Understanding quadratics

IB Mathematics: Analysis and Approaches SL

Two worked examples ▪ Ten practice questions ▪ Detailed solutions

Learning goal	Turn a clue into an equation, choose a useful quadratic form, and explain what each answer means.
Syllabus link	AA SL 2.6 and 2.7: quadratic functions, equations and inequalities.
Structure	Method guide; the two photographed examples; 10 questions; worked solutions.
Calculator	Use a calculator to check your work. In Question 9(c), use it to approximate the boundary values. Always show your algebra.

How to use this worksheet. Read the examples, then write directly in the spaces provided. Give exact values unless an approximation is requested. Include units in context questions. Try each question before checking its solution.

Start here: what does the information tell me?

Information in the question	What to write or do
A point (p, q) lies on the graph.	Substitute both coordinates: $f(p) = q$.
An x-intercept is $(p, 0)$.	Substitute $x = p$ and set the output to zero: $f(p) = 0$.
The axis of symmetry is $x = h$.	For $f(x) = ax^2 + bx + c$, write $-\dfrac{b}{2a} = h$.
Two distinct points have the same height.	The axis is halfway between their x-coordinates: $x = \dfrac{x_1 + x_2}{2}$.
The function is $a(x - h)^2 + k$.	This is **vertex form**: vertex (h, k), axis $x = h$.
A quantity "does not exceed" B.	Write quantity $\leq B$. The answer may be an interval.
The graph meets the ground.	Set the height equal to zero. Keep only roots that fit the situation.

Keep these ideas separate. The axis is a line, $x = h$. The vertex is a point, (h, k). If $a > 0$, the minimum value is k and it occurs at $x = h$. If $a < 0$, the maximum value is k and it occurs at $x = h$.

The equal-height clue in your photo. The points $(0, 5)$ and $(-4, 5)$ have the same height. Their midpoint in the horizontal direction gives

$$x = \frac{0 + (-4)}{2} = \boxed{-2}.$$

So the axis is $x = -2$. This does **not** tell us that the vertex is $(-2, 5)$: we still do not know the vertex's height.

Syllabus reference: *Mathematics: analysis and approaches guide*, sections SL 2.6–2.7, printed p. 39. The two worked examples follow the supplied photographs. All ten practice questions are original.

Worked example A: the symmetry-and-coefficients question

The quadratic $f(x) = ax^2 + bx + 22$ has axis of symmetry $x = 1.75$.

(a) Using only the axis, write an equation in a and b.

(b) The graph meets the x-axis at $(-2, 0)$. Write a second equation in a and b.

(c) Hence find a and b.

(d) Find the x-coordinate of the other x-intercept.

Solution

(a) Turn the axis into a relationship between a and b.

For a quadratic in standard form, the axis is $x = -\dfrac{b}{2a}$. Therefore,

$$\boxed{-\frac{b}{2a} = 1.75} \implies -b = 1.75(2a) = 3.5a \implies \boxed{b = -3.5a}.$$

The first boxed equation is already a correct answer. We rearrange it so that we can substitute for b later. One equation does not yet give both coefficients.

(b) Turn the point $(-2, 0)$ into a second equation.

Every point on the x-axis has $y = 0$, so this point tells us that $f(-2) = 0$:

$$a(-2)^2 + b(-2) + 22 = 0 \implies \boxed{4a - 2b + 22 = 0}.$$

Notice that $(-2)^2 = 4$, while $b(-2) = -2b$.

(c) Use the two equations together.

Replace b by $-3.5a$ in the second equation:

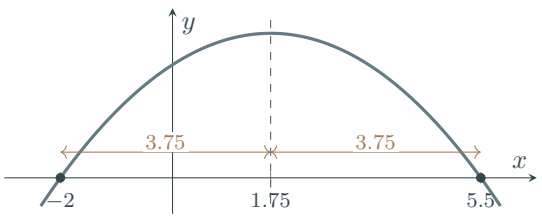
$$\begin{aligned} 4a - 2(-3.5a) + 22 &= 0 \\ 4a + 7a + 22 &= 0 \\ 11a &= -22 \\ a &= \boxed{-2}. \end{aligned} \qquad \begin{aligned} b &= -3.5a \\ &= -3.5(-2) \\ &= \boxed{7}. \end{aligned}$$

Thus $f(x) = -2x^2 + 7x + 22$.

(d) Reflect the known root across the axis.

The root -2 is $1.75 - (-2) = 3.75$ units to the left of the axis. The other root is the same distance to the right:

$$x = 1.75 + 3.75 = \boxed{5.5}.$$



Check both original clues.

$$-\frac{7}{2(-2)} = 1.75,$$

$$f(-2) = -8 - 14 + 22 = 0.$$

Both agree with the question.
Keep the numbers separate: 3.5 appears in $b = -3.5a$; 3.75 is the distance from each root to the axis.

Worked example B: the shirt-production cost question

The cost, in Fijian dollars (FJD), of producing x shirts is modelled by

$$C(x) = (x - 75)^2 + 100.$$

(a) Find the cost of producing 70 shirts.

(b) The cost should not exceed 500 FJD. The factory must produce at least 55 shirts and at most s shirts. Find s.

(c) Find the number of shirts produced when the cost is lowest.

Solution

(a) A given number of shirts is an input: substitute $x = 70$.

$$C(70) = (70 - 75)^2 + 100 = (-5)^2 + 100 = 25 + 100 = \boxed{125 \text{ FJD}}.$$

The answer is a cost, so we give it in FJD. Remember that $(-5)^2 = 25$.

(b) A budget is a limit: translate it into an inequality.

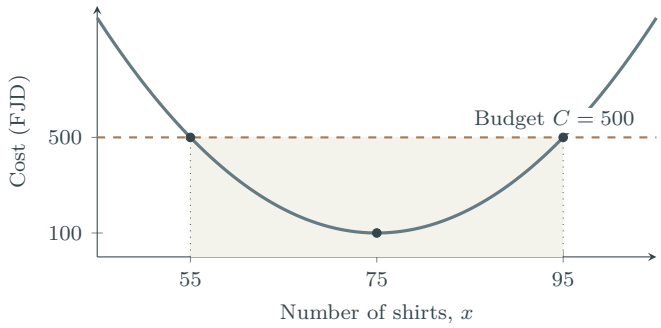
"Should not exceed 500" means "less than or equal to 500":

$$\begin{aligned} C(x) &\leq 500 \\ (x - 75)^2 + 100 &\leq 500 \\ (x - 75)^2 &\leq 400. \end{aligned}$$

A number has a square of at most 400 when it lies between -20 and 20. So

$$-20 \leq x - 75 \leq 20 \quad \Longrightarrow \quad \boxed{55 \leq x \leq 95}.$$

The allowed quantities are **all whole numbers from 55 to 95, inclusive**. The greatest is 95, so $\boxed{s = 95}$.



Why an interval?
Solving $C(x) = 500$ gives the two boundary values: 55 and 95. Between them, the curve lies below the budget line.
The endpoints are included because a cost of exactly 500 is allowed.
Symmetry also checks the answer. The axis is $x = 75$. Since 55 is 20 below 75, the other boundary is $75 + 20 = 95$.
The curve helps us see the interval; the number of shirts must be a whole number.

(c) A square cannot be negative: use the vertex form.

The smallest value of $(x - 75)^2$ is 0. This happens when $x = 75$, giving

$$C(75) = 0 + 100 = 100 \text{ FJD}.$$

The question asks for the **number of shirts**, so the answer is $\boxed{75 \text{ shirts}}$. The minimum *cost* is 100 FJD. There is no need to expand the square.

1. Symmetry and unknown coefficients

Question 1

Start with a known quadratic

Let

$$f(x) = 2x^2 - 12x + 10.$$

Helpful starting point: compare the expression with $ax^2 + bx + c$. Keep the sign of each coefficient.

(a) State a, b and c. Hence write down the coordinates of the y-intercept.

(b) Substitute into $x = -\dfrac{b}{2a}$ to find the equation of the axis of symmetry.

(c) Calculate $f(1)$ to verify that $(1, 0)$ lies on the graph.

(d) Use symmetry to find the coordinates of the other x-intercept. Explain your reasoning.

(e) Find the coordinates of the vertex. State, with a reason, whether it is a minimum or a maximum.

1. Symmetry and unknown coefficients

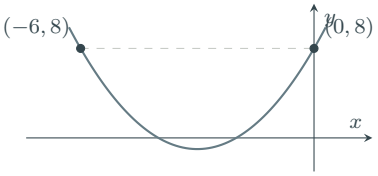
Question 2

Equal heights give a midpoint

The points $(-6, 8)$ and $(0, 8)$ lie on the graph of

$$f(x) = x^2 + bx + c.$$

Think first: these points have the same y-coordinate. How does a parabola's symmetry help? The sketch shows the given points.



(a) Explain why the axis lies halfway between the two given x-coordinates. Find its equation.

(b) Use $(0, 8)$ to find c.

(c) Use the axis formula to find b.

(d) Find the coordinates of the vertex.

(e) Check your function by calculating $f(-6)$.

1. Symmetry and unknown coefficients

Question 3

A guided coefficient question

The quadratic

$$f(x) = ax^2 + bx + 16$$

has axis of symmetry $x = 1$ and meets the x-axis at $(-2, 0)$.

Use worked example A if needed. Turn each clue into an equation before you try to solve for the coefficients.

(a) Use the axis of symmetry to show that $b = -2a$.

(b) Use $(-2, 0)$ to write a second equation in a and b. Remember that an x-intercept has $y = 0$.

(c) Solve your two equations to find a and b. Show the substitution clearly.

(d) Use symmetry to find the coordinates of the other x-intercept.

(e) Find the maximum value of f. State the value of x at which it occurs.

1. Symmetry and unknown coefficients

Question 4

The same method, less guidance

The graph of

$$f(x) = ax^2 + bx - 12$$

has axis of symmetry $x = 2$ and passes through $(-2, 0)$.

(a) Write down two equations in a and b using the information given.

(b) Find a and b.

(c) Find the coordinates of the other x-intercept.

(d) Find the vertex. State, with a reason, whether it is a minimum or a maximum.

1. Symmetry and unknown coefficients

Question 5

An axis with a decimal value

Let

$$f(x) = ax^2 + bx + 8.$$

Its axis of symmetry is $x = 1.5$, and $(-1, 0)$ is an x-intercept.

(a) Find a and b, showing the two equations you use and how you solve them.

(b) Find the coordinates of the other x-intercept without solving a quadratic equation.

(c) Find the maximum value of f and state where it occurs.

1. Symmetry and unknown coefficients

Question 6

Combine three pieces of information

The graph of

$$f(x) = ax^2 + bx + c$$

passes through $(0, 6)$, $(4, 6)$ and $(1, 12)$.

(a) Use the two points at the same height to find the equation of the axis of symmetry.

(b) Find c.

(c) Use the axis of symmetry to write an equation connecting a and b.

(d) Use $(1, 12)$ and your earlier answers to find a and b.

(e) Find the maximum value of f and state where it occurs.

2. Vertex form and cost models

Question 7

Understand vertex form

The graph of

$$g(x) = a(x-3)^2 + 4$$

passes through $(1, 12)$.

(a) Write down the coordinates of the vertex and the equation of the axis of symmetry.

(b) Use the given point to find a.

(c) Find the minimum value of g and state the value of x at which it occurs. Explain why this is a minimum.

(d) Solve $g(x) = 22$. Keep the expression in vertex form and include both square-root signs.

(e) Explain, using symmetry, why part (d) has two answers.

2. Vertex form and cost models

Question 8

A guided budget question

A workshop models its daily cost, in euros, by

$$C(n) = \frac{1}{2}(n - 40)^2 + 80,$$

where n is the whole number of items produced and $20 \leq n \leq 60$.

(a) Find the cost of producing 34 items.

(b) Find the minimum cost and the number of items that gives this cost.

(c) Solve $C(n) = 130$. These are the *boundary values* for the next part.

(d) Hence find all possible numbers of items if the cost must not exceed 130 euros.

(e) Explain why giving only the two values from part (c) would not fully answer part (d).

2. Vertex form and cost models

Question 9

Independent practice [8 marks]

The cost, in euros, of producing n items is modelled by

$$C(n) = 2(n - 30)^2 + 50,$$

where n is a whole number and $10 \leq n \leq 50$.

(a) Find $C(26)$ and interpret your answer. [2 marks]

(b) State the minimum cost and the number of items for which it occurs. [2 marks]

(c) The cost must not exceed 180 euros. Find the least and greatest possible numbers of items. Show your algebra, give the real-valued boundary points to three decimal places, and explain how the whole-number condition affects your answer. [4 marks]

3. A quadratic model in context

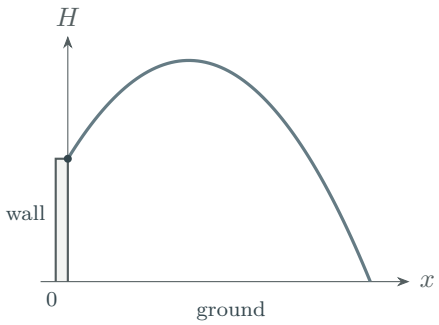
Question 10

Independent practice [10 marks]

A ball is thrown from the top of a wall. Its height above the ground, in metres, is modelled by

$$H(x) = -\frac{1}{2}(x-2)^2 + \frac{9}{2},$$

where x is the horizontal distance in metres from the base of the wall, measured in the direction of the throw. The model applies from release until the ball first hits the ground.



(a) Find the height of the wall. [2 marks]

(b) Find the maximum height of the ball and the horizontal distance from the wall at which it occurs. [2 marks]

(c) Find the horizontal distance from the base of the wall to where the ball hits the ground. Explain why one algebraic solution must be rejected. [3 marks]

(d) Find the horizontal distances for which the ball is at least 4 metres above the ground. [3 marks]

Detailed worked solutions: Questions 1–2

Use the reasoning, not only the final answer. If your answer is different, compare the equation you formed from each clue. Then check the algebra and the interpretation.

1. Symmetry and unknown coefficients

Question 1. Start with a known quadratic

(a) Comparing $2x^2 - 12x + 10$ with $ax^2 + bx + c$ gives

$$\boxed{a = 2, \quad b = -12, \quad c = 10}.$$

At the y-axis, $x = 0$, so $f(0) = 10$. The y-intercept is $\boxed{(0, 10)}$.

(b) Use the coefficient $b = -12$, including its sign:

$$x = -\frac{-12}{2(2)} = \frac{12}{4} = 3.$$

The axis of symmetry is the line $\boxed{x = 3}$.

(c) $f(1) = 2(1)^2 - 12(1) + 10 = 2 - 12 + 10 = \boxed{0}$, so $(1, 0)$ lies on the graph.

(d) The root 1 is $3 - 1 = 2$ units left of the axis. Reflecting it gives $3 + 2 = 5$, so the other intercept is $\boxed{(5, 0)}$.

(e) The vertex is on $x = 3$. Its height is

$$f(3) = 2(3)^2 - 12(3) + 10 = 18 - 36 + 10 = -8.$$

The vertex is $\boxed{(3, -8)}$. Since $a = 2 > 0$, the parabola opens upwards and this is a **minimum**.

Question 2. Equal heights give a midpoint

(a) The points have the same height, so they are equally far from the axis:

$$x = \frac{-6 + 0}{2} = \boxed{-3}.$$

(b) The point $(0, 8)$ gives $f(0) = 8$, so $\boxed{c = 8}$.

(c) Here $a = 1$. Use the axis $x = -3$:

$$-\frac{b}{2(1)} = -3 \quad \Longrightarrow \quad -b = -6 \quad \Longrightarrow \quad \boxed{b = 6}.$$

(d) $f(-3) = (-3)^2 + 6(-3) + 8 = 9 - 18 + 8 = -1$, so the vertex is $\boxed{(-3, -1)}$.

(e) $f(-6) = (-6)^2 + 6(-6) + 8 = 36 - 36 + 8 = \boxed{8}$, as required.

Detailed worked solutions: Questions 3–4

1. Symmetry and unknown coefficients

Question 3. A guided coefficient question

(a) $-\dfrac{b}{2a} = 1 \Rightarrow -b = 2a \Rightarrow \boxed{b = -2a}$.

(b) An x-intercept has output 0. Substitute $x = -2$:

$$a(-2)^2 + b(-2) + 16 = 0 \quad \Longrightarrow \quad \boxed{4a - 2b + 16 = 0}.$$

(c) Substitute $b = -2a$:

$$4a - 2(-2a) + 16 = 0 \Rightarrow 8a = -16 \Rightarrow \boxed{a = -2}, \qquad b = -2(-2) = \boxed{4}.$$

(d) The axis $x = 1$ is halfway between the roots. The known root -2 is 3 units left of it, so the other root is $1 + 3 = 4$. The intercept is $\boxed{(4, 0)}$.

(e) Since $a < 0$, the vertex gives a maximum. At $x = 1$,

$$f(1) = -2(1)^2 + 4(1) + 16 = \boxed{18}.$$

The maximum value is 18, and it occurs at $\boxed{x = 1}$.

Question 4. The same method, less guidance

(a) The axis and the point give two different equations:

$$-\frac{b}{2a} = 2 \Rightarrow \boxed{b = -4a}, \qquad f(-2) = 0 \Rightarrow \boxed{4a - 2b - 12 = 0}.$$

(b) Replace b in the second equation:

$$\begin{aligned} 4a - 2(-4a) - 12 &= 0 \\ 12a - 12 &= 0 \\ a = \boxed{1}, \qquad b &= -4(1) = \boxed{-4}. \end{aligned}$$

Thus $f(x) = x^2 - 4x - 12$.

(c) The root -2 is $2 - (-2) = 4$ units left of the axis $x = 2$. The reflected root is $2 + 4 = 6$, so the other intercept is $\boxed{(6, 0)}$.

(d) $f(2) = 2^2 - 4(2) - 12 = 4 - 8 - 12 = -16$. The vertex is $\boxed{(2, -16)}$. Since $a = 1 > 0$, it is a **minimum**.

Detailed worked solutions: Questions 5–6

1. Symmetry and unknown coefficients

Question 5. An axis with a decimal value

(a) The two clues give

$$-\frac{b}{2a} = 1.5 \Rightarrow \boxed{b = -3a}, \qquad f(-1) = 0 \Rightarrow \boxed{a - b + 8 = 0}.$$

Substitute $b = -3a$ into the second equation:

$$a - (-3a) + 8 = 0 \Rightarrow 4a = -8 \Rightarrow \boxed{a = -2}, \qquad b = -3(-2) = \boxed{6}.$$

Notice that $1.5 \times 2 = 3$ when rearranging the axis formula.

(b) The distance from -1 to the axis is $1.5 - (-1) = 2.5$. The reflected root is $1.5 + 2.5 = 4$, so the other intercept is $\boxed{(4, 0)}$.

(c) Since $a = -2 < 0$, the vertex gives the maximum:

$$f(1.5) = -2(1.5)^2 + 6(1.5) + 8 = -4.5 + 9 + 8 = \boxed{\frac{25}{2}}.$$

The maximum value is $\frac{25}{2} = 12.5$ and it occurs at $\boxed{x = 1.5}$.

Question 6. Combine three pieces of information

(a) The points $(0, 6)$ and $(4, 6)$ have equal heights, so the axis is

$$x = \frac{0 + 4}{2} = \boxed{2}.$$

(b) The point $(0, 6)$ lies on the y-axis, so $\boxed{c = 6}$.

(c) Use the axis: $-\frac{b}{2a} = 2 \Rightarrow \boxed{b = -4a}$.

(d) The point $(1, 12)$ gives $f(1) = 12$:

$$a + b + 6 = 12 \Rightarrow a + b = 6.$$

Using $b = -4a$ gives

$$a - 4a = 6 \Rightarrow -3a = 6 \Rightarrow \boxed{a = -2}, \qquad b = -4(-2) = \boxed{8}.$$

(e) The function is $f(x) = -2x^2 + 8x + 6$. It opens downwards, so its maximum is at the axis $x = 2$:

$$f(2) = -2(4) + 8(2) + 6 = -8 + 16 + 6 = \boxed{14}.$$

The maximum value is 14 and it occurs at $\boxed{x = 2}$.

Detailed worked solutions: Questions 7–8

2. Vertex form and cost models

Question 7. Understand vertex form

(a) Compare $a(x-3)^2+4$ with $a(x-h)^2+k$. The vertex is $\boxed{(3,4)}$ and the axis is $\boxed{x=3}$.

(b) A point on the graph satisfies the equation. Using $(1,12)$:

$$\begin{aligned} 12 &= a(1-3)^2+4 \\ 12 &= 4a+4 \\ 8 &= 4a \\ a &= \boxed{2}. \end{aligned}$$

(c) Now $g(x)=2(x-3)^2+4$. Since $(x-3)^2 \geq 0$ and its coefficient is positive, $g(x) \geq 4$. The square is zero when $x=3$. Hence the minimum value is $\boxed{4}$ and it occurs at $\boxed{x=3}$.

(d) Set the output equal to 22:

$$\begin{aligned} 2(x-3)^2+4 &= 22 \\ 2(x-3)^2 &= 18 \\ (x-3)^2 &= 9 \\ x-3 &= \pm 3 \\ x &= \boxed{0 \text{ or } 6}. \end{aligned}$$

(e) The line $y=22$ is above the minimum. It meets both sides of the parabola. The two answers 0 and 6 are each 3 units from the axis $x=3$.

Question 8. A guided budget question

(a) Substitute $n=34$:

$$C(34) = \frac{1}{2}(34-40)^2+80 = \frac{1}{2}(36)+80 = \boxed{98}.$$

Producing 34 items costs **98 euros** according to the model.

(b) The squared term is smallest when $n-40=0$, so the minimum cost is **80 euros**, at $\boxed{n=40}$ items.

(c) Find the quantities that cost exactly 130 euros:

$$\begin{aligned} \frac{1}{2}(n-40)^2+80 &= 130 \\ \frac{1}{2}(n-40)^2 &= 50 \\ (n-40)^2 &= 100 \\ n-40 &= \pm 10 \\ n &= \boxed{30 \text{ or } 50}. \end{aligned}$$

(d) "Must not exceed" means $C(n) \leq 130$. The curve opens upwards, so it is at or below the budget between these boundaries:

$$\boxed{30 \leq n \leq 50, \quad n \in \mathbb{Z}}.$$

That is, **any whole number from 30 to 50, inclusive**. These numbers also satisfy the model's domain $20 \leq n \leq 60$.

(e) The two values in part (c) give costs *equal* to 130 euros. Whole numbers between them give lower costs, which are allowed too. For example, $C(40)=80<130$.

Detailed worked solutions: Questions 9–10

2–3. Independent applications

Question 9. Independent budget question

(a) $C(26) = 2(26-30)^2 + 50 = 2(16) + 50 = \boxed{82}$. Producing 26 items costs **82 euros** according to the model.

(b) The square is zero at $n = 30$. The minimum cost is therefore **50 euros**, when $\boxed{30 \text{ items}}$ are produced.

(c) Translate the budget into an inequality and keep the square intact:

$$\begin{aligned} 2(n-30)^2 + 50 &\leq 180 \\ (n-30)^2 &\leq 65 \\ -\sqrt{65} \leq n - 30 &\leq \sqrt{65} \\ 30 - \sqrt{65} \leq n &\leq 30 + \sqrt{65}. \end{aligned}$$

The boundary values are 21.938 and 38.062 to three decimal places. The whole numbers inside this interval are

$$\boxed{22 \leq n \leq 38, \quad n \in \mathbb{Z}}.$$

So the least possible quantity is **22 items** and the greatest is **38 items**. We choose integers *inside* the interval: round the lower boundary upwards and the upper boundary downwards. The model's domain is also satisfied.

Question 10. The ball and the wall

(a) At release, the horizontal distance is $x = 0$. Therefore,

$$H(0) = -\frac{1}{2}(0-2)^2 + \frac{9}{2} = -2 + \frac{9}{2} = \boxed{\frac{5}{2} \text{ m}}.$$

The wall is 2.5 metres high. We substitute $x = 0$; we do not set the height to zero.

(b) The negative coefficient means that the vertex is a maximum. From vertex form, the maximum height is $\boxed{\frac{9}{2} \text{ m}}$ (4.5 m), at a horizontal distance of $\boxed{2 \text{ m}}$ from the wall.

(c) The ball reaches the ground when its height is zero:

$$-\frac{1}{2}(x-2)^2 + \frac{9}{2} = 0 \Rightarrow (x-2)^2 = 9 \Rightarrow x - 2 = \pm 3 \Rightarrow x = -1 \text{ or } 5.$$

The model starts at release, where $x = 0$, and follows the forward motion. The root $x = -1$ is outside this domain. The ball hits the ground $\boxed{5 \text{ m}}$ from the base of the wall. The physical domain is $0 \leq x \leq 5$.

(d) "At least 4 metres" means $H(x) \geq 4$:

$$-\frac{1}{2}(x-2)^2 + \frac{9}{2} \geq 4 \Rightarrow -\frac{1}{2}(x-2)^2 \geq -\frac{1}{2} \Rightarrow (x-2)^2 \leq 1.$$

The inequality reverses when we multiply by -2. Therefore,

$$-1 \leq x - 2 \leq 1 \Rightarrow \boxed{1 \leq x \leq 3}.$$

The ball is at least 4 m high from **1 m to 3 m horizontally from the wall**, including both endpoints. Here x is distance, not time.