

Extrema, optimisation and inflexion

IB Mathematics AA SL/HL

50 shared questions + 10 HL challenges | Graphs, reasoning and applications

Learning goal	Find and test extrema, optimize in context, and justify points of inflexion.
Syllabus link	SL 5.8: local extrema, derivative tests, optimisation and inflexion. HL challenges deepen this shared content.
Structure	15 short + 15 extended + 20 exam-style questions + 10 HL challenges.
Technology	Non-calculator throughout. All exam-style questions are Paper 1 practice.

Instructions. Give exact answers. Show derivatives, justify classifications and state physical domains in optimisation. At an inflexion point, verify a change of concavity. SL students complete Questions 1–50; HL students also complete 51–60. Work on separate paper.

1. Short response

- Find the stationary point of $f(x) = x^2 - 6x + 5$ and classify it.
- Find the local maximum point of $g(x) = -2x^2 + 8x + 1$.
- Given $f'(x) = x(x - 4)$, find the stationary x -values and classify each using a change of sign.
- It is known that $f'(2) = 0$ and $f''(2) = -5$. State what this tells you about the graph at $x = 2$.
- Given $f'(x) = (x + 1)^2$, explain why the stationary point at $x = -1$ is neither a maximum nor a minimum.
- For $f(x) = x^3$, find the point of inflexion and state whether its tangent is horizontal.
- For $f(x) = x^4$, explain why $(0, 0)$ is not a point of inflexion despite $f''(0) = 0$.
- A function satisfies $f'''(x) = 6x - 12$. State where its graph is concave down and where it is concave up.
- Find and classify the stationary points of $f(x) = x^3 - 6x^2 + 9x + 1$.
- Find the point of inflexion of $f(x) = x^3 - 3x + 2$ and the gradient there.
- Given $f'(x) = 3(x - 2)^2$, classify the stationary point at $x = 2$. Justify the word “inflexion if you use it.
- Find the absolute maximum and absolute minimum values of $f(x) = x^3 - 3x$ on $[-2, 3]$.
- A rectangle has perimeter 20 cm and one side x cm. Write its area as a function of x and state the physical domain.
- A profit model is $P(q) = -q^2 + 12q - 20$, where $q \geq 0$. Find the value of q that maximizes profit and state the maximum profit.

15. If $f'(a) = f''(a) = 0$, can the point be classified from these two facts alone? Give two contrasting examples at $a = 0$.

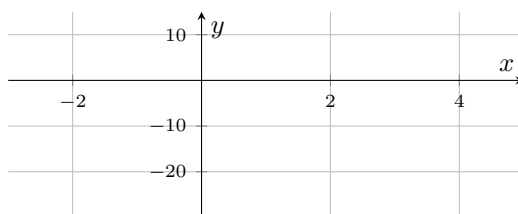
2. Extended response

16. Let $f(x) = x^4 - 8x^2$.

- Find and classify all stationary points.
- Find all points of inflexion.
- State where the graph is concave up and where it is concave down.

17. For $f(x) = x^3 - 3x^2 - 9x + 2$,

- find the local extrema;
- find the point of inflexion and the tangent gradient there;
- sketch the curve, labelling these three points (intercepts are not required).



18. A function has derivative $f'(x) = (x + 2)(x - 1)^2$.

- Find and classify the stationary points using the first derivative.
- Show that $x = 1$ gives a horizontal point of inflexion.
- Find the other x -value at which the concavity changes.

19. Let $f(x) = x + \frac{4}{x}$, for $x > 0$. Find its minimum point, prove it is an absolute minimum and describe its concavity.

20. Let $f(x) = xe^{-x}$, for $x \geq 0$. Find its maximum point, its point of inflexion and the intervals of concavity.

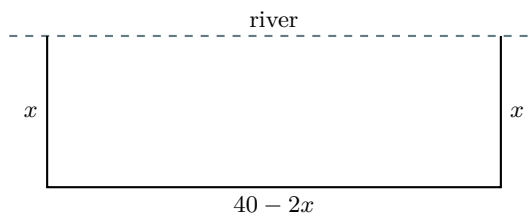
21. Let $f(x) = \ln x - \frac{1}{2}x$, for $x > 0$. Find its maximum point and explain why it has no point of inflexion.

22. The cubic $f(x) = x^3 + ax^2 + bx + 1$ has stationary points at $x = -1$ and $x = 2$. Find a and b and classify both points.

23. A closed cuboid has a square base of side x cm and volume 64 cm^3 .

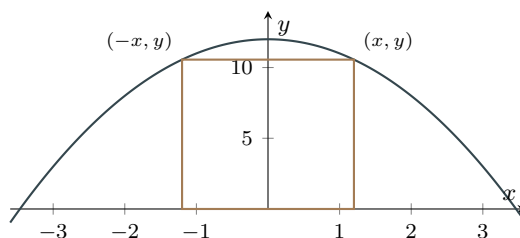
- Express its surface area in terms of x .
- Find the dimensions that minimize the surface area and justify your answer.

- 24.** A rectangular field borders a straight river. Only the other three sides require fencing. There are 40 m of fencing. Find the dimensions giving maximum area.



- 25.** An open box is made by cutting squares of side x cm from each corner of a 12 cm by 12 cm sheet and folding the sides up. Find the largest possible volume.

- 26.** A rectangle has vertices $(\pm x, 0)$ and $(\pm x, y)$ on the curve $y = 12 - x^2$, where $x > 0$ and $y > 0$. Find its largest area.



- 27.** A firm sells q units at price $p = 30 - q$ euros per unit. Its cost is $C(q) = 4q + 25$ euros. Treat q as continuous, with $0 \leq q \leq 30$. Find the output, price and profit at maximum profit.

- 28.** Find the points on the curve $y = x^2$ closest to $(0, 3)$. Explain why minimizing squared distance is valid.

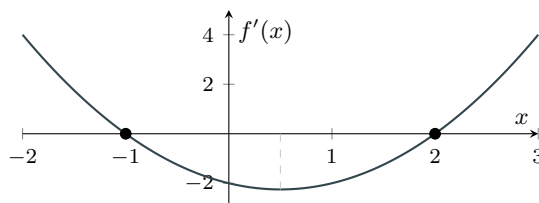
- 29.** For $f(x) = x^3 - 6x^2 + 9x + 2$ on $[0, 5]$, find the absolute maximum and minimum, and explain why finding local extrema alone is insufficient.

- 30.** The graph of f' is shown. Its zeros are -1 and 2 , and its vertex has $x = \frac{1}{2}$.

(a) Classify the stationary points of f .

(b) State where f is increasing.

(c) State where f is concave up or concave down, and give the x -coordinate of its point of inflexion.



3. IB exam-style practice — Paper 1

31. Let $f(x) = x^3 + 3x^2 - 9x + 4$.

- (a) Find $f'(x)$ and hence the coordinates of the stationary points. [4 marks]
- (b) Classify these points using the second derivative. [2 marks]
- (c) Find the point of inflexion and show that its tangent is not horizontal. [3 marks]
- (d) Find the equation of that tangent. [2 marks]

32. Consider $f(x) = x^4 - 4x^3$.

- (a) Find all stationary points. [3 marks]
- (b) Use the sign of f' to classify them. [3 marks]
- (c) Find both points of inflexion and state the intervals on which the graph is concave up. [4 marks]

33. The sign table for f' is shown. The function is differentiable for every real x .

x	$-\infty$	-2	3	$+\infty$		
$f'(x)$		$+$	0	$-$	0	$+$

- (a) State the intervals on which f decreases. [1 marks]
- (b) Classify the stationary points at $x = -2$ and $x = 3$, giving reasons. [2 marks]
- (c) Given $f'(x) = (x + 2)(x - 3)$, find the x -coordinate of the point of inflexion and justify it. [3 marks]

34. A rectangle has perimeter 48 cm. One side is x cm.

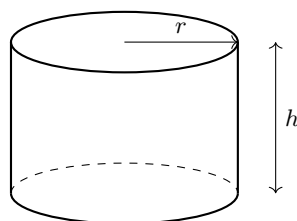
- (a) Show that its area is $A(x) = 24x - x^2$ and state the domain. [2 marks]
- (b) Find the dimensions giving maximum area. [3 marks]
- (c) Justify that the area is a maximum and state its value. [2 marks]

35. An open container has a square base of side x cm and height h cm. Its volume is 108 cm^3 .

- (a) Show that its material area is $S(x) = x^2 + 432/x$. [3 marks]
- (b) Find x and h when the material area is minimized. [3 marks]
- (c) Justify the minimum and find its value. [2 marks]

36. A closed cylindrical can has volume $54\pi \text{ cm}^3$, radius r cm and height h cm.

- (a) Show that its surface area is $S(r) = 2\pi r^2 + 108\pi/r$. [3 marks]
- (b) Find the radius and height giving the least surface area. [3 marks]
- (c) Prove that your result gives a minimum. [2 marks]

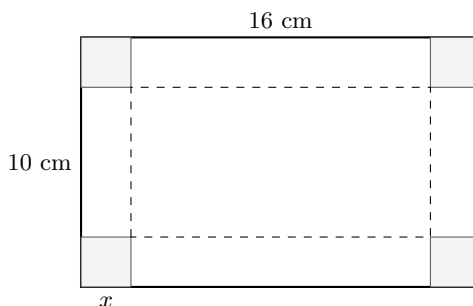


37. An open box is formed by removing a square of side x cm from each corner of a 16 cm by 10 cm card and folding up the sides.

(a) State the physical domain of x . [1 marks]

(b) Show that $V(x) = 4x^3 - 52x^2 + 160x$. [2 marks]

(c) Find the maximum volume, justifying your answer. [5 marks]



schematic, not to scale

38. A poster must contain a rectangular printed area of 100 cm^2 . A margin of 1 cm is required on all four sides. Let x cm be the width of the printed area.

(a) Express the total poster area in terms of x . [3 marks]

(b) Find the dimensions of the poster using the least paper. [3 marks]

(c) Justify that this is a minimum. [2 marks]

39. The average cost per item in a production model is $C(x) = x^2 + 54/x$, where $x > 0$ is a continuous production index.

(a) Find the stationary value of x . [3 marks]

(b) Show that it gives the absolute minimum cost. [2 marks]

(c) Find the minimum cost and explain why $x = 0$ is not a candidate. [2 marks]

40. Let $f(x) = xe^{-2x}$ for $x \geq 0$.

(a) Show that $f'(x) = e^{-2x}(1 - 2x)$. [2 marks]

(b) Find the maximum point. [3 marks]

(c) Find the point of inflexion and state the intervals of concavity. [4 marks]

41. Let $f(x) = \frac{\ln x}{x}$, for $x > 0$.

(a) Find $f'(x)$. [2 marks]

(b) Find and classify the stationary point. [3 marks]

(c) Given $f''(x) = \frac{2 \ln x - 3}{x^3}$, find the point of inflexion and justify it. [3 marks]

42. For $f(x) = \sin x + \cos x$ on $[0, 2\pi]$,

(a) find all stationary points and classify them; [4 marks]

(b) find the points of inflexion in the interior of the interval; [3 marks]

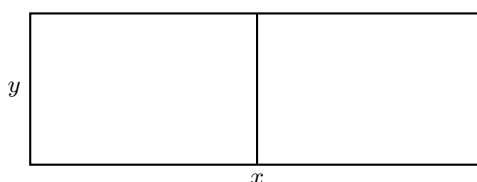
(c) state the absolute maximum and minimum values. [2 marks]

- 43.** Let $f(x) = x^3 + ax^2 + bx + c$. Its graph has a point of inflexion at $(1, 4)$ and a stationary point at $x = 0$.
- (a) Find a , b and c . [5 marks]
- (b) Find and classify both stationary points. [4 marks]

- 44.** The derivative of f is $f'(x) = x(x - 2)^2$.
- (a) Find all stationary x -values and classify them using the first derivative. [4 marks]
- (b) Find all x -values at which the concavity changes. [3 marks]
- (c) Identify which of these gives a horizontal point of inflexion. [1 marks]

- 45.** A rectangle in the first quadrant has vertices $(0, 0)$, $(x, 0)$, (x, y) and $(0, y)$, where (x, y) lies on $y = 8 - 2x$.
- (a) Express its area in terms of x and state the physical domain. [2 marks]
- (b) Find the maximum area and the dimensions that achieve it. [4 marks]
- (c) Find the gradient of the area function when $x = 1$ and interpret its sign. [2 marks]

- 46.** A fence of total length 60 m encloses two adjacent rectangular pens of equal width. The dividing fence is parallel to the sides of length y . Let x be the total width of both pens.
- (a) Write the fencing constraint and express the total area in terms of x . [3 marks]
- (b) Find the dimensions maximizing the total area. [3 marks]
- (c) Find the area of each pen. [2 marks]



- 47.** A theatre models the price of a ticket by $p = 40 - 0.1q$ euros, where q is the number of tickets sold. The cost is $C(q) = 10q + 500$ euros.
- (a) Form the profit function. [2 marks]
- (b) Treating q as continuous, find the number of tickets giving maximum profit and the corresponding price. [4 marks]
- (c) Find the maximum profit. [1 marks]

- 48.** The curve $y = \frac{x}{x^2 + 1}$ has derivative $y' = \frac{1 - x^2}{(x^2 + 1)^2}$.
- (a) Find and classify its stationary points. [4 marks]
- (b) Given $y'' = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$, find all points of inflexion. [4 marks]
- (c) State whether any point of inflexion has a horizontal tangent. [1 marks]

- 49.** Consider $f(x) = x^4 - 4x^2$ on $[-1, 3]$.
- (a) Find the stationary points in the interval and classify them. [4 marks]
- (b) Find the absolute maximum and minimum values on the interval. [3 marks]
- (c) Explain why a local maximum need not be the absolute maximum. [1 marks]

50. A smooth function is $f(x) = x^5 - 5x$.

- Find and classify the stationary points. [4 marks]
- Find the point of inflexion and state its gradient. [3 marks]
- A student claims every point of inflexion is stationary. Use your results to assess this claim. [1 marks]

4. AA HL exclusive challenge

For HL students only. These questions deepen SL 5.8 using more demanding algebra and linked reasoning.

51. Consider the family $f_a(x) = x^3 - 3ax$, where $a \in \mathbb{R}$.

- For $a > 0$, find and classify the stationary points in terms of a . [4 marks]
- Describe the stationary-point behaviour when $a = 0$ and when $a < 0$. [3 marks]
- Find the point of inflexion for every value of a and determine when it is horizontal. [3 marks]

52. Let $f(x) = x^4 - 2ax^2$, where $a > 0$.

- Find all stationary points and classify them. [4 marks]
- Find the points of inflexion in terms of a . [4 marks]
- The vertical difference between the local maximum and each local minimum is 9. Find a . [2 marks]

53. Let $f(x) = \frac{x}{(x+a)^2}$, where $x > 0$ and $a > 0$.

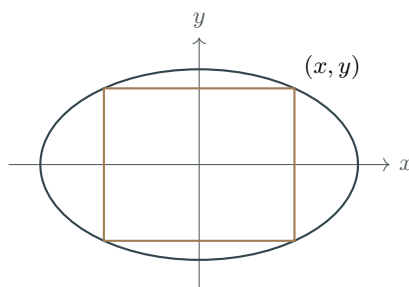
- Find the maximum point in terms of a . [4 marks]
- Show that $f''(x) = \frac{2(x-2a)}{(x+a)^4}$ and find the point of inflexion. [4 marks]
- The maximum value of f is $1/12$. Find a . [2 marks]

54. A closed cylinder has volume $32\pi \text{ cm}^3$. Material for its curved side costs $\text{€}c$ per cm^2 , while each circular end costs $\text{€}2c$ per cm^2 , with $c > 0$.

- Express the total material cost in terms of radius r . [3 marks]
- Find the dimensions minimizing the cost. [4 marks]
- Explain why the usual minimum-surface-area ratio $h = 2r$ does not apply. [2 marks]

55. A rectangle has vertices $(\pm x, \pm y)$ on the ellipse $x^2/25 + y^2/9 = 1$, with $x, y > 0$.

- Express the square of its area in terms of $u = x^2$. [3 marks]
- Hence find the maximum area and the rectangle dimensions. [4 marks]
- Justify both the maximum and the use of squared area. [2 marks]



56. Two positive real numbers satisfy $x + y = 12$. Find the largest value of x^2y and prove that it is the global maximum on the permitted domain. [7 marks]

57. Points on a parabola are parametrized by $P(t^2, 2t)$, where $t \in \mathbb{R}$. Let $A = (3, 0)$.

(a) Show that $AP^2 = t^4 - 2t^2 + 9$. [2 marks]

(b) Find both points of the parabola nearest to A . [4 marks]

(c) Find the shortest distance and justify that there are no nearer points. [3 marks]

58. Let $f(x) = x^3 - 3x^2 + kx + 1$, where $k \in \mathbb{R}$.

(a) Find the set of values of k for which there are two distinct stationary points. [3 marks]

(b) Find the set of values of k for which both stationary x -coordinates are positive. [3 marks]

(c) Classify the stationary point in the boundary case $k = 3$. [3 marks]

59. For $f(x) = x^3 - 3x$, the gradient of a chord joining $(-a, f(-a))$ to $(b, f(b))$ is zero, where $a, b > 0$.

(a) Show that $a^2 - ab + b^2 = 3$. [3 marks]

(b) Use this relation to express $s = a + b$ as a function of $d = a - b$. [2 marks]

(c) Find the largest possible value of s and the values of a, b that achieve it. [4 marks]

60. Let $f(x) = x^4 + ax^3 + bx^2 + cx + d$. The graph has a horizontal point of inflexion at $(1, 0)$ and a stationary point at $x = -2$.

(a) Find a, b, c, d . [6 marks]

(b) Classify the stationary point at $x = -2$. [2 marks]

(c) Find the other point of inflexion and its gradient. [4 marks]

Answer key and solution outlines

1. Short response

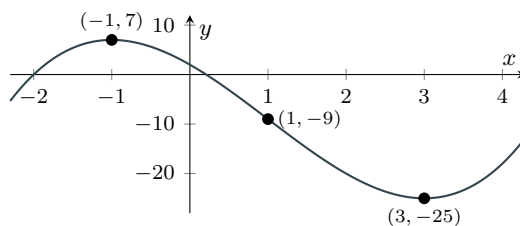
- $f'(x) = 2x - 6 = 0 \Rightarrow x = 3$. Since $f'' = 2 > 0$, $(3, -4)$ is a local minimum.
- $g' = -4x + 8 = 0 \Rightarrow x = 2$. $g'' = -4 < 0$; maximum at $(2, 9)$.
- Zeros 0, 4. Signs $+, -, +$ give a local maximum at $x = 0$ and a local minimum at $x = 4$.

x	$-\infty$	0	4	$+\infty$		
$f'(x)$		+	0	-	0	+
$f(x)$			Max		Min	

- The graph has a local maximum and is concave down at $x = 2$.
- $f' > 0$ on both sides of -1 , so the function increases on both sides and there is no sign change.
- $f'' = 6x$ changes from negative to positive at 0. The point is $(0, 0)$ and $f'(0) = 0$, so the tangent is horizontal.
- $f'' = 12x^2 > 0$ on both sides of 0: concavity does not change. It is a minimum, not an inflexion point.
- Concave down for $x < 2$; concave up for $x > 2$. The concavity changes at $x = 2$.
- $f' = 3(x - 1)(x - 3)$; $f'' = 6x - 12$. Maximum $(1, 5)$; minimum $(3, 1)$.
- $f'' = 6x$ changes sign at 0; inflexion $(0, 2)$ with gradient $f'(0) = -3$.
- f' stays positive on both sides, while $f'' = 6(x - 2)$ changes from negative to positive. Hence it is a horizontal point of inflexion.
- Check endpoints and stationary values: $f(-2) = -2$, $f(-1) = 2$, $f(1) = -2$, $f(3) = 18$. Maximum 18 at 3; minimum -2 at -2 and 1.
- The other side is $10 - x$; $A = x(10 - x)$, with $0 < x < 10$.
- $P' = -2q + 12 = 0 \Rightarrow q = 6$. $P'' = -2 < 0$; maximum profit $P(6) = 16$.
- No. For x^4 it is a minimum; for x^3 it is a horizontal inflexion. Both have first and second derivatives zero at 0.

2. Extended response

- $f' = 4x(x^2 - 4)$ and $f'' = 12x^2 - 16$. Minima $(\pm 2, -16)$; maximum $(0, 0)$. Inflexions $(\pm 2/\sqrt{3}, -80/9)$, since f'' changes sign. Concave up for $|x| > 2/\sqrt{3}$; down for $|x| < 2/\sqrt{3}$.
- $f' = 3(x + 1)(x - 3)$; $f'' = 6x - 6$. Maximum $(-1, 7)$, minimum $(3, -25)$; inflexion $(1, -9)$ with gradient -12 . Sketch: rising to the maximum, falling through the inflexion to the minimum, then rising. Concave down for $x < 1$, up for $x > 1$.



- Signs of f' are $-, +, +$ across $-2, 1$: minimum at -2 ; neither extremum at 1. Expanding gives $f' = x^3 - 3x + 2$, so $f'' = 3(x - 1)(x + 1)$. It changes sign at both -1 and 1 ; $f'(1) = 0$, so 1 is a horizontal inflexion. The other concavity change is at $x = -1$.

- 19.** $f' = 1 - 4/x^2 = 0 \Rightarrow x = 2$. It is negative for $0 < x < 2$, positive for $x > 2$, so $(2, 4)$ is the absolute minimum. $f'' = 8/x^3 > 0$: concave up throughout.
- 20.** $f' = e^{-x}(1 - x)$ changes $+$ $\rightarrow$ $-$ at 1, giving maximum $(1, e^{-1})$. $f'' = e^{-x}(x - 2)$ changes sign at 2, giving inflexion $(2, 2e^{-2})$. Concave down for $0 < x < 2$, up for $x > 2$.
- 21.** $f' = 1/x - 1/2 = 0$ at $x = 2$. $f'' = -1/x^2 < 0$ for all $x > 0$, so the maximum is $(2, \ln 2 - 1)$ and concavity never changes.
- 22.** $f' = 3(x + 1)(x - 2) = 3x^2 - 3x - 6$. Comparing with $3x^2 + 2ax + b$ gives $a = -3/2$, $b = -6$. Since $f'' = 6x - 3$, $x = -1$ gives a maximum and $x = 2$ a minimum.
- 23.** $h = 64/x^2$, so $S = 2x^2 + 4xh = 2x^2 + 256/x$. $S' = 4x - 256/x^2 = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$, $h = 4$. $S'' = 4 + 512/x^3 > 0$; minimum area 96 cm^2 .
- 24.** Let x be each perpendicular side; the parallel side is $40 - 2x$. $A = 40x - 2x^2$, $0 < x < 20$. $A' = 40 - 4x = 0$ at 10; $A'' = -4 < 0$. Dimensions 10 m by 20 m; area 200 m^2 .
- 25.** $V = x(12 - 2x)^2$, $0 < x < 6$. $V' = 12(x - 2)(x - 6)$; it changes from $+$ to $-$ at $x = 2$. Dimensions $8 \times 8 \times 2 \text{ cm}$; maximum volume 128 cm^3 .
- 26.** $A = 2x(12 - x^2) = 24x - 2x^3$, $0 < x < \sqrt{12}$. $A' = 24 - 6x^2 = 0 \Rightarrow x = 2$; $A'' = -12x < 0$. Height 8, width 4; maximum area 32 square units.
- 27.** Revenue $R = q(30 - q)$; profit $P = R - C = -q^2 + 26q - 25$. $P' = 26 - 2q = 0$ at $q = 13$; $P'' = -2 < 0$. Price €17; maximum profit €144.
- 28.** $D^2 = x^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$. Its derivative is $2x(2x^2 - 5)$; minima at $x = \pm\sqrt{5/2}$. Points $(\pm\sqrt{5/2}, 5/2)$; distance $\sqrt{11}/2$. The square root is increasing, so minimizing D^2 minimizes D .
- 29.** $f' = 3(x - 1)(x - 3)$. Candidate values $f(0) = 2$, $f(1) = 6$, $f(3) = 2$, $f(5) = 22$. Absolute maximum 22 at 5; minimum 2 at 0, 3. Endpoints can exceed the local extrema.
- 30.** f' changes $+$ $\rightarrow$ $-$ at -1 (maximum), then $- \rightarrow +$ at 2 (minimum). f increases for $x < -1$ and $x > 2$. The derivative graph decreases for $x < 1/2$ and increases for $x > 1/2$; hence f is concave down then up, with inflexion at $x = 1/2$.

3. IB exam-style practice

- 31.** (a) $f' = 3(x + 3)(x - 1)$; points $(-3, 31)$ and $(1, -1)$. (b) $f'' = 6x + 6$ is negative at -3 and positive at 1 : maximum then minimum. (c) f'' changes sign at -1 , giving $(-1, 15)$; $f'(-1) = -12 \neq 0$. (d) $y - 15 = -12(x + 1)$, so $y = -12x + 3$.
- 32.** (a) $f' = 4x^2(x - 3)$: $(0, 0)$, $(3, -27)$. (b) f' has signs $-$, $-$, $+$, so 0 is neither extremum; 3 is a minimum. (c) $f'' = 12x(x - 2)$ changes sign at 0, 2: inflexions $(0, 0)$ and $(2, -16)$. Concave up for $x < 0$ and $x > 2$.
- 33.** (a) $-2 < x < 3$. (b) Maximum at -2 ($+$ $\rightarrow$ $-$); minimum at 3 ($- \rightarrow +$). (c) $f'' = 2x - 1$ changes sign at $x = 1/2$, so this is the inflexion abscissa.

x	$-\infty$	-2	3	$+\infty$		
$f'(x)$		$+$	0	$-$	0	$+$
$f(x)$			Max	Min		

- 34.** (a) $2x + 2y = 48 \Rightarrow y = 24 - x$; $A = xy = x(24 - x)$, $0 < x < 24$. (b) $A' = 24 - 2x = 0 \Rightarrow x = 12$, $y = 12$. (c) $A'' = -2 < 0$; maximum 144 cm^2 .
- 35.** (a) $x^2h = 108 \Rightarrow h = 108/x^2$. Without a lid, $S = x^2 + 4xh = x^2 + 432/x$. (b) $S' = 2x - 432/x^2 = 0 \Rightarrow x^3 = 216$, so $x = 6$, $h = 3$. (c) $S'' = 2 + 864/x^3 > 0$; minimum $S = 108 \text{ cm}^2$.
- 36.** (a) $\pi r^2h = 54\pi \Rightarrow h = 54/r^2$; $S = 2\pi r^2 + 2\pi rh$. (b) $S' = 4\pi r - 108\pi/r^2 = 0 \Rightarrow r^3 = 27$. Thus $r = 3$, $h = 6 \text{ cm}$. (c) $S'' = 4\pi + 216\pi/r^3 > 0$ for $r > 0$; minimum $54\pi \text{ cm}^2$.
- 37.** (a) $0 < x < 5$. (b) $V = x(16 - 2x)(10 - 2x)$. (c) $V' = 12x^2 - 104x + 160 = 4(3x - 20)(x - 2)$. Only $x = 2$ lies in the domain; the derivative changes $+$ $\rightarrow$ $-$. Maximum $V = 2(12)(6) = 144 \text{ cm}^3$.

- 38.** (a) Printed height $100/x$; $A = (x+2)(100/x+2) = 104 + 2x + 200/x$. (b) $A' = 2 - 200/x^2 = 0 \Rightarrow x = 10$. Poster dimensions 12 cm by 12 cm. (c) $A'' = 400/x^3 > 0$.
- 39.** (a) $C' = 2x - 54/x^2 = 0 \Rightarrow x^3 = 27$, hence $x = 3$. (b) $C' < 0$ for $0 < x < 3$ and $C' > 0$ for $x > 3$; absolute minimum. (c) $C(3) = 27$; the model is undefined at 0.
- 40.** (a) Product rule: $f' = e^{-2x} - 2xe^{-2x}$. (b) The derivative changes $+$ $\rightarrow$ $-$ at $x = 1/2$; maximum $(1/2, 1/(2e))$. (c) $f'' = 4e^{-2x}(x-1)$ changes sign at 1; inflexion $(1, e^{-2})$. Concave down for $0 < x < 1$, up for $x > 1$.
- 41.** (a) Quotient rule gives $f' = (1 - \ln x)/x^2$. (b) $x = e$; sign changes $+$ $\rightarrow$ $-$, so maximum $(e, 1/e)$. (c) $2 \ln x - 3 = 0 \Rightarrow x = e^{3/2}$. The numerator increases through zero and $x^3 > 0$, so concavity changes; point $(e^{3/2}, 3/(2e^{3/2}))$.
- 42.** (a) $f' = \cos x - \sin x = 0$ at $\pi/4, 5\pi/4$. Since $f'' = -\sin x - \cos x$, maximum $(\pi/4, \sqrt{2})$, minimum $(5\pi/4, -\sqrt{2})$. (b) $f'' = 0$ and changes sign at $3\pi/4, 7\pi/4$, giving $(3\pi/4, 0), (7\pi/4, 0)$. (c) Endpoint values are 1; absolute maximum $\sqrt{2}$, minimum $-\sqrt{2}$.
- 43.** (a) $f''(1) = 6 + 2a = 0 \Rightarrow a = -3$. $f'(0) = b = 0$. $f(1) = 1 - 3 + c = 4 \Rightarrow c = 6$. (b) $f' = 3x(x-2)$, $f'' = 6x - 6$; maximum $(0, 6)$, minimum $(2, 2)$.
- 44.** (a) Zeros 0, 2; signs $-$, $+$, $+$, so minimum at 0, neither extremum at 2. (b) $f'' = (x-2)(3x-2)$ changes sign at $2/3, 2$. (c) $x = 2$, since $f'(2) = 0$; $f'(2/3) \neq 0$.
- 45.** (a) $A = x(8-2x)$, $0 < x < 4$. (b) $A' = 8 - 4x = 0 \Rightarrow x = 2$, $y = 4$; $A'' = -4 < 0$; maximum 8 square units. (c) $A'(1) = 4 > 0$: a small increase in x increases the area.
- 46.** (a) $2x + 3y = 60$, so $y = (60 - 2x)/3$; $A = 20x - \frac{2}{3}x^2$. (b) $A' = 20 - \frac{4}{3}x = 0 \Rightarrow x = 15$, $y = 10$. $A'' = -4/3 < 0$. (c) Total 150 m^2 ; each pen 75 m^2 .
- 47.** (a) $P = q(40 - 0.1q) - (10q + 500) = -0.1q^2 + 30q - 500$. (b) $P' = 30 - 0.2q = 0 \Rightarrow q = 150$. $P'' = -0.2 < 0$; price €25. (c) Profit $P(150) = €1750$.
- 48.** (a) Denominator positive; signs $-$, $+$, $-$ across $-1, 1$. Minimum $(-1, -1/2)$; maximum $(1, 1/2)$. (b) y'' has signs $-$, $+$, $-$, $+$ across $-\sqrt{3}, 0, \sqrt{3}$. Inflexions $(-\sqrt{3}, -\sqrt{3}/4), (0, 0), (\sqrt{3}, \sqrt{3}/4)$. (c) None: the gradients are $-1/8, 1, -1/8$.
- 49.** (a) $f' = 4x(x^2 - 2)$; within the interval, $x = 0$ gives a local maximum $(0, 0)$ and $x = \sqrt{2}$ a minimum $(\sqrt{2}, -4)$. (b) Endpoint values $f(-1) = -3$, $f(3) = 45$. Absolute maximum 45 at 3; minimum -4 at $\sqrt{2}$. (c) The endpoint value 45 exceeds the local maximum value 0.
- 50.** (a) $f' = 5(x^4 - 1) = 5(x^2 - 1)(x^2 + 1)$; signs $+$, $-$, $+$. Maximum $(-1, 4)$, minimum $(1, -4)$. (b) $f'' = 20x^3$ changes sign at 0; inflexion $(0, 0)$ with gradient -5 . (c) False: this inflexion has a nonzero gradient.

4. AA HL exclusive challenge

- 51.** $f'_a = 3(x^2 - a)$, $f''_a = 6x$. For $a > 0$: maximum $(-\sqrt{a}, 2a^{3/2})$ and minimum $(\sqrt{a}, -2a^{3/2})$. For $a = 0$: a horizontal inflexion at $(0, 0)$, no extremum. For $a < 0$: $f'_a > 0$ everywhere, no stationary point. For all a , concavity changes at $(0, 0)$; its gradient is $-3a$, horizontal only for $a = 0$.
- 52.** $f' = 4x(x^2 - a)$, $f'' = 12x^2 - 4a$. Maximum $(0, 0)$; minima $(\pm\sqrt{a}, -a^2)$. Inflexions at $x = \pm\sqrt{a/3}$, where $y = -5a^2/9$; f'' changes sign at both. Vertical difference $a^2 = 9$; since $a > 0$, $a = 3$.
- 53.** $f' = (a - x)/(x + a)^3$ changes $+$ $\rightarrow$ $-$ at $x = a$; maximum $(a, 1/(4a))$. Differentiating f' gives $[-(x + a) - 3(a - x)]/(x + a)^4 = 2(x - 2a)/(x + a)^4$. Thus inflexion $(2a, 2/(9a))$, with down-to-up concavity change. $1/(4a) = 1/12 \Rightarrow a = 3$.
- 54.** $h = 32/r^2$; cost $C = 2c\pi rh + 4c\pi r^2 = 64c\pi/r + 4c\pi r^2$. $C' = -64c\pi/r^2 + 8c\pi r = 0 \Rightarrow r = 2$, $h = 8 \text{ cm}$. $C'' = 128c\pi/r^3 + 8c\pi > 0$. Here the ends cost twice as much per unit area, so the optimum is taller: $h = 4r$.
- 55.** $A = 4xy$, $y^2 = 9(1 - x^2/25)$; $A^2 = 144u - \frac{144}{25}u^2$, $0 < u < 25$. Derivative $144 - \frac{288}{25}u = 0$ at $u = 25/2$; second derivative $-288/25 < 0$. Thus $x = 5/\sqrt{2}$, $y = 3/\sqrt{2}$; dimensions $5\sqrt{2}$ by $3\sqrt{2}$, area 30. Since $A > 0$, maximizing A^2 also maximizes A .
- 56.** Let $F = x^2(12 - x) = 12x^2 - x^3$, $0 < x < 12$. $F' = 3x(8 - x)$, positive for $0 < x < 8$, negative for $8 < x < 12$. Hence the unique global maximum occurs at $x = 8$, $y = 4$, and equals 256. The boundary limits are both zero.

57. $AP^2 = (t^2 - 3)^2 + (2t)^2 = t^4 - 2t^2 + 9$. Derivative $4t(t^2 - 1)$ gives $t = -1, 0, 1$. Second derivative $12t^2 - 4$ gives minima at $t = \pm 1$, a maximum at 0. Nearest points $(1, \pm 2)$; distance $\sqrt{8} = 2\sqrt{2}$. Also $AP^2 = (t^2 - 1)^2 + 8 \geq 8$, proving global minimality.

58. $f' = 3x^2 - 6x + k$. Two distinct roots require $\Delta = 36 - 12k > 0$, hence $k < 3$. Their sum is $2 > 0$ and product $k/3$, so both are positive exactly when $0 < k < 3$. For $k = 3$, $f' = 3(x - 1)^2$ and $f'' = 6(x - 1)$ changes sign: horizontal inflexion at $(1, 2)$.

59. Equal heights imply $a^3 + b^3 = 3(a + b)$. Divide by $a + b > 0$ to obtain $a^2 - ab + b^2 = 3$. With $s = a + b$, $d = a - b$, this becomes $(s^2 + 3d^2)/4 = 3$, so $s = \sqrt{12 - 3d^2}$; positivity of a, b gives $|d| < \sqrt{3}$. Then $ds/dd = -3d/\sqrt{12 - 3d^2}$, positive for $d < 0$ and negative for $d > 0$. Maximum $s = 2\sqrt{3}$ at $d = 0$, hence $a = b = \sqrt{3}$.

60. A horizontal inflexion at 1 requires $f'(1) = f''(1) = 0$, so 1 is a repeated root of f' . Thus $f' = 4(x - 1)^2(x + 2) = 4x^3 - 12x + 8$. Comparing coefficients gives $a = 0$, $b = -6$, $c = 8$. From $f(1) = 0$, $d = -3$. At -2 , $f'' = 12x^2 - 12 = 36 > 0$; minimum $(-2, -27)$. The other zero of f'' is -1 , with a concavity change; point $(-1, -16)$ and gradient $f'(-1) = 16$.