Sequences and kinematics

IB Mathematics: Analysis and Approaches SL

Exam-style practice: two original questions

Learning goal	Connect sequences with logarithms; interpret motion using calculus.
Syllabus link	AA SL 1.2, 1.3, 1.7, 1.8 and 5.9.
Structure	2 questions, 22 marks. Worked solutions begin on page 4.
Technology	A graphical calculator is required. Use radian mode.

Instructions Show your reasoning. Give exact answers where possible; otherwise give final answers to 3 significant figures. Keep full calculator precision during working.

Exam-style questions

1. Geometric sequences and logarithms

[Maximum mark: 15]

Consider a geometric sequence $a_1, a_2, a_3, \ldots$ with common ratio r. The sum to infinity of this sequence, A_∞, is 7.

(a) Find an expression for a_1 in terms of r. [1]

Working space

Consider a second geometric sequence $a_1^2, a_2^2, a_3^2, \ldots$ with common ratio q.

(b) Show that $q = r^2$. [2]

Working space

The sum to infinity of this second sequence, B_∞, is 21.

(c) **(i)** Show that $5r^2 - 7r + 2 = 0$. [3]

(ii) Find the value of r. [2]

(iii) Hence, find the value of a_1. [1]

Working space

Question 1 continues on the next page.

Question 1 Continued

Consider a third sequence whose nth term is given by

$$c_n = \log_k(a_n) + k, \qquad k \in \mathbb{Z}^+, \quad k > 1.$$

(d) Show that this third sequence is arithmetic. [2]

Working space

Let C_{24} be the sum of the first 24 terms of this third sequence.

(e) Find the greatest value of k for which $C_{24} < 0$. [4]

Working space

2. Motion along a straight line

[Maximum mark: 7]

A particle moves along a straight line and is initially at the point O. Its velocity, $v\,\text{m}\,\text{s}^{-1}$, at time t seconds is given by

$$v(t) = (2t^2 - 6t + 3)e^{-0.5t} - 3\cos t, \qquad 0 \leq t \leq 6.$$

(a) Find the maximum displacement of the particle. [4]

Working space

(b) Determine the time intervals during which the acceleration of the particle is negative. [3]

Working space

End of questions. Worked solutions follow on a new page.

Worked solutions 1. Sequences and logarithms

(a) The first sum to infinity gives

$$\frac{a_1}{1-r} = 7 \quad \Longrightarrow \quad \boxed{a_1 = 7(1-r)}, \qquad |r| < 1.$$

(b) The ratio of consecutive squared terms is

$$q = \frac{a_2^2}{a_1^2} = \left(\frac{a_2}{a_1}\right)^2 = \boxed{r^2}.$$

(c)(i) The second sum to infinity is 21, so

$$\begin{aligned} \frac{a_1^2}{1-r^2} = 21 \quad &\Longrightarrow \quad 49(1-r)^2 = 21(1-r^2), \\ 49 - 98r + 49r^2 &= 21 - 21r^2, \\ 70r^2 - 98r + 28 &= 0. \end{aligned}$$

Dividing by 14 gives $\boxed{5r^2 - 7r + 2 = 0}$.

(ii) Factorizing gives $(5r-2)(r-1) = 0$. Reject $r = 1$, since $|r| < 1$. Thus

$$\boxed{r = \frac{2}{5}}.$$

(iii) Substitute into the expression from part (a):

$$a_1 = 7\left(1 - \frac{2}{5}\right) = \boxed{\frac{21}{5}}.$$

(d) Using $a_n = a_1 r^{n-1}$ and the logarithm laws,

$$c_n = \log_k(a_1) + k + (n-1)\log_k(r).$$

This is arithmetic, with

$$c_1 = \log_k\left(\frac{21}{5}\right) + k, \qquad d = \log_k\left(\frac{2}{5}\right).$$

(e) Sum the first 24 terms:

$$\begin{aligned} C_{24} &= \frac{24}{2}(2c_1 + 23d) \\ &= 24\left[k + \frac{\ln(21/5) + \frac{23}{2}\ln(2/5)}{\ln k}\right] \\ &= 24\left(k - \frac{9.102258891\ldots}{\ln k}\right). \end{aligned}$$

Since $\ln k > 0$,

$$C_{24} < 0 \quad \Longleftrightarrow \quad k \ln k < 9.102258891\ldots$$

k	$k \ln k$	C_{24}
5	8.04719	−15.7332
6	10.7506	22.0784

Since $k \ln k$ increases for $k > 1$, the greatest possible integer is $\boxed{k_{\max} = 5}$.

Worked solutions 2. Kinematics

(a) The displacement from the initial point is

$$s(t) = \int_0^t \left[(2u^2 - 6u + 3)e^{-0.5u} - 3\cos u\right] \mathrm{d}u.$$

The velocity zeros are $t = 0$, $1.760509476\ldots$ and $5.417736878\ldots$. Compare the numerical integrals at the stationary points and at both endpoints:

t (s)	$s(t)$ (m)	Candidate
0	0	Initial endpoint
$1.760509476\ldots$	$-2.812822077\ldots$	Local minimum
$5.417736878\ldots$	$6.087275790\ldots$	Local maximum
6	$5.777000051\ldots$	Final endpoint

The maximum displacement is therefore $\boxed{6.09\,\text{m}}$, at $t = 5.42\,\text{s}$. Here, the greatest distance from O also happens to be 6.09 m.

(b) Differentiate using the product rule:

$$\begin{aligned} a(t) &= (4t - 6)e^{-0.5t} - \tfrac{1}{2}(2t^2 - 6t + 3)e^{-0.5t} + 3\sin t \\ &= (-t^2 + 7t - 7.5)e^{-0.5t} + 3\sin t. \end{aligned}$$

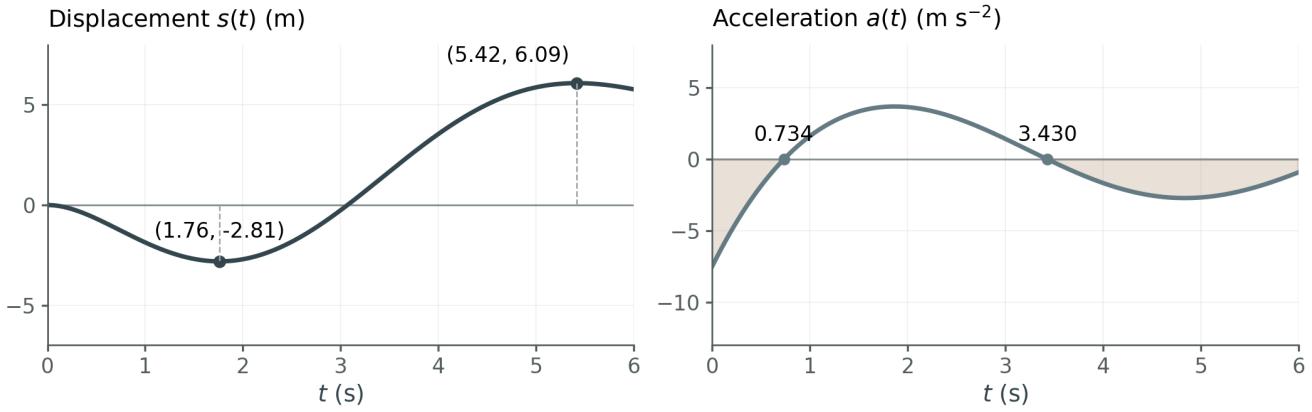
The acceleration zeros are

$$\alpha = 0.734019358\ldots, \qquad \beta = 3.430204012\ldots$$

The signs of a are $-, +, -$, so

$$\boxed{0 \le t < \alpha \quad \text{or} \quad \beta < t \le 6}, \qquad \alpha \approx 0.734, \quad \beta \approx 3.43.$$

Exclude the acceleration zeros and include the domain endpoints, where acceleration is negative.



A reliable calculator method for Question 2

1. Graph $v(t)$ over the *whole* interval and find every zero.
2. Evaluate $\int_0^t v(u)\,\mathrm{d}u$ at each interior zero and at both endpoints.
3. Compare these displacement values to identify the greatest value of $s(t)$.
4. Graph $a(t) = v'(t)$ and find where it is below the time axis.

Use numerical integration on a GDC: an algebraic antiderivative is not required.

Original practice questions and worked solutions prepared for Radford Mathematics. Suggested marks are provided for practice. Syllabus references: *Mathematics: analysis and approaches guide*, SL 1.2, 1.3, 1.7, 1.8 and 5.9.