



The Second Derivative

IB Mathematics: Analysis and Approaches SL / HL | SL 5.7

50 shared questions + 10 AA HL challenges

Learning goal	Calculate second derivatives; connect changing gradients, concavity and the graphs of f , f' and f'' .
Syllabus link	AA SL 5.7: second derivatives and graphical relationships, with linked SL 5.8 concave-up / concave-down terminology. HL challenges deepen this content.
Structure	1–15 short response; 16–30 extended response; 31–50 IB exam-style. 51–60: AA HL exclusive challenge.
Technology	 marks GDC questions (Paper 2 style). Work without a calculator on all other questions (Paper 1 style).

Instructions. Work on separate paper and show your method. Give exact answers unless a decimal is requested or a numerical method is used; give numerical results to 3 significant figures. Use radians. State derivative rules clearly and retain domain restrictions.

The HL section deepens second-derivative and graphical reasoning; no additional AHL techniques are assumed.

1. Short response

15 questions

- Find $f''(x)$ when $f(x) = 3x^4 - 5x^2 + 7$.
- Given $y = 2x^3 - 4x + 1$, find $\frac{d^2y}{dx^2}$.
- Find $f''(x)$ for $f(x) = 6/x$, $x \neq 0$.
- Find $f''(x)$ for $f(x) = \sqrt{x}$, $x > 0$.
- Given $f'(x) = 5x^4 - 3x^2 + 2$, find $f''(x)$.
- Find $f''(x)$ for $f(x) = e^{3x}$.
- Find $f''(x)$ for $f(x) = \sin x$.
- Find $f''(x)$ for $f(x) = \ln x$, $x > 0$.
- The gradient of the graph of $y = f'(x)$ at $x = 2$ is -3 . Write this information using second-derivative notation.
- On an interval, $f''(x) > 0$. What does this say about the gradient of $y = f(x)$ as x increases? Is the graph concave up or concave down on this interval?
- For $f(x) = x^3 - 3x$, find $f'(1)$ and $f''(1)$.
- The graph of $y = f'(x)$ is the straight line through $(-1, 4)$ and $(3, -4)$. Find $f''(x)$.
- A student claims that $f''(a) = 0$ always implies $f'(a) = 0$. Use $f(x) = x^3 + x$ and $a = 0$ to test the claim.



14. If $g(x) = f(x) + 7$, compare $g'(x)$ with $f'(x)$ and $g''(x)$ with $f''(x)$.

15. Given $f(x) = ax^2 + bx + c$ and $f''(x) = 10$, find a .

2. Extended response

15 questions

16. For $f(x) = x^4 - 4x^3 + 2x + 5$, find $f'(x)$ and $f''(x)$. Solve $f''(x) = 0$.

17. For $f(x) = (2x - 1)^3$, find and factorize $f''(x)$. Find where the gradient of f is increasing. State where the graph of f is concave up and where it is concave down.

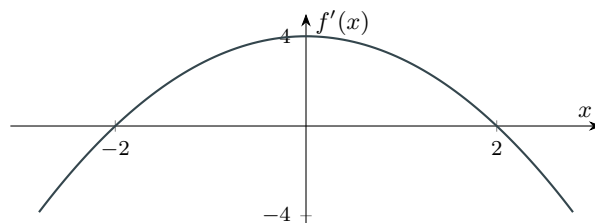
18. For $f(x) = xe^x$, find $f'(x)$ and $f''(x)$. Find the gradient of $y = f'(x)$ at $x = 0$.

19. For $f(x) = (x + 1)/x$, $x \neq 0$, simplify before finding $f''(x)$. State its sign for positive and negative x . Hence describe the concavity of each branch of the graph of f .

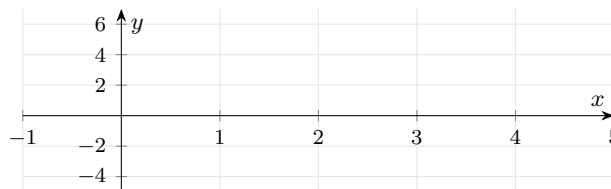
20. Let $f(x) = \cos(2x)$. Find $f''(x)$ and solve $f''(x) = 0$ for $0 \leq x \leq \pi$.

21. For $f(x) = \ln(3x + 1)$, $x > -1/3$, find $f'(x)$ and $f''(x)$. Describe the sign of the gradient and how it changes. State the concavity of the graph throughout its domain.

22. The graph shows $y = f'(x) = 4 - x^2$. State the intervals where f is increasing. Then find where the gradient of f is increasing, and state where the graph of f is concave up or concave down.



23. For $f(x) = x^3/3 - 2x^2 + 3x$, show that $f'(x) = (x - 1)(x - 3)$. Find the equation of $y = f''(x)$ and sketch it on the supplied axes.



24. The table gives information about a function with a continuous second derivative.

x	-1	0	2
$f'(x)$	3	-2	0
$f''(x)$	-4	5	-1

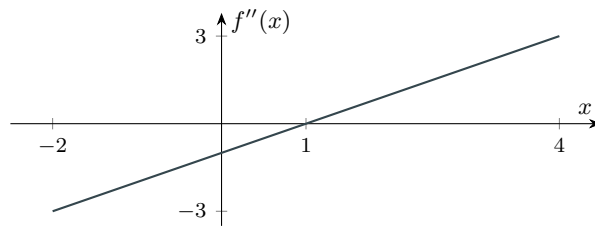
At each listed value, state whether f has positive, negative or zero gradient, and whether that gradient is increasing or decreasing. Also state whether the graph is locally concave up or concave down at each listed value.

25. A cubic is $f(x) = x^3 + ax^2 + bx + 4$. Given $f'(1) = 0$ and $f''(1) = 8$, find a and b .

26. Let $f(x) = x^3$ and $g(x) = x^3 + 4x - 2$. Show that their second derivatives are equal. Explain why their gradients are not equal.

27. For $f(x) = x^4/4 - x^2/2$, find where $f'(x)$ is increasing and where it is decreasing. Hence state the intervals on which the graph of f is concave up or concave down.

28. The graph shows $y = f''(x) = x - 1$. State where f' is increasing and where it is decreasing. Hence state where the graph of f is concave up or concave down. Can the zeros of f be determined from this graph alone? Explain.



29. For $f(x) = e^{-x} + x^2$, find the tangent to the graph of $y = f'(x)$ at $x = 0$.

30.  Use technology to calculate $f''(1)$ for $f(x) = x^2 \sin x$. Verify your result using an algebraic expression for $f''(x)$.

3. IB exam-style practice

20 questions

31.

[Maximum mark: 8]

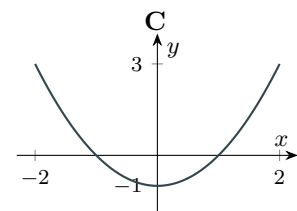
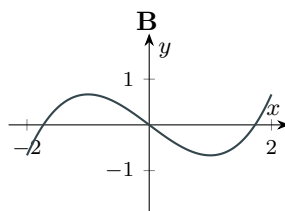
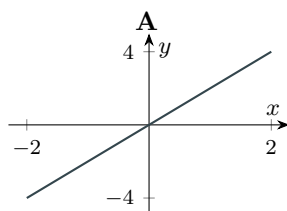
Consider $f(x) = x^3 - 6x^2 + 9x + 2$.

- Find $f'(x)$ and $f''(x)$. [3]
- Find the values of x where the graph of f has zero gradient. [2]
- Find where the gradient of f is increasing. State where the graph of f is concave up and concave down, giving a reason. [3]

32.

[Maximum mark: 8]

The three graphs below are $y = f(x)$, $y = f'(x)$ and $y = f''(x)$ in some order.

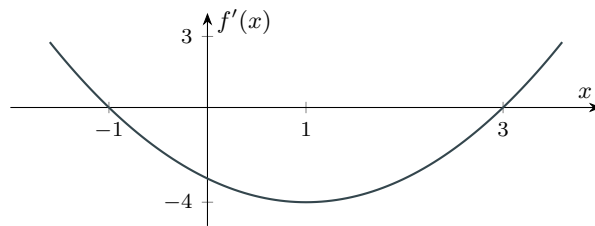


- Identify the graph representing each function. Give two graphical reasons. [4]
- Use the graphs to find the values of x where f has zero gradient. [2]
- State where f' is decreasing and give the corresponding concavity of f . [2]

33.

[Maximum mark: 7]

The graph shows $y = f'(x) = (x - 1)^2 - 4$.



- Find the intervals where f is increasing. [2]
- Find $f''(x)$ and the value of x where the graph of f' has a horizontal tangent. [3]
- Explain why this horizontal tangent is not a stationary point of f . [2]

34.

[Maximum mark: 9]

Let $f(x) = (x^2 + 1)^2$.

- Find $f'(x)$ and show that $f''(x) = 12x^2 + 4$. [4]
- Explain why the gradient of f is increasing for every real x , and state the concavity of its graph. [2]
- Find the tangent to $y = f'(x)$ at $x = 1$. [3]

35.

[Maximum mark: 9]

A curve has equation $y = x^2 + 8/x$, for $x > 0$.

- Find dy/dx and d^2y/dx^2 . [4]
- Find the point where the curve has zero gradient. [3]
- Show that the gradient is increasing throughout the domain, and hence state the concavity of the curve. [2]

36.

[Maximum mark: 8]

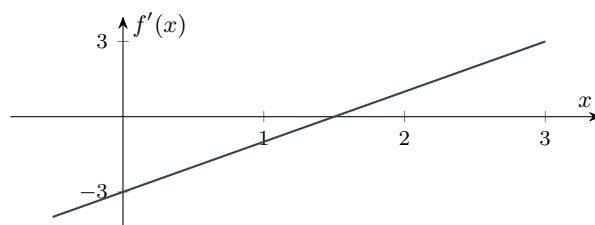
Let $f(x) = \sin(2x) + x$, for $0 \leq x \leq \pi$.

- Find the first and second derivatives. [3]
- Find all zeros of f'' in this interval. [2]
- State where the gradient of f is increasing and where it is decreasing. Hence describe the concavity of the graph on these intervals. [3]

37.

[Maximum mark: 7]

The derivative graph is the line $f'(x) = 2x - 3$. A quadratic $f(x) = ax^2 + bx + c$ also satisfies $f(0) = 4$.

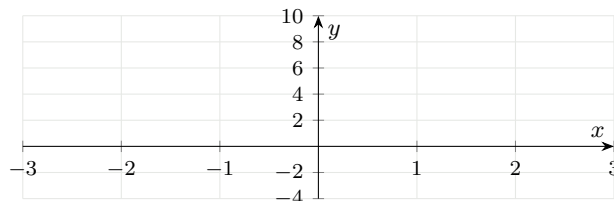


- Find a , b and c . [3]
- Find $f''(x)$ and explain its graphical meaning. [2]
- State where f is decreasing while its gradient is increasing, and describe the concavity of its graph there. [2]

38.

[Maximum mark: 7]

A function has derivative $f'(x) = x^3 - 3x$. The supplied axes are for a sketch of its second derivative.



- Find $f''(x)$. [1]
- Sketch $y = f''(x)$, labelling its intercepts. [3]
- Find where f' is decreasing and explain how the sketch shows this. Hence state where the graph of f is concave down. [3]

39.

[Maximum mark: 8]

Let $f(x) = e^{2x} - 4x^2$.

- Find $f'(x)$ and $f''(x)$. [3]
- Solve $f''(x) = 0$, giving an exact answer. [2]
- Find the sign of f'' on either side of this value. Interpret it in terms of the changing gradient and concavity of f . [3]

40.

[Maximum mark: 6]

A cubic $f(x) = ax^3 + bx^2 + cx + d$ satisfies the following conditions.

Condition	Value
$f(0)$	5
$f'(0)$	-2
$f''(0)$	6
$f''(2)$	18

- Find a , b , c and d . [4]
- Find where the gradient of f is decreasing, and state the concavity of its graph there. [2]

41.

[Maximum mark: 8]

For $x > -2$, let $f(x) = \ln(x+2) - x/2$.

- Find the first and second derivatives. [3]
- Find the value of x where $f'(x) = 0$. [2]
- A student says that $f''(x) < 0$ means f is decreasing throughout its domain. Explain the error using this function, and state its correct concavity. [3]

42.

[Maximum mark: 8]

Let $f(x) = x^2 e^{-x}$.

- Show that $f'(x) = e^{-x}(2x - x^2)$. [2]
- Find $f''(x)$ in factorized or expanded form. [3]
- Find the exact values of x at which the graph of f' has horizontal tangents. [3]

43.

[Maximum mark: 7]

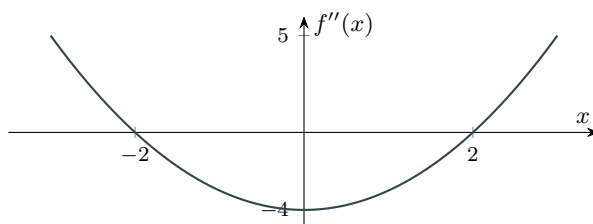
For $f(x) = x^3 - 2x$, consider the graphs of f' and f'' .

- Find both derivative functions. [2]
- Find the exact x -coordinates of their intersections. [3]
- Explain the meaning of the equality $f'(a) = f''(a)$ in terms of the slopes of two graphs. [2]

44.

[Maximum mark: 8]

The graph shows $f''(x) = x^2 - 4$.



- Find where f' is increasing and decreasing. Hence state where the graph of f is concave up and concave down. [4]
- State the x -coordinates at which f' has horizontal tangents. [2]
- Is this information sufficient to find the stationary points of f ? Explain. [2]

45.

[Maximum mark: 7]

Two models are $f(x) = x^4 + ax^2 + bx + c$ and $g(x) = x^4 - 3x^2 + 2x + 7$. Their second derivatives agree for every real x .

- Find a . [2]
- Given that $f'(1) = g'(1)$, find b . [2]
- Given that $f(0) = 1$, find c . Explain why the first condition alone could not determine b or c . [3]

46.

[Maximum mark: 9]

An experimental response is modelled by $R(t) = 20(1 - e^{-t/2})$, where $t \geq 0$ is time in minutes and R is measured in units.

- Find $R'(t)$ and $R''(t)$. [3]
- State their signs and describe what happens to the response and to its rate of change. State whether the response graph is concave up or concave down. [4]
- Find $R'(0)$ and $R''(0)$, including units. [2]

47. 

[Maximum mark: 9]

Let $f(x) = x^2 \sin x$, for $0 \leq x \leq 3$.

- Show that $f''(x) = (2 - x^2) \sin x + 4x \cos x$. [4]
- Use technology to find the only zero of f'' in $0 < x < 3$. [2]
- State where the gradient of f is increasing and where it is decreasing in $0 < x < 3$. Describe the concavity of the graph on these intervals. [3]

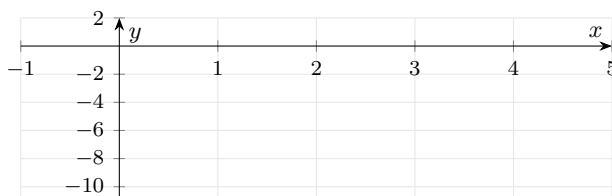


48.

[Maximum mark: 7]

A twice-differentiable function has $f'(x) = -(x-2)^2 - 1$ for all real x .

- (a) Show that f is decreasing everywhere. [2]
(b) Find $f''(x)$. [1]
(c) Sketch f' on the supplied axes and use it to state where the gradient of f is increasing. State where the graph of f is concave up or concave down. [4]



49.

[Maximum mark: 9]

A curve is $y = \sqrt{2x+1}$, for $x > -1/2$.

- (a) Find dy/dx and d^2y/dx^2 . [4]
(b) Find the tangent to the graph of dy/dx at $x = 0$. [3]
(c) Explain how the sign of the second derivative describes the changing gradient and concavity of the original curve. [2]

50.

[Maximum mark: 8]

The table gives exact information at three points of a twice-differentiable function.

x	$f(x)$	$f'(x)$	$f''(x)$
-1	4	-2	3
1	0	0	-4
3	-2	5	0

- (a) Find the tangent to $y = f(x)$ at $x = -1$. [2]
(b) Find the tangent to $y = f'(x)$ at $x = 1$. [2]
(c) At $x = 3$, identify which of these two graphs has a horizontal tangent and justify your answer. [2]
(d) Explain why the values in the last row do not determine whether f'' changes sign at $x = 3$. [2]

4. AA HL exclusive challenge

10 questions

AA SL students: the shared worksheet ends at Question 50.

51.

[Maximum mark: 8]

Let $f(x) = (x^2 + 1)e^{-x}$. Find $f''(x)$ in factorized form, then determine the intervals where the gradient of f is increasing. Also state the intervals of concave-up and concave-down behaviour of f .

52.

[Maximum mark: 9]

For $f(x) = \ln(1 + x^2)$, find $f''(x)$. Sketch its sign table and hence state where the gradient of f is increasing and where its graph is concave up or concave down. Your answer should also explain why $f''(\pm 1) = 0$ does not imply $f'(\pm 1) = 0$.

53.

[Maximum mark: 8]

A quartic is $f(x) = ax^4 + bx^3 + cx^2 + dx + e$. The following conditions hold.

Condition	Value
$f(0)$	2
$f'(0)$	-3
$f''(0)$	4
$f''(1)$	10
$f''(-1)$	22

Find $f(x)$ and the exact zeros of its second derivative, if any.

54.

[Maximum mark: 7]

A twice-differentiable function is $F(x) = f(2x - 1)$. Derive expressions for $F'(x)$ and $F''(x)$ in terms of derivatives of f . Given $f'(3) = -2$ and $f''(3) = 5$, find the tangent to $y = F'(x)$ at $x = 2$.

55.

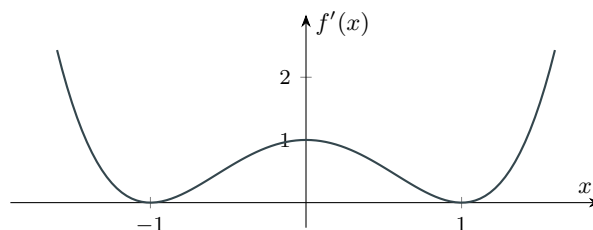
[Maximum mark: 9]

For $x > 0$, $f(x) = x^2 \ln x$. Show that $f''(x) = 2 \ln x + 3$. Find the coordinates of the horizontal-tangent point on $y = f'(x)$, and state where the gradient of f is decreasing. State where the graph of f is concave up and concave down.

56.

[Maximum mark: 9]

A derivative graph is $f'(x) = (x^2 - 1)^2$, as shown.



Find every zero of f'' and the intervals where it is positive. Hence state where the graph of f is concave up or concave down. Explain why f can be increasing on both sides of a point where $f' = 0$.

57.

[Maximum mark: 8]

Let $f(x) = e^{ax}(x + b)$, where $a \neq 0$. Find $f''(x)$ in terms of a and b . Given that $f''(0) = 0$ and $f'(0) = -1$, determine the relation between a and b . If also $f(0) = 2$, find $f(x)$.

**58.**

[Maximum mark: 8]

Let $f(x) = x/(x+1)$, $x \neq -1$. Show that $(x+1)f''(x) + 2f'(x) = 0$. Find the exact point on $y = f'(x)$ where the tangent has gradient $-1/4$.

59.

[Maximum mark: 10]

For $f(x) = \sin x + \frac{1}{4}\sin(2x)$ on $0 \leq x \leq 2\pi$, show that $f''(x) = -\sin x(1 + 2\cos x)$. Find its zeros and construct a sign table. Hence state every interval where the gradient of f is increasing. Also state the intervals on which the graph of f is concave up or concave down.

60.

[Maximum mark: 11]

A twice-differentiable function has $f''(x) = (x-1)^2(x+2)$. A student claims that the gradient of f changes from decreasing to increasing at both $x = -2$ and $x = 1$.

- (a) Construct a sign table for f'' and assess the claim. At which of these two values does the graph of f change concavity? [5]
- (b) For $G(x) = f(2x-3) + 5x$, find $G''(x)$ and its zeros. [3]
- (c) State where the gradient of G is decreasing and its graph is concave down. Explain the effect of the term $5x$ on its second derivative. [3]



Answer key and solution outlines

AA SL/HL 5.7

Equivalent forms are accepted. Numerical coordinates are rounded to 3 significant figures; use unrounded values for comparisons.

Section 1: Short response

1. $f'(x) = 12x^3 - 10x$; $f''(x) = 36x^2 - 10$.
2. $dy/dx = 6x^2 - 4$; $d^2y/dx^2 = 12x$.
3. $f' = -6x^{-2}$; $f'' = 12x^{-3} = 12/x^3$.
4. $f' = \frac{1}{2}x^{-1/2}$; $f'' = -\frac{1}{4}x^{-3/2}$.
5. $f''(x) = 20x^3 - 6x$.
6. $f' = 3e^{3x}$; $f'' = 9e^{3x}$.
7. $f' = \cos x$; $f'' = -\sin x$.
8. $f' = 1/x$; $f'' = -1/x^2$.
9. $f''(2) = -3$.
10. The gradient $f'(x)$ is increasing. This does not by itself determine whether f is increasing or decreasing. The graph is concave up because $f'' > 0$.
11. $f' = 3x^2 - 3$, $f'' = 6x$; $f'(1) = 0$, $f''(1) = 6$.
12. $f'' = (-4 - 4)/(3 - (-1)) = -2$.
13. $f' = 3x^2 + 1$, $f'' = 6x$. At 0, $f''(0) = 0$ but $f'(0) = 1$, so the claim is false.
14. $g' = f'$ and $g'' = f''$; a vertical translation changes neither derivative.
15. $f'' = 2a = 10$, so $a = 5$.

Section 2: Extended response

16. $f' = 4x^3 - 12x^2 + 2$; $f'' = 12x^2 - 24x = 12x(x - 2)$. Zeros: $x = 0, 2$.
17. $f' = 6(2x - 1)^2$; $f'' = 24(2x - 1)$. The gradient is increasing for $x > \frac{1}{2}$. Concave up for $x > 1/2$; concave down for $x < 1/2$.
18. $f' = (x + 1)e^x$; $f'' = (x + 2)e^x$. The requested gradient is $f''(0) = 2$.
19. $f = 1 + x^{-1}$; $f' = -x^{-2}$; $f'' = 2x^{-3}$. Positive for $x > 0$, negative for $x < 0$. The graph is concave up for $x > 0$ and concave down for $x < 0$.
20. $f' = -2\sin(2x)$; $f'' = -4\cos(2x)$. Zeros: $x = \pi/4, 3\pi/4$.
21. $f' = 3/(3x + 1) > 0$; $f'' = -9/(3x + 1)^2 < 0$. The function increases, but its gradient decreases. The graph is concave down throughout $x > -1/3$.
22. f increases for $-2 < x < 2$, since $f' > 0$. $f'' = -2x > 0$ for $x < 0$, so its gradient increases there (within the displayed interval when reading the graph). The graph of f is concave up for $x < 0$ and concave down for $x > 0$.
23. $f' = x^2 - 4x + 3 = (x - 1)(x - 3)$; $f'' = 2x - 4$. Sketch a straight line through $(0, -4)$ and $(2, 0)$.
24. At -1 : positive gradient, decreasing. At 0 : negative gradient, increasing. At 2 : zero gradient, decreasing. At $x = -1$: concave down; at $x = 0$: concave up; at $x = 2$: concave down, from the signs of f'' .
25. $f' = 3x^2 + 2ax + b$; $f'' = 6x + 2a$. $6 + 2a = 8 \Rightarrow a = 1$; $3 + 2 + b = 0 \Rightarrow b = -5$.
26. $f' = 3x^2$, $g' = 3x^2 + 4$; both second derivatives are $6x$. The gradients differ by the constant 4.
27. $f' = x^3 - x$; $f'' = 3x^2 - 1$. f' increases for $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$, and decreases for $-1/\sqrt{3} < x < 1/\sqrt{3}$. Concave up for $x < -1/\sqrt{3}$ or $x > 1/\sqrt{3}$; concave down for $-1/\sqrt{3} < x < 1/\sqrt{3}$.
28. f' decreases for $x < 1$ and increases for $x > 1$. The zeros of f cannot be determined: adding any constant to f leaves f'' unchanged but can alter those zeros. Concave down for $x < 1$ and concave up for $x > 1$.
29. $f' = -e^{-x} + 2x$; $f'' = e^{-x} + 2$. Point $(0, -1)$ and gradient 3 give $y = 3x - 1$.
30. $f' = 2x \sin x + x^2 \cos x$; $f'' = 2 \sin x + 4x \cos x - x^2 \sin x$. At 1: $\sin 1 + 4 \cos 1 \approx 3.00$.



Section 3: IB exam-style practice

- 31.** (a) $f' = 3x^2 - 12x + 9 = 3(x-1)(x-3)$; $f'' = 6x - 12$. (b) $x = 1, 3$. (c) Gradient increases for $x > 2$, since $f'' > 0$. The graph is concave up for $x > 2$ and concave down for $x < 2$, from the signs of $6x - 12$.
- 32.** (a) f : B; f' : C; f'' : A. B has horizontal tangents at the zeros ± 1 of C; C decreases for $x < 0$ and increases for $x > 0$, matching the sign of A. (b) $x = -1, 1$. (c) $x < 0$ (within the displayed domain). The graph of f is concave down for $x < 0$ in the displayed domain, because $f'' < 0$.
- 33.** (a) $f' > 0$ for $x < -1$ or $x > 3$. (b) $f'' = 2(x-1)$; horizontal tangent at $x = 1$. (c) $f'(1) = -4 \neq 0$, so the graph of f is not horizontal there.
- 34.** (a) $f' = 4x(x^2 + 1)$; $f'' = 4(x^2 + 1) + 8x^2 = 12x^2 + 4$. (b) $12x^2 + 4 > 0$. The graph of f is concave up for all real x . (c) $f'(1) = 8$, $f''(1) = 16$; $y - 8 = 16(x - 1)$, or $y = 16x - 8$.
- 35.** (a) $y' = 2x - 8/x^2$; $y'' = 2 + 16/x^3$. (b) $x^3 = 4 \Rightarrow x = \sqrt[3]{4}$, $y = 3 \cdot 4^{2/3}$. (c) $2 + 16/x^3 > 0$ for $x > 0$. The curve is concave up throughout $x > 0$.
- 36.** (a) $f' = 2\cos(2x) + 1$; $f'' = -4\sin(2x)$. (b) $x = 0, \pi/2, \pi$. (c) Gradient decreases on $(0, \pi/2)$ and increases on $(\pi/2, \pi)$. Concave down on $(0, \pi/2)$ and concave up on $(\pi/2, \pi)$.
- 37.** (a) $2ax + b = 2x - 3$, so $a = 1$, $b = -3$, $c = 4$. (b) $f'' = 2$, the constant slope of the derivative graph. (c) $x < 3/2$: $f' < 0$ but $f'' > 0$. The graph is concave up on this interval; a decreasing function can be concave up.
- 38.** (a) $f'' = 3x^2 - 3$. (b) Upward-opening parabola, vertex $(0, -3)$ and zeros $(-1, 0)$, $(1, 0)$. (c) $-1 < x < 1$, where the second-derivative graph lies below the x -axis. The graph of f is concave down for $-1 < x < 1$.
- 39.** (a) $f' = 2e^{2x} - 8x$; $f'' = 4e^{2x} - 8$. (b) $x = \frac{1}{2} \ln 2$. (c) Negative below this value, positive above it; the gradient decreases then increases. Concave down for $x < \frac{1}{2} \ln 2$ and concave up for $x > \frac{1}{2} \ln 2$.
- 40.** (a) $d = 5$, $c = -2$, $2b = 6 \Rightarrow b = 3$. Then $12a + 6 = 18 \Rightarrow a = 1$. (b) $f'' = 6x + 6 < 0$ for $x < -1$. The graph is concave down for $x < -1$.
- 41.** (a) $f' = 1/(x+2) - 1/2$; $f'' = -1/(x+2)^2$. (b) $x = 0$. (c) A negative second derivative means the gradient decreases. Here f increases on $(-2, 0)$ since $f' > 0$, then decreases for $x > 0$. The graph is concave down throughout $x > -2$; this describes changing gradient, not the sign of the gradient.
- 42.** (a) Product rule gives $2xe^{-x} - x^2e^{-x}$. (b) $f'' = e^{-x}(x^2 - 4x + 2)$. (c) $x^2 - 4x + 2 = 0 \Rightarrow x = 2 \pm \sqrt{2}$, since the exponential never vanishes.
- 43.** (a) $f' = 3x^2 - 2$, $f'' = 6x$. (b) $3x^2 - 6x - 2 = 0$, so $x = 1 \pm \sqrt{15}/3$. (c) The slope of $y = f(x)$ at a equals the slope of $y = f'(x)$ at a ; it is not an equality of their heights.
- 44.** (a) f' increases for $x < -2$ and $x > 2$; decreases for $-2 < x < 2$. Concave up for $x < -2$ and $x > 2$; concave down for $-2 < x < 2$. (b) $x = -2, 2$. (c) No: the values or zeros of f' are not fixed by f'' alone; an arbitrary vertical shift of f' preserves its derivative.
- 45.** (a) $12x^2 + 2a = 12x^2 - 6 \Rightarrow a = -3$. (b) $4 - 6 + b = 4 - 6 + 2 \Rightarrow b = 2$. (c) $c = 1$. A linear term $bx + c$ disappears after two differentiations.
- 46.** (a) $R' = 10e^{-t/2}$; $R'' = -5e^{-t/2}$. (b) $R' > 0$, $R'' < 0$: response increases while its rate of increase decreases. The response graph is concave down for $t > 0$. (c) 10 units/min and -5 units/min².
- 47.** (a) Differentiate $f' = 2x \sin x + x^2 \cos x$ to get $2 \sin x + 4x \cos x - x^2 \sin x$. (b) The interior zero is $x = 1.519855 \dots \approx 1.52$. (c) The gradient increases for $0 < x < 1.519855 \dots$ and decreases for $1.519855 \dots < x < 3$, from the signs of f'' . Concave up for $0 < x < 1.519855 \dots$; concave down for $1.519855 \dots < x < 3$.
- 48.** (a) $f' = -(x-2)^2 - 1 \leq -1 < 0$. (b) $f'' = -2(x-2)$. (c) Downward-opening parabola, vertex $(2, -1)$, y -intercept $(0, -5)$, no zeros. Gradient of f increases for $x < 2$. The graph of f is concave up for $x < 2$ and concave down for $x > 2$.
- 49.** (a) $y' = (2x+1)^{-1/2}$; $y'' = -(2x+1)^{-3/2}$. (b) Point $(0, 1)$, gradient -1 : $y = 1 - x$ in the derivative graph. (c) The second derivative is negative, so the positive gradient decreases as x increases. The original curve is concave down throughout $x > -1/2$.
- 50.** (a) $y - 4 = -2(x+1)$, so $y = -2x + 2$. (b) $y = -4(x-1) = -4x + 4$. (c) $y = f'(x)$ is horizontal because $f''(3) = 0$; the original graph has gradient 5. (d) A value at one point gives no sign information on either side.

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51. $f' = e^{-x}(-x^2 + 2x - 1)$; $f'' = e^{-x}(2x - 4x + 3) = e^{-x}(x - 1)(x - 3)$. Since the exponential is positive, the gradient increases for $x < 1$ and $x > 3$. Concave up for $x < 1$ and $x > 3$; concave down for $1 < x < 3$.

52. $f' = 2x/(1 + x^2)$; $f'' = 2(1 - x^2)/(1 + x^2)^2$. Signs are $-$, 0 , $+$, 0 , $-$ across -1 , 1 . The gradient increases for $-1 < x < 1$. $f'(-1) = -1$ and $f'(1) = 1$, both nonzero. Concave up for $-1 < x < 1$; concave down for $x < -1$ and $x > 1$.

x	$-\infty$	-1	1	$+\infty$	
$f''(x)$	$-$	0	$+$	0	$-$

53. $e = 2$, $d = -3$, $c = 2$. From $12a + 6b + 4 = 10$ and $12a - 6b + 4 = 22$, $a = 1$, $b = -1$. Thus $f = x^4 - x^3 + 2x^2 - 3x + 2$; $f'' = 12x^2 - 6x + 4$. Discriminant $36 - 192 = -156 < 0$, so there are no real zeros.

54. Chain rule: $F'(x) = 2f'(2x - 1)$ and $F''(x) = 4f''(2x - 1)$. Thus $F'(2) = -4$ and $F''(2) = 20$; tangent $y + 4 = 20(x - 2)$, or $y = 20x - 44$.

55. $f' = 2x \ln x + x$, so $f'' = 2 \ln x + 3$. This is zero at $x = e^{-3/2}$; $f'(x) = -2e^{-3/2}$. Point $(e^{-3/2}, -2e^{-3/2})$. Gradient decreases for $0 < x < e^{-3/2}$. Concave down for $0 < x < e^{-3/2}$; concave up for $x > e^{-3/2}$.

56. $f'' = 4x(x^2 - 1) = 4x(x - 1)(x + 1)$; zeros $-1, 0, 1$. Signs $-$, $+$, $-$, $+$, so positive on $(-1, 0)$ and $(1, \infty)$. $f' = (x^2 - 1)^2 > 0$ on both sides of ± 1 ; its isolated zeros do not change the increasing behaviour of f . Concave up on $(-1, 0)$ and $(1, \infty)$; concave down on $(-\infty, -1)$ and $(0, 1)$.

57. $f' = e^{ax}[a(x + b) + 1]$; $f'' = e^{ax}[a^2(x + b) + 2a]$. At zero, $a^2b + 2a = 0 \Rightarrow ab = -2$; then $f'(0) = ab + 1 = -1$ adds no new condition. $b = f(0) = 2$, so $a = -1$ and $f(x) = e^{-x}(x + 2)$.

58. $f' = 1/(x + 1)^2$; $f'' = -2/(x + 1)^3$. The expression is $-2/(x + 1)^2 + 2/(x + 1)^2 = 0$. Solve $-2/(x + 1)^3 = -1/4$: $(x + 1)^3 = 8$, so $x = 1$ and $f'(1) = 1/4$. Point $(1, 1/4)$.

59. $f' = \cos x + \frac{1}{2} \cos(2x)$; $f'' = -\sin x - \sin(2x)$. Zeros: $0, 2\pi/3, \pi, 4\pi/3, 2\pi$. Signs on the four intervening intervals: $-$, $+$, $-$, $+$. Gradient increases on $(2\pi/3, \pi)$ and $(4\pi/3, 2\pi)$. Concave up on $(2\pi/3, \pi)$ and $(4\pi/3, 2\pi)$; concave down on $(0, 2\pi/3)$ and $(\pi, 4\pi/3)$.

60. (a) Signs $-$, 0 , $+$, 0 , $+$ across $-2, 1$. Only at -2 does the gradient change from decreasing to increasing; the double zero at 1 gives no sign change. The graph changes from concave down to concave up only at $x = -2$. At $x = 1$, it is concave up on both sides; the zero alone does not imply a change of concavity. (b) $G'' = 4f'''(2x - 3) = 4(2x - 4)^2(2x - 1)$, with zeros $x = 1/2, 2$. (c) Gradient decreases for $x < 1/2$. The second derivative of $5x$ is zero. The graph of G is concave down for $x < 1/2$.