e^{kx})$, where $k > 0$. The tangent at $x = 0$ has gradient 3.

- Find k . [3]
- Show that $0 < f'(x) < 6$ for all real x . [2]
- Find the exact point where the gradient is 4. [3]

59.

[Maximum mark: 10]

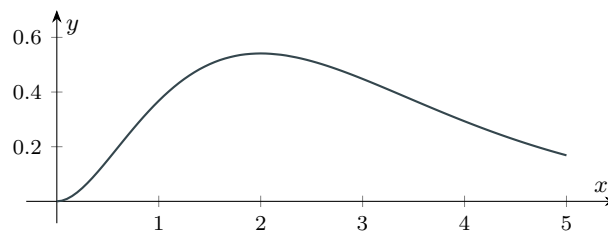
Let $f(x) = e^x \cos x$.

- Find $f'(x)$ and all horizontal-tangent x -coordinates in $0 \leq x \leq 2\pi$. [4]
- Find the exact equations of the tangents at $x = 0$ and $x = \pi$. [3]
- Find the exact coordinates of their intersection. [3]

60.

[Maximum mark: 10]

The curve $y = x^2 e^{-x}$ is shown. A tangent drawn at $x = a$, where $a > 0$, passes through the origin.



- Find $\frac{dy}{dx}$. [2]
- Find the exact value of a and the equation of the tangent. [4]
- Find the exact equation of the normal at this point. [2]
- This normal meets the x -axis at N . Find N . [2]

Answer key and solution outlines

AA SL/HL 5.6

Equivalent forms are accepted. Numerical coordinates are rounded to 3 significant figures; use unrounded values for comparisons.

Section 1: Short response

1. $f'(x) = \frac{3}{2\sqrt{x}} + 6x^{-4}$.
2. $\frac{dy}{dx} = 6\sqrt{x} - \frac{10}{3}x^{-1/3}$.
3. $f'(x) = \cos x - 2\sin x$.
4. $f'(x) = 3e^x - 4/x$.
5. $\frac{dy}{dx} = 3\cos(3x)$.
6. $\frac{dy}{dx} = -2\sin(2x - 1)$.
7. $f'(x) = 2xe^{x^2}$.
8. $f'(x) = \frac{5}{5x + 2}$.
9. $\frac{dy}{dx} = 12(2x - 1)^5$.
10. $\frac{dy}{dx} = -4x(x^2 + 1)^{-3}$.
11. $f' = 2xe^x + x^2e^x = x(x + 2)e^x$.
12. $\frac{dy}{dx} = \sin x + (x + 1)\cos x$.
13. $f'(x) = \frac{(x - 1) - (x + 2)}{(x - 1)^2} = -\frac{3}{(x - 1)^2}$.
14. (i) $(-1)(4) + (3)(5) = 11$. (ii) $\frac{(-1)(4) - (3)(5)}{4^2} = -19/16$.
15. The inner derivative $2x$ is missing. By the chain rule, the derivative is $\frac{2x}{x^2 + 1}$.

Section 2: Extended response

16. $f' = \frac{1}{2}(3x^2 + 1)^{-1/2}(6x) = \frac{3x}{\sqrt{3x^2 + 1}}$. Hence $f'(1) = 3/2$.
17. $f' = 3x^2 \sin(2x) + 2x^3 \cos(2x)$. At $x = \pi/2$, the sine is 0 and the cosine is -1 , so $f'(\pi/2) = -\pi^3/4$.
18. Quotient rule: $f' = [x(1/x) - \ln x]/x^2 = (1 - \ln x)/x^2$. This is zero at $x = e$; the point is $(e, 1/e)$ and the tangent is $y = 1/e$.
19. $f' = \frac{(x + 1)e^x - e^x}{(x + 1)^2} = \frac{xe^x}{(x + 1)^2}$. At $x = 0$, $f = 1$ and $f' = 0$, so the tangent is $y = 1$.
20. $f' = 3\sin^2 x \cos x - 3x^2 \sin(x^3)$. The inner functions are $\sin x$ and x^3 , with derivatives $\cos x$ and $3x^2$.
21. $y' = 2/(2x + 1)$; at $x = 0$ the gradient is 2. Hence $y = 2x$.
22. $y' = -e^{-x}$; at $(0, 1)$ the tangent gradient is -1 , so the normal gradient is 1. Thus $y = x + 1$.
23. $f' = 2xe^{-x} - (x^2 + 1)e^{-x} = -(x - 1)^2e^{-x}$. It is zero only at $x = 1$, where $f(1) = 2/e$, so $y = 2/e$. Since $e^{-x} > 0$ and $(x - 1)^2 \geq 0$, $f' \leq 0$.
24. $f' = \frac{2(x^2 + 1) - 2x(2x + 1)}{(x^2 + 1)^2} = \frac{-2x^2 - 2x + 2}{(x^2 + 1)^2}$. Zeros $x = (-1 \pm \sqrt{5})/2$: points $(-1.62, -0.618)$ and $(0.618, 1.62)$.
25. (i) $f'(g(1))g'(1) = f'(2)(-3) = -12$. (ii) $g'(f(1))f'(1) = g'(3)(2) = -2$.



26. $f' = ae^{ax} + b$. For $a = 2, b = 3$, $f'(0) = 2 + 3 = 5$ and $f(1) = e^2 + 3$. Also $f(0) = 1$, so the tangent is $y = 5x + 1$.
27. $f' = 2x \ln x + x = x(2 \ln x + 1)$. As $x > 0$, $\ln x = -1/2$, so $x = e^{-1/2}$ and $f(x) = -1/(2e)$. Point: $(e^{-1/2}, -1/(2e))$.
28. For $x \neq 1$, $f(x) = x + 1$, so $f'(x) = 1$. The original function is undefined at $x = 1$; cancellation does not restore that missing point.
29. Chain rule: $f' = \frac{1}{\sqrt{x^2 + 4}} \frac{x}{\sqrt{x^2 + 4}} = \frac{x}{x^2 + 4}$. Alternatively $f = \frac{1}{2} \ln(x^2 + 4)$, so $f' = \frac{1}{2} \frac{2x}{x^2 + 4}$.
30. $f' = e^{-x}(1 - x) + 0.1$. Solve $f' = 0$ on $[0, 6]$: $(1.41, 0.485)$, $(2.99, 0.449)$. Both solutions must be included.

Section 3: IB exam-style practice

31. (a) $f' = 15(3x - 2)^4$. (b) $f(1) = 1$, $f'(1) = 15$; $y - 1 = 15(x - 1)$, so $y = 15x - 14$.
32. (a) $f' = 2/\sqrt{4x + 5}$. (b) At $(1, 3)$, gradient $2/3$, so $y - 3 = \frac{2}{3}(x - 1)$. Set $y = 0$: $x = -7/2$; $P = (-7/2, 0)$.
33. (a) $f' = \ln x + 1$. (b) $\ln x + 1 = 2 \Rightarrow x = e$; point (e, e) . (c) $y - e = 2(x - e)$, so $y = 2x - e$.
34. (a) $f' = e^{-x} - xe^{-x}$. (b) $e^{-x} \neq 0$, so $x = 1$; point $(1, 1/e)$. (c) $f(0) = 0$, $f'(0) = 1$, so $y = x$.
35. (a) $f' = \frac{2x(x+2) - (x^2+1)}{(x+2)^2} = \frac{x^2+4x-1}{(x+2)^2}$. (b) $x = -2 \pm \sqrt{5}$. (c) $f(0) = 1/2$, $f'(0) = -1/4$, so normal gradient 4; $y = 4x + 1/2$.
36. (a) $f' = 2 \cos(2x) - \sin x$. (b) $2 \cos(\pi/3) - \sin(\pi/6) = 1/2$. (c) $f(\pi/6) = \sqrt{3}$; $y - \sqrt{3} = \frac{1}{2}(x - \pi/6)$.
37. (a) $f' = 2x/(x^2 + 1)$. (b) $2x = x^2 + 1 \Rightarrow (x - 1)^2 = 0$, so the only point is $(1, \ln 2)$. (c) $y = x + \ln 2 - 1$.
38. (a) $f' = e^{\sin x} \cos x$. (b) 1 and -1 . (c) Both points have $y = 1$; tangents $y = x + 1$ and $y = -x + \pi + 1$. Equating gives $2x = \pi$ and $y = 1 + \pi/2$.
39. (a) $\frac{d}{dx} \ln(2x) = 1/x$; quotient rule gives $[x(1/x) - \ln(2x)]/x^2$. (b) $\ln(2x) = 1 \Rightarrow x = e/2$, and $f(x) = 2/e$; point $(e/2, 2/e)$.
40. (a) $f' = 6x(x^2 - 1)^2$. (b) $x = -1, 0, 1$: points $(-1, 0)$, $(0, -1)$, $(1, 0)$. (c) The factor x also gives a zero, at $x = 0$.
41. (a) $y' = (x \cos x - \sin x)/x^2$. (b) The point is $(\pi, 0)$ and the gradient is $-1/\pi$, so $y = 1 - x/\pi$.
42. (a) $h(1) = f(4) = 1$; $h'(1) = f'(4)g'(1) = 6(-1) = -6$. (b) $p(1) = 8$, $p'(1) = 3(4) + 2(-1) = 10$; $y - 8 = 10(x - 1)$, so $y = 10x - 2$.
43. (a) $f(0) = b = 2$. (b) $f' = (ax + a + b)e^x$; $a + b = 5$, so $a = 3$. (c) $(3x + 5)e^x = 0 \Rightarrow x = -5/3$; point $(-5/3, -3e^{-5/3})$.
44. (a) $f' = 2xe^{-x/2} - \frac{1}{2}x^2e^{-x/2} = \frac{1}{2}x(4 - x)e^{-x/2}$. (b) $(4, 16e^{-2})$. (c) Solve $\frac{1}{2}x(4 - x)e^{-x/2} = 0.5$ on $[0, 8]$: $(0.319, 0.0865)$, $(2.59, 1.84)$.
45. (a) $f' = \frac{\ln x}{2\sqrt{x}} + \frac{\sqrt{x}}{x} = \frac{\ln x + 2}{2\sqrt{x}}$. (b) $\ln x = -2 \Rightarrow x = e^{-2}$; point $(e^{-2}, -2/e)$.
46. (a) $f' = \frac{2xe^{2x} - e^{2x}}{x^2} = \frac{(2x - 1)e^{2x}}{x^2}$. (b) $x = 1/2$, $f(1/2) = 2e$, so $y = 2e$. (c) $0 < x < 1/2$.
47. (a) $f' = -2x \sin(x^2)$. (b) $x = 0$, $\sqrt{\pi}$, $\sqrt{2\pi}$. (c) $-2\sqrt{\pi/2} = -\sqrt{2\pi}$.
48. (a) $f' = 1/(x+1) - 1/(x+1)^2 = x/(x+1)^2$. (b) $4x = (x+1)^2 \Rightarrow (x-1)^2 = 0$: only $(1, \ln 2 - 1/2)$. (c) $f(0) = 0$, $f'(0) = 0$, so $y = 0$.
49. (a) $f' = e^{-0.2x}(\cos x - 0.2 \sin x)$. (b) $(1.37, 0.745)$, $(4.51, -0.397)$. (c) $f'(\pi) = -e^{-\pi/5}$.
50. (a) Write $y = x(x^2 + 1)^{-1/2}$. Then $y' = (x^2 + 1)^{-1/2} - x^2(x^2 + 1)^{-3/2} = (x^2 + 1)^{-3/2}$. (b) This is strictly positive for all real x . (c) $(x^2 + 1)^{3/2} = 2\sqrt{2} \Rightarrow x^2 + 1 = 2$, so points $(\pm 1, \pm 1/\sqrt{2})$, with matching signs.



Section 4: AA HL exclusive challenge

51. (a) $f' = e^x(2x \sin x + \cos x)$. (b) Point $(\pi/2, e^{\pi^2/4})$ and gradient $\pi e^{\pi^2/4}$; $y - e^{\pi^2/4} = \pi e^{\pi^2/4}(x - \pi/2)$. (c) $e^{\pi^2/4}(1 - \pi^2/2)$.

52. (a) $f = \ln(x^2 + 1) - 2 \ln(x + 1)$; $f' = \frac{2x}{x^2 + 1} - \frac{2}{x + 1} = \frac{2(x - 1)}{(x^2 + 1)(x + 1)}$. (b) $x = 1$, $f(1) = \ln(1/2) = -\ln 2$, so $y = -\ln 2$. (c) $-1 < x < 1$.

53. (a) $f(0) = b = 2$; $f' = e^{-x}[-x^2 + (2 - a)x + a - b]$, so $a - b = 3$ and $a = 5$. (b) $-x^2 - 3x + 3 = 0$: $x = (-3 \pm \sqrt{21})/2$. (c) The quadratic factor is positive between its roots: $(-3 - \sqrt{21})/2 < x < (-3 + \sqrt{21})/2$.

54. (a) The quotient rule gives numerator $(\sin x + x \cos x)(1 + \cos x) + x \sin^2 x$. Since $\sin^2 x + \cos^2 x = 1$, this becomes $(1 + \cos x)(\sin x + x)$. Cancel one factor. (b) Numerator and denominator are positive for $0 < x < \pi$, so $f' > 0$. (c) $1 + \pi/2$.

55. (a) $h(1) = f(3)/g(2) = 4/5$. (b) Let $u = f \circ g$, $v = g \circ f$. Then $u(1) = 4$, $u'(1) = (-2)(2) = -4$, $v(1) = 5$, $v'(1) = (-1)(3) = -3$. Thus $h'(1) = [(-4)(5) - 4(-3)]/25 = -8/25$. (c) $y - 4/5 = -\frac{8}{25}(x - 1)$.

56. (a) $f' = \frac{2 \ln x - (\ln x)^2}{x^2} = \frac{\ln x(2 - \ln x)}{x^2}$. (b) $(1, 0)$ and $(e^2, 4/e^2)$. (c) A tangent at $x = c$ passes through $(0, 0)$ when $f(c) = cf'(c)$. Thus $(\ln c)^2 = \ln c(2 - \ln c)$, giving $2 \ln c(\ln c - 1) = 0$. Points $(1, 0)$ and $(e, 1/e)$; tangents $y = 0$ and $y = x/e^2$.

57. (a) $f' = \frac{e^x(x + a - 1)}{(x + a)^2}$. (b) $f(0) = 1/a$, $f'(0) = (a - 1)/a^2$. Passing through $(1, 0)$ gives $0 = 1/a + (a - 1)/a^2$, so $2a - 1 = 0$ and $a = 1/2$. Tangent $y = -2x + 2$. (c) $x = 1/2$ and $f(1/2) = \sqrt{e}$: $(1/2, \sqrt{e})$.

58. (a) $f' = \frac{ke^{kx}}{1 + e^{kx}}$; $f'(0) = k/2 = 3$, so $k = 6$. (b) For $u = e^{6x} > 0$, $0 < u/(1 + u) < 1$, hence $0 < 6u/(1 + u) < 6$. (c) $6u/(1 + u) = 4 \Rightarrow u = 2$; point $(\ln 2/6, \ln 3)$.

59. (a) $f' = e^x(\cos x - \sin x)$; zeros $x = \pi/4, 5\pi/4$. (b) At 0: $y = x + 1$. At π : $y + e^\pi = -e^\pi(x - \pi)$. (c) $x = \frac{e^\pi(\pi - 1) - 1}{1 + e^\pi}$, $y = \frac{\pi e^\pi}{1 + e^\pi}$.

60. (a) $y' = x(2 - x)e^{-x}$. (b) $f(a) = af'(a)$ gives $a^2e^{-a} = a^2(2 - a)e^{-a}$. Since $a > 0$, cancel to get $a = 1$. Point $(1, 1/e)$, gradient $1/e$, tangent $y = x/e$. (c) Normal gradient $-e$: $y - 1/e = -e(x - 1)$. (d) Set $y = 0$: $N = (1 + e^{-2}, 0)$.