



Complete calculus revision

IB Mathematics AA SL Topic 5

200 questions | Progressive practice and worked solution outlines

Learning goal	Connect calculus techniques, graphs and applications; prepare for both non-calculator and calculator examinations.
Syllabus link	All shared AA calculus: SL 5.1–5.11. This is an SL-only pack.
Structure	60 short + 60 extended + 40 Paper 1-style + 40 Paper 2-style questions.
Technology	Non-calculator unless marked  GDC. Use radians; give exact answers unless decimals are requested. GDC answers: 3 significant figures.

Instructions. Show your method and use correct mathematical notation. State units in context. Retain unrounded values during GDC calculations. Work on separate paper. For areas and total distance, write the correct integral expression before calculating. Check sign changes and endpoints where relevant.

Revision map

Syllabus	Focus	Suggested questions
SL 5.1	Limits, gradients and rates	1, 4, 62, 70
SL 5.2	Increasing and decreasing functions	7, 9, 12, 126
SL 5.3	Integer-power differentiation	13, 61, 79, 169
SL 5.4	Tangents and normals	19, 67, 81, 200
SL 5.5	Antiderivatives and initial conditions	23, 78, 82, 186
SL 5.6	Standard derivatives; chain, product, quotient rules	25, 75, 161, 200
SL 5.7	Second derivatives and linked graphs	31, 84, 129, 199
SL 5.8	Extrema, concavity, inflexion and optimization	33, 89, 137, 200
SL 5.9	Kinematics, speed and total distance	41, 99, 142, 198
SL 5.10	Standard integrals and substitution	47, 108, 150, 192
SL 5.11	Definite integrals and areas between curves	53, 118, 166, 200

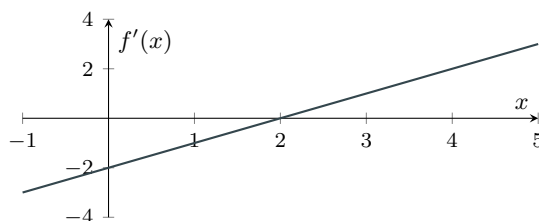
Work by topic first, then use Q121–160 for mixed Paper 1 practice and Q161–200 for mixed Paper 2 practice. Calculator icons also identify technology tasks wherever they appear.

1. Short response (Q1–60)

1. The table samples $g(x)$ near $x = 1$. Estimate $\lim_{x \rightarrow 1} g(x)$. Does the table determine $g(1)$?

x	0.90	0.99	1.01	1.10
y	1.90	1.99	2.01	2.10

2. A table gives $f(1.99) = 6.9601$ and $f(2.01) = 7.0401$. Estimate $f'(2)$ using the secant through these points.
3. A cost function $C(q)$ is measured in euros and q is the number of items. Interpret $C'(50) = 3.2$, including units.
4. The volume of water $V(t)$ is measured in litres and time t in minutes. Interpret $V'(8) = -4.5$.
5. The tangent at $(2, 7)$ passes through $(5, 13)$. Find the gradient of the curve at $(2, 7)$.
6. A function approaches 4 as x approaches 2 from the left, and approaches 7 from the right. Does a two-sided limit exist at 2? Explain.
7. Given $f'(x) = x + 2$, state where f is increasing and decreasing.
8. Given $f'(x) = -(x - 1)(x - 5)$, state where f is increasing.
9. The derivative is positive on both sides of $x = 3$, but $f'(3) = 0$. Must f have a local maximum or minimum there?
10. Can a function have a negative value and still be increasing? Give an example.
11. The graph shows $y = f'(x)$. State where f is decreasing.



12. If $f'(x) < 0$ throughout $2 < x < 6$, compare $f(3)$ and $f(5)$.
13. Differentiate $f(x) = 2x^5$.
14. Find $f'(1)$ for $f(x) = -3x^4 + 5x + 7$.
15. Differentiate $f(x) = 6x^{-1}$.



16. Find $f'(1)$ for $f(x) = 3x + 4x^{-2}$.

17. Differentiate $f(x) = 5x^2 - 2x^{-3}$.

18. Find $f'(1)$ for $f(x) = 9$.

19. Find the tangent to $y = x^2 + 1$ at $x = 2$.

20. Find the tangent to $y = x^3 - 2x$ at $x = 1$.

21. A curve passes through $(1, 4)$ with tangent gradient 2. Find the normal there.

22. A curve has a horizontal tangent at $(3, 8)$. Write the equations of its tangent and normal.

23. Find $\int (6x^2) dx$.

24. Find $\int (6x^{-2}) dx$.

25. Differentiate $f(x) = 3\sqrt{x} - 2x^{-1/2}$, for $x > 0$.

26. Differentiate $y = 4 \sin x - 3 \cos x$.

27. Differentiate $f(x) = 2e^x + 5 \ln x$, for $x > 0$.

28. Differentiate $y = (2x - 1)^6$.

29. Differentiate $f(x) = x^2 e^x$.

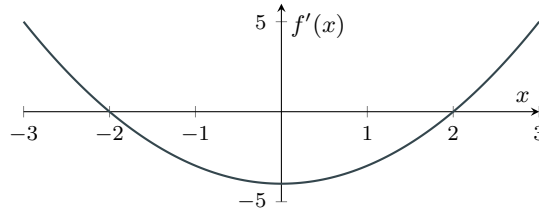
30. Differentiate $f(x) = (x + 1)/(x - 1)$, $x \neq 1$.

31. Find d^2y/dx^2 for $y = x^4 - 4x^3$.

32. Find $f''(x)$ for $f(x) = e^{-2x}$.

33. For $f(x) = \ln(2x + 1)$, $x > -1/2$, find $f''(x)$ and state the concavity.

34. The graph shows $f'(x) = x^2 - 4$. State where the gradient of f is increasing and where f is concave down.



35. Find the stationary point of $f(x) = x^2 - 6x + 5$ and classify it.

36. For $f(x) = x^3$, find the point of inflexion and state whether its tangent is horizontal.

37. For $f(x) = x^4$, explain why $(0, 0)$ is not a point of inflexion despite $f''(0) = 0$.

38. A function satisfies $f''(x) = 6x - 12$. State where its graph is concave down and where it is concave up.

39. Find the point of inflexion of $f(x) = x^3 - 3x + 2$ and the gradient there.

40. If $f'(a) = f''(a) = 0$, can the point be classified from these two facts alone? Give two contrasting examples at $a = 0$.

41. A particle has displacement $s(t) = t^3 - 4t^2 + 5t$ metres. Find $v(t)$ and $a(t)$.

42. At an instant a particle has velocity -7 m/s and acceleration 2 m/s². State its speed and whether it is speeding up or slowing down.

43. A particle has acceleration $a(t) = 6t$ m/s² and $v(0) = 4$ m/s. Find $v(t)$.

44. Given $v(t) = (t - 2)^2$ for $t \geq 0$, does the particle change direction at $t = 2$? Explain.

45. Write integrals for the displacement and total distance of a particle between times a and b , where $a < b$ and its velocity is $v(t)$.

46. The signed areas under a velocity graph are $+12$ m, -5 m and $+3$ m on successive intervals. Find the displacement and total distance.

47. Find $\int (3x^{2/3} - 4x^{-3/2} + 2) dx$.

48. Find $\int \frac{8}{x} dx$.

49. Find $\int (2x - 1)^4 dx$.

50. Find $\int e^{4-2x} dx$.

51. Find $\int 6x(x^2 + 1)^2 dx$.

52. Find $\int \frac{\sin x}{\cos x} dx$.

53. Evaluate $\int_{-1}^2 (3x^2 - 2x + 1) dx$.

54. Evaluate $\int_0^{\ln 3} e^{2x} dx$.

55. Evaluate $\int_0^{\pi/6} \cos(2x) dx$.

56. Evaluate $\int_1^0 (2x + 1) dx$. Explain the effect of reversing the limits.

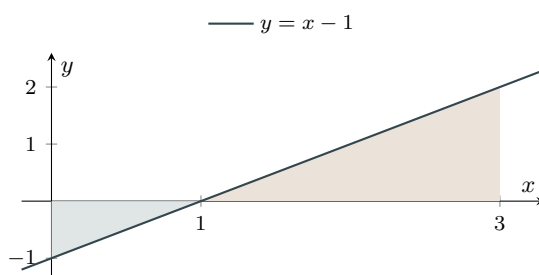
57. The table gives values of a differentiable function g .

x	-1	1	3
$g(x)$	4	-2	7

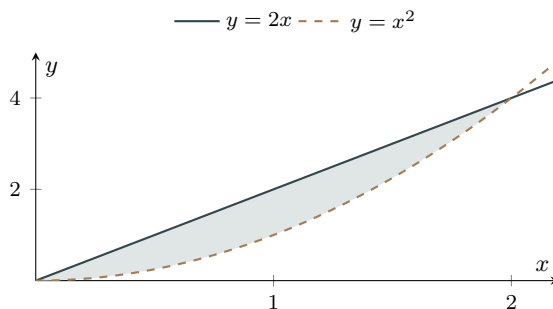
Find (a) $\int_{-1}^3 g'(x) dx$; (b) $\int_3^1 g'(x) dx$.

58. It is given that $\int_0^2 f(x) dx = 5$ and $\int_2^5 f(x) dx = -3$. Find $\int_0^5 [2f(x) + 1] dx$.

59. For $f(x) = x - 1$, find (a) $\int_0^3 f(x) dx$; (b) the total area between the graph and the x -axis on $0 \leq x \leq 3$.



60. Write an integral for the area enclosed by $y = 2x$ and $y = x^2$, then evaluate it.



2. Extended response (Q61–120)

61. The function $g(x) = \frac{x^3 - 8}{x - 2}$ is undefined at $x = 2$. Use the table to estimate its limit there. Explain why substituting $x = 2$ directly does not give the limit.

x	1.90	1.99	2.01	2.10
y	11.4100	11.9401	12.0601	12.6100

62. A tank volume is modelled by $F(t) = -t^2 + 12t + 50$. The input and output units make F' have units of L/min.
- (a) Find the average rate of change from $t = 3$ to $t = 4$.
- (b) Find the instantaneous rate at $t = 3$ and interpret its sign.

63. Given $f(4) = 12$ and $f'(4) = 2.5$, use the tangent line to estimate $f(4.08)$. Explain why the estimate is local.

64. A smooth function is sampled near $x = 3$.

x	2.9	2.99	3	3.01	3.1
y	5.61	5.9601	6	6.0401	6.41

- (a) Estimate $f'(3)$ from the two nearest points on either side.
- (b) State whether the function is increasing near 3, giving evidence.

65. Let $f(x) = x^2 \ln x$, $x > 0$.

- (a) Find $f'(x)$ and $f''(x)$.
- (b) Find the tangent at $x = 1$.

66. For $f(x) = e^{2x}(x - 1)$, find f' and f'' , then find the stationary point and classify it.

67. Let $f(x) = x/(x + 2)$, $x \neq -2$. Find f' and the tangent and normal at $x = 0$.

68. For $f(x) = \sin(3x) + \cos(2x)$, find f' , f'' and the tangent at $x = 0$.

69. Let $f(x) = \sqrt{2x + 3}$, $x > -3/2$. Find its first two derivatives and describe both its monotonicity and concavity.

70. Find the tangent to $y = \ln(3x + 1)$ at $x = 0$. Use it to estimate $\ln(1.06)$ and explain whether the estimate is above or below the true value.

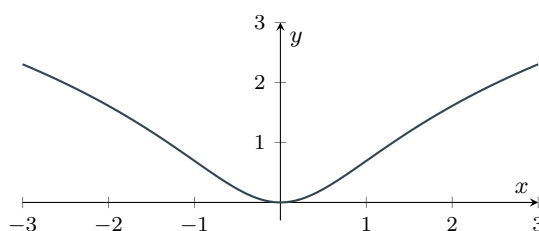
71. Let $f(x) = x^2/(x + 1)$, $x \neq -1$.

(a) Find and simplify f' .

(b) Find all points where the tangent is horizontal.

(c) Find f'' by first writing $f = x - 1 + 1/(x + 1)$.

72. For $f(x) = \ln(x^2 + 1)$, find its first two derivatives, stationary point and intervals of concavity.

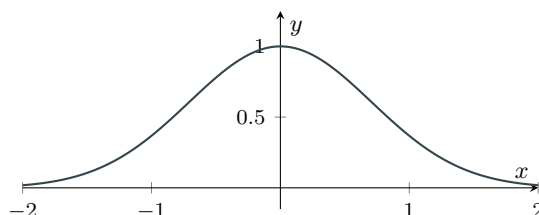


73. Let $f(x) = (x^2 + 1)^3$. Find f' and show that $f'' = 6(x^2 + 1)(5x^2 + 1)$. What does this imply about concavity?

74. A curve has equation $y = ae^x + bx$, where a, b are constants. It passes through $(0, 2)$ and has a horizontal tangent at $x = \ln 2$. Find a, b and the tangent equation there.

75. Let $f(x) = \sin x \cos x$. Find f' using the product rule, then verify it by rewriting f using a double-angle identity. Find f'' .

76. For $f(x) = e^{-x^2}$, find f', f'' and the coordinates of every point of inflexion.



77. For $f(x) = x^3 - 2x^2 + 4$,

(a) find $f(2)$ and $f'(2)$;

(b) find the tangent and normal at $x = 2$.

78. For $f(x) = 3x^2 + 2x^{-1}$,

(a) find $f(1)$ and $f'(1)$;

(b) find the tangent and normal at $x = 1$.

79. The curve $y = ax^2 + bx + 1$ passes through $(1, 6)$ and has tangent gradient 7 there. Find a and b .

80. The tangent to $y = x^2 + 6$ at $x = 2$ meets the axes at A and B .

- (a) Find its equation.
 - (b) Find the area of triangle OAB , where O is the origin.
-

81. Given $f'(x) = 6x^2 + 2$ and $f(0) = 3$,

- (a) find $f(x)$;
 - (b) find $f(2)$.
-

82. Given $f'(x) = 3 + 8x^{-2}$ and $f(1) = 4$,

- (a) find $f(x)$;
 - (b) find $f(2)$.
-

83. Let $f(x) = x^4 - 8x^2$.

- (a) Find and classify all stationary points.
 - (b) Find all points of inflexion.
 - (c) State where the graph is concave up and where it is concave down.
-

84. A function has derivative $f'(x) = (x + 2)(x - 1)^2$.

- (a) Find and classify the stationary points using the first derivative.
 - (b) Show that $x = 1$ gives a horizontal point of inflexion.
 - (c) Find the other x -value at which the concavity changes.
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85. Let $f(x) = x + \frac{4}{x}$, for $x > 0$. Find its minimum point, prove it is an absolute minimum and describe its concavity.

86. Let $f(x) = xe^{-x}$, for $x \geq 0$. Find its maximum point, its point of inflexion and the intervals of concavity.

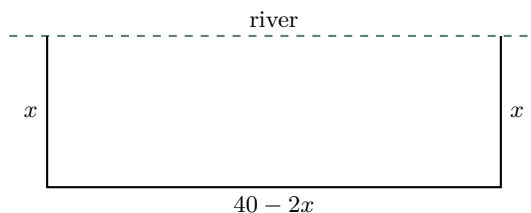
87. Let $f(x) = \ln x - \frac{1}{2}x$, for $x > 0$. Find its maximum point and explain why it has no point of inflexion.

88. The cubic $f(x) = x^3 + ax^2 + bx + 1$ has stationary points at $x = -1$ and $x = 2$. Find a and b and classify both points.

89. A closed cuboid has a square base of side x cm and volume 64 cm^3 .

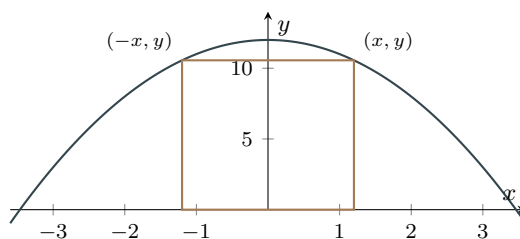
- (a) Express its surface area in terms of x .
- (b) Find the dimensions that minimize the surface area and justify your answer.

- 90.** A rectangular field borders a straight river. Only the other three sides require fencing. There are 40 m of fencing. Find the dimensions giving maximum area.



- 91.** An open box is made by cutting squares of side x cm from each corner of a 12 cm by 12 cm sheet and folding the sides up. Find the largest possible volume.

- 92.** A rectangle has vertices $(\pm x, 0)$ and $(\pm x, y)$ on the curve $y = 12 - x^2$, where $x > 0$ and $y > 0$. Find its largest area.



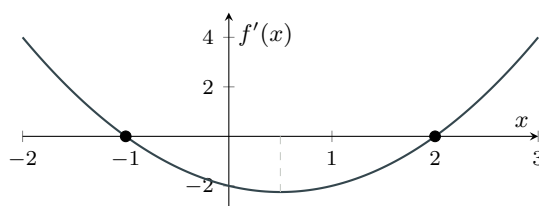
- 93.** Find the points on the curve $y = x^2$ closest to $(0, 3)$. Explain why minimizing squared distance is valid.

- 94.** The graph of f' is shown. Its zeros are -1 and 2 , and its vertex has $x = \frac{1}{2}$.

(a) Classify the stationary points of f .

(b) State where f is increasing.

(c) State where f is concave up or concave down, and give the x -coordinate of its point of inflexion.

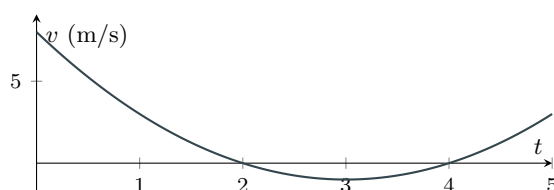


- 95.** A particle has velocity $v(t) = t^2 - 6t + 8$ m/s for $0 \leq t \leq 5$ and $s(0) = 3$ m.

(a) Find its times of rest and state its direction between them.

(b) Find its final position and displacement.

(c) Find the total distance travelled.



96. A particle has acceleration $a(t) = 4t - 6$ m/s², $v(0) = 4$ m/s and $s(0) = 2$ m.

- (a) Find $v(t)$ and $s(t)$.
 - (b) Find the times of rest.
 - (c) Find its displacement and total distance from $t = 0$ to $t = 3$.
-

97. A particle moves with $v(t) = 3 \cos t$ m/s for $0 \leq t \leq \pi$, with $s(0) = 2$ m.

- (a) Find its position function and time of direction change.
 - (b) Find its displacement and total distance.
 - (c) Find its average speed.
-

98. A particle has velocity $v(t) = e^t - 2$ m/s for $0 \leq t \leq \ln 4$, with $s(0) = 0$.

- (a) Find its position and time of rest.
 - (b) Find its displacement and total distance, giving exact answers.
-

99. For $v(t) = t^2 - 4t + 3$ m/s on $0 \leq t \leq 4$,

- (a) find when the speed is increasing;
 - (b) find the greatest speed on the interval;
 - (c) explain why zero acceleration alone does not imply zero velocity.
-

100. A particle has displacement $s(t) = t(t - 3)^2$ metres for $0 \leq t \leq 4$.

- (a) Find all times it is at the origin and all times it is at rest.
 - (b) Describe what happens at $t = 3$.
 - (c) Find the total distance travelled.
-

101. Particles P and Q move on the same straight line with $s_P = t^2$ and $s_Q = 6t - 5$, in metres, for $t \geq 0$.

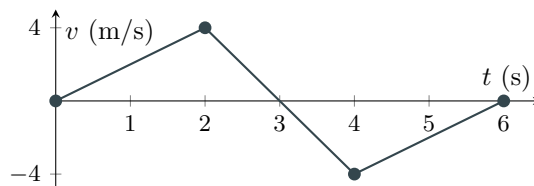
- (a) Find when they meet.
 - (b) Find their velocities at each meeting.
 - (c) Find their separation at $t = 3$.
-

102. A particle has constant acceleration a , initial velocity u and initial position 0. It has $s(2) = 10$ m and $s(4) = 28$ m.

- (a) Use integration to express $s(t)$ in terms of u, a .
- (b) Find u, a and the velocity at $t = 3$.

103. The velocity graph is made of straight segments joining $(0, 0)$, $(2, 4)$, $(4, -4)$ and $(6, 0)$, with seconds and m/s on the axes.

- Find the time of direction change.
- Find the displacement and total distance over the six seconds.
- Find the acceleration on $2 < t < 4$.



104. A particle has $s(t) = t^3 - 6t^2 + 12t - 1$ metres, for $t \geq 0$.

- Show that it is at rest at $t = 2$ but does not change direction.
- Find its acceleration at $t = 2$.
- Find the distance travelled from 0 to 4.

105. A function satisfies $f'(x) = 3\sqrt{x} - 2x^{-2}$ for $x > 0$, and $f(1) = 5$.

- Find the general antiderivative.
- Find $f(x)$ and verify the initial condition.

106. Given $f'(x) = 4/(2x - 3)$ for $x > 3/2$ and $f(2) = 7$,

- find $f(x)$;
- explain why absolute-value bars may be omitted from the logarithm on this domain.

107. A function satisfies $f'(x) = 5\cos(2x) - 3\sin x$ and $f(0) = 1$. Find $f(x)$ and the exact value of $f(\pi/2)$.

108. A student attempts the integral shown below.

Integrand	$\frac{4(2x+5)^3}{(2x+5)^4 + C}$
Proposed antiderivative	$\frac{4(2x+5)^3}{(2x+5)^4 + C}$

- Differentiate the proposed antiderivative.
- Identify the error and give the correct integral.

109. Use the substitution $u = x^2 + 3$ to find $\int \frac{4x}{(x^2 + 3)^2} dx$. Show the integral in terms of u before evaluating it.

110. Find $\int 3x^2 \sqrt{x^3 + 2} dx$ using a suitable substitution. State your substitution and differential.

111. Find $\int \frac{2x-1}{x^2-x+5} dx$. Explain why the logarithm can be written without absolute-value bars for all real x .

112. Find $\int \sin x(2 + \cos x)^4 dx$. Verify the sign of your answer by differentiation.

113. Use $u = 3x - 1$ to evaluate $\int_0^2 (3x - 1)^2 dx$. State the transformed limits.

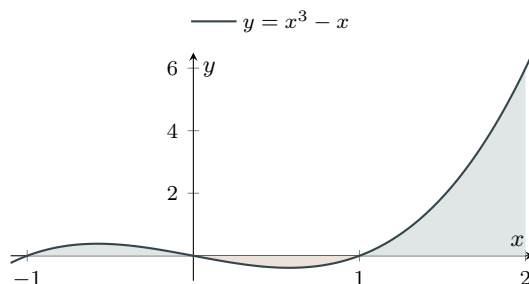
114. Evaluate $\int_0^1 \frac{4x}{2x^2 + 1} dx$ by substitution. Check that the logarithm argument stays positive.

115. Let $f(x) = x^3 - x$ on $-1 \leq x \leq 2$.

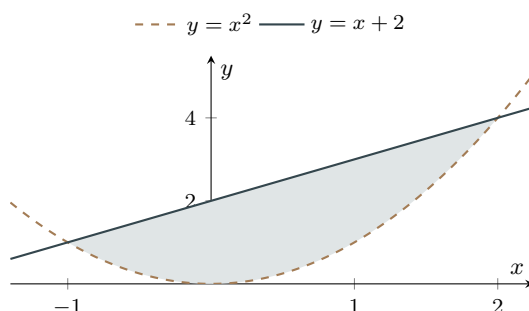
(a) Find all zeros of f in this interval.

(b) Find $\int_{-1}^2 f(x) dx$.

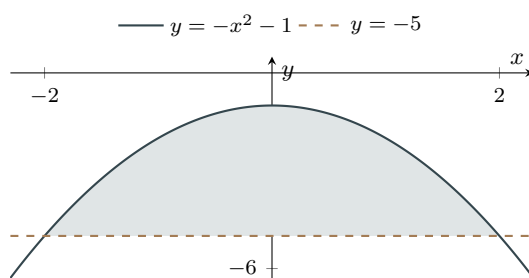
(c) Find the total area between the curve and the x -axis on the interval.



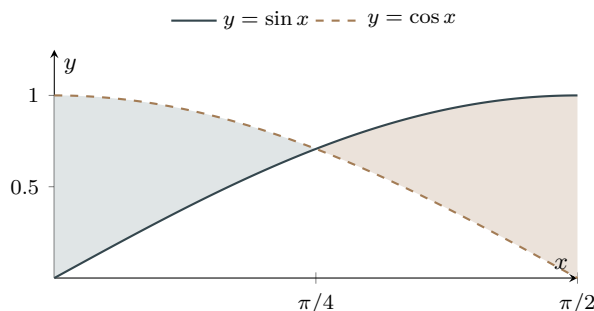
116. Find the points of intersection of $y = x^2$ and $y = x + 2$. Hence find the exact area enclosed by the two curves.



117. Find the area enclosed by $y = -x^2 - 1$ and $y = -5$. Explain why the fact that both graphs are below the x -axis does not change the upper-minus-lower method.



118. Find the total area between $y = \sin x$ and $y = \cos x$ on $0 \leq x \leq \pi/2$. State where the upper curve changes.



119. The vertical line $x = c$, where $0 < c < 4$, divides the region under $y = x$ above the x -axis on $[0, 4]$ into two equal areas. Find c exactly.

120. F is differentiable with continuous derivative. The table gives some values of F .

x	-1	0	2	4
$F(x)$	2	5	1	6

- (a) Find $\int_{-1}^4 F'(x) dx$ and $\int_0^4 F'(x) dx$.
- (b) Explain why these data do not determine the total area between $y = F'(x)$ and the x -axis on $[-1, 4]$.

3. Paper 1-style examination practice (Q121–160)

121. Let $f(x) = x^2 \ln x$, $x > 0$.

- (a) Find f' and f'' . [4 marks]
- (b) Find and classify the stationary point. [4 marks]
- (c) Find the point of inflexion. [4 marks]

122. Let $f(x) = e^x/(x+1)$ for $x > -1$.

- (a) Find $f'(x)$. [3 marks]
- (b) Find and classify the stationary point. [3 marks]
- (c) Find the tangent at $x = 1$. [3 marks]

123. For $0 \leq x \leq \pi$, let $f(x) = \sin x + \frac{1}{2} \sin(2x)$.

- (a) Find f' and use a double-angle identity to solve $f' = 0$ in the interior. [4 marks]
- (b) Find the greatest value of f on the interval. [4 marks]

124. Let $f(x) = ax^3 + bx^2 + cx + 2$. The graph has a stationary point at $(1, 0)$ and $f''(1) = 6$.

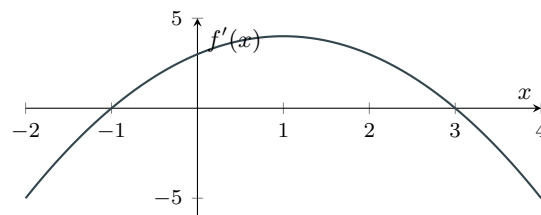
- (a) Form three equations in a, b, c . [3 marks]
- (b) Find a, b, c . [4 marks]
- (c) Classify the given stationary point. [1 mark]

125. Let $f(x) = \sqrt{x+3}$, $x > -3$.

- (a) Find the tangent and normal at $x = 1$. [5 marks]
 (b) Find where the normal meets the x -axis. [2 marks]
 (c) Explain why the tangent at $x = 1$ lies above the curve near that point. [2 marks]

126. The graph shows $f'(x) = -(x+1)(x-3)$, and $f(0) = 2$.

- (a) State the intervals where f increases and decreases. [3 marks]
 (b) Find $f(x)$. [4 marks]
 (c) Find the point of inflexion of f . [3 marks]



127. A particle has position $s(t) = 2\ln(t+1) - t$ metres, for $t \geq 0$.

- (a) Find velocity and acceleration. [3 marks]
 (b) Find when the particle reverses direction. [2 marks]
 (c) Find the total distance travelled for $0 \leq t \leq 3$. [4 marks]

128. Let $f'(x) = 4x(x^2 + 1)$ and $f(0) = 1$.

- (a) Find $f(x)$. [3 marks]
 (b) Find the tangent to f at $x = 1$. [3 marks]
 (c) Find the area between f and this tangent on $[0, 1]$. [4 marks]

129. Let $f(x) = x^3 + 3x^2 - 9x + 4$.

- (a) Find $f'(x)$ and hence the coordinates of the stationary points. [4 marks]
 (b) Classify these points using the second derivative. [2 marks]
 (c) Find the point of inflexion and show that its tangent is not horizontal. [3 marks]
 (d) Find the equation of that tangent. [2 marks]

130. Consider $f(x) = x^4 - 4x^3$.

- (a) Find all stationary points. [3 marks]
 (b) Use the sign of f' to classify them. [3 marks]
 (c) Find both points of inflexion and state the intervals on which the graph is concave up. [4 marks]

131. The sign table for f' is shown. The function is differentiable for every real x .

x	$-\infty$		-2		3		$+\infty$
$f'(x)$		$+$	0	$-$	0	$+$	

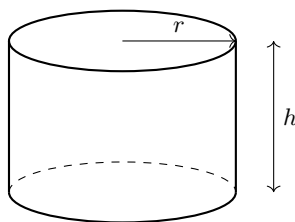
- (a) State the intervals on which f decreases. [1 mark]
- (b) Classify the stationary points at $x = -2$ and $x = 3$, giving reasons. [2 marks]
- (c) Given $f'(x) = (x + 2)(x - 3)$, find the x -coordinate of the point of inflexion and justify it. [3 marks]

132. An open container has a square base of side x cm and height h cm. Its volume is 108 cm^3 .

- (a) Show that its material area is $S(x) = x^2 + 432/x$. [3 marks]
- (b) Find x and h when the material area is minimized. [3 marks]
- (c) Justify the minimum and find its value. [2 marks]

133. A closed cylindrical can has volume $54\pi \text{ cm}^3$, radius r cm and height h cm.

- (a) Show that its surface area is $S(r) = 2\pi r^2 + 108\pi/r$. [3 marks]
- (b) Find the radius and height giving the least surface area. [3 marks]
- (c) Prove that your result gives a minimum. [2 marks]



134. A poster must contain a rectangular printed area of 100 cm^2 . A margin of 1 cm is required on all four sides. Let $x \text{ cm}$ be the width of the printed area.

- (a) Express the total poster area in terms of x . [3 marks]
- (b) Find the dimensions of the poster using the least paper. [3 marks]
- (c) Justify that this is a minimum. [2 marks]

135. Let $f(x) = xe^{-2x}$ for $x \geq 0$.

- (a) Show that $f'(x) = e^{-2x}(1 - 2x)$. [2 marks]
- (b) Find the maximum point. [3 marks]
- (c) Find the point of inflexion and state the intervals of concavity. [4 marks]

136. Let $f(x) = \frac{\ln x}{x}$, for $x > 0$.

- (a) Find $f'(x)$. [2 marks]
- (b) Find and classify the stationary point. [3 marks]
- (c) Given $f''(x) = \frac{2 \ln x - 3}{x^3}$, find the point of inflexion and justify it. [3 marks]

- 137.** Let $f(x) = x^3 + ax^2 + bx + c$. Its graph has a point of inflexion at $(1, 4)$ and a stationary point at $x = 0$.
- (a) Find a , b and c . [5 marks]
- (b) Find and classify both stationary points. [4 marks]

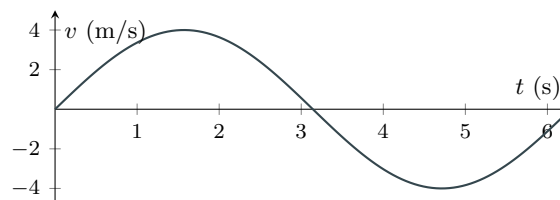
- 138.** The curve $y = \frac{x}{x^2 + 1}$ has derivative $y' = \frac{1 - x^2}{(x^2 + 1)^2}$.
- (a) Find and classify its stationary points. [4 marks]
- (b) Given $y'' = \frac{2x(x^2 - 3)}{(x^2 + 1)^3}$, find all points of inflexion. [4 marks]
- (c) State whether any point of inflexion has a horizontal tangent. [1 mark]

- 139.** A particle moves on a straight line with velocity $v(t) = 3t^2 - 18t + 24$ m/s, for $0 \leq t \leq 5$. Its initial position is 2 m.
- (a) Find its acceleration and all times of rest. [3 marks]
- (b) Find its position function. [3 marks]
- (c) Find its displacement and total distance during the first 5 seconds. [5 marks]

- 140.** A particle has acceleration $a(t) = 12t - 12$ m/s², initial velocity 6 m/s and initial position 1 m, for $0 \leq t \leq 3$.
- (a) Find $v(t)$ and $s(t)$. [5 marks]
- (b) Find all times it is at rest and state whether it changes direction. [3 marks]
- (c) Find its total distance travelled. [2 marks]

- 141.** For $0 \leq t \leq 4$, a particle has displacement $s(t) = t^3 - 6t^2 + 9t + 4$ metres.
- (a) Find its times of rest and corresponding positions. [4 marks]
- (b) Find the intervals on which speed is increasing. [4 marks]
- (c) Find the greatest speed on the interval. [3 marks]

- 142.** A particle has velocity $v(t) = 4 \sin t$ m/s for $0 \leq t \leq 2\pi$, with $s(0) = 3$ m.
- (a) Find its acceleration and position. [4 marks]
- (b) State the time of direction change in the interior. [2 marks]
- (c) Find the total distance and average speed over the interval. [4 marks]



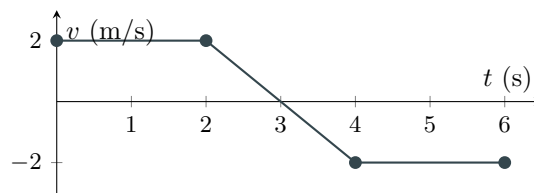
- 143.** A particle has velocity $v(t) = 3e^{-t} - 1$ m/s for $0 \leq t \leq \ln 9$, with initial position 0.
- (a) Find its position and time of direction change. [4 marks]
- (b) Find its displacement. [2 marks]
- (c) Find its total distance. [4 marks]

144. Two particles have positions $s_P = t^2 - 2t$ and $s_Q = 4t - 5$ metres, for $t \geq 0$.

- Find the two times when they meet. [3 marks]
- Find their velocities at each meeting. [3 marks]
- Find their greatest separation between the two meetings. [4 marks]

145. A velocity graph joins the points $(0, 2)$, $(2, 2)$, $(4, -2)$ and $(6, -2)$ with straight segments. Time is in seconds and velocity in m/s.

- Find when the particle reverses direction. [2 marks]
- Find the displacement and total distance over the six seconds. [4 marks]
- Find its acceleration on each open segment and explain why no single acceleration is assigned at the two corners. [3 marks]



146. A particle travels on a straight line with $v(t) = 6/(t+1)^2 - 1$ m/s for $0 \leq t \leq 3$, with $s(0) = 0$.

- Find its position function. [3 marks]
- Find its time of direction change. [2 marks]
- Find its total distance travelled, giving an exact answer. [4 marks]

147. For $x > 0$, a curve has $f'(x) = 4x^3 - 3x^{-1/2} + 2/x$ and passes through $(1, 2)$.

- Find $f(x)$. [5 marks]
- Find the equation of the tangent at $(1, 2)$. [3 marks]

148. A function satisfies $f'(x) = 6/(3x-1)$ for $x > 1/3$, and $f(1) = 4$.

- Find $f(x)$. [4 marks]
- Solve $f(x) = 6$, giving an exact answer. [3 marks]

149. A curve passes through $(0, 1)$ and has gradient $dy/dx = 8e^{-2x} + 3\cos(3x)$.

- Find the equation of the curve. [5 marks]
- Find the tangent at $x = 0$. [3 marks]

150. A function satisfies $F'(x) = (3x^2 - 2)e^{x^3 - 2x}$ and $F(0) = 3$.

- Find $F(x)$ by inspection or substitution. [4 marks]
- Find the exact value of $F(2)$. [2 marks]
- Verify your expression for F by differentiation. [2 marks]

151. A function satisfies $F'(x) = e^{2x}/(1 + e^{2x})$ and $F(0) = 2$.

- (a) Find $F(x)$. [5 marks]
- (b) Find $F(\ln 2)$, expressing your answer using a single logarithm. [2 marks]

152. For $x > -1$, $F'(x) = a/(x+1) + 2x$, where a is a constant. The table gives two values of F .

x	0	1
$F(x)$	1	4

- (a) Find $F(x)$ in terms of a and an integration constant. [3 marks]
- (b) Use the table to find both constants. [4 marks]

153. Let $f(x) = 2x^2 - 4x + 3$.

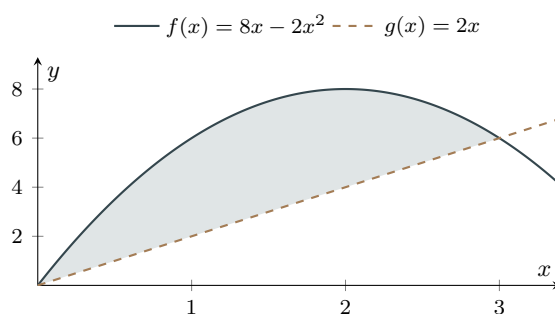
- (a) Express $f(x)$ in completed-square form and state its minimum value. [3 marks]
- (b) Evaluate $\int_{-1}^2 f(x) dx$. [4 marks]
- (c) Explain why this integral equals the total area between the curve and the x -axis on the given interval. [1 mark]

154. The curve $y = x^2 - 9$ encloses a finite region with the x -axis.

- (a) Find the points where the curve meets the axis. [2 marks]
- (b) Write a definite integral for the area of this region. [2 marks]
- (c) Find the exact area. [3 marks]

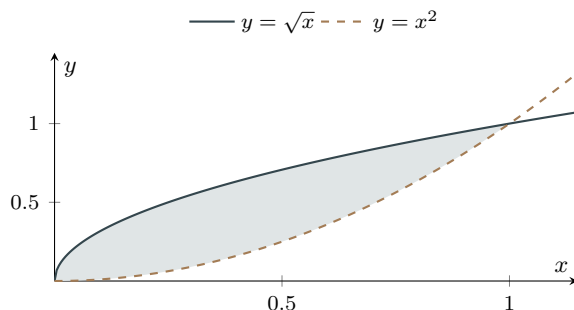
155. The diagram shows $f(x) = 8x - 2x^2$ and $g(x) = 2x$.

- (a) Find the coordinates of their intersections. [3 marks]
- (b) Find the exact area of the shaded region. [4 marks]
- (c) Find the equation of the tangent to f that is parallel to g . [4 marks]



156. Consider $y = \sqrt{x}$ and $y = x^2$, where $x \geq 0$.

- (a) Find all intersection points. [3 marks]
- (b) Determine which curve is upper between these points. [1 mark]
- (c) Find the exact area enclosed by the curves. [4 marks]

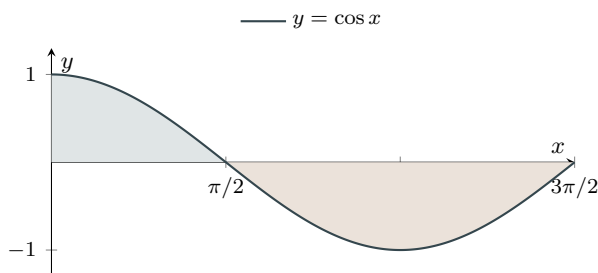


157. For $k > 0$, let $f(x) = x^2 - k$. It is given that $\int_0^2 f(x) dx = 0$.

- (a) Find k . [3 marks]
- (b) Find the zero of f in $[0, 2]$. [2 marks]
- (c) Find the total area between f and the x -axis on this interval. [5 marks]

158. The shaded regions lie between $y = \cos x$ and the x -axis for $0 \leq x \leq 3\pi/2$.

- (a) Evaluate the signed integral of $\cos x$ over the interval. [2 marks]
- (b) Write an expression for the total shaded area. [2 marks]
- (c) Find this area exactly. [3 marks]



159. A student evaluates $\int_0^3 (x^2 - 2x) dx = 0$ and concludes that the total area between the curve and the x -axis is zero.

- (a) Verify the value of the definite integral. [3 marks]
- (b) Explain the error in the conclusion. [2 marks]
- (c) Calculate the correct total area. [4 marks]

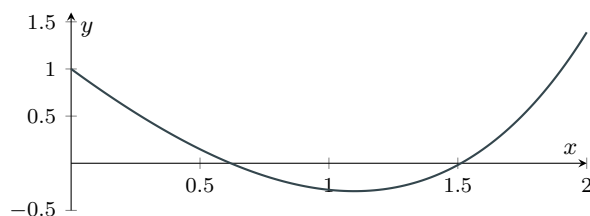
160. For $k > 0$, the curves $y = k - x^2$ and $y = x^2$ enclose a region of area $64/3$.

- (a) Find the intersection x -coordinates in terms of k . [2 marks]
- (b) Show that the enclosed area is $\frac{4}{3}k\sqrt{k/2}$. [4 marks]
- (c) Find k . [3 marks]

4. Paper 2-style examination practice (Q161–200)

161.  *GDC* Let $f(x) = e^x - 3x$ on $0 \leq x \leq 2$.

- (a) Find the stationary point and classify it. [4 marks]
- (b) Find both zeros of f . [3 marks]
- (c) Write a suitable integral expression and find the total area between the curve and the x -axis. [4 marks]

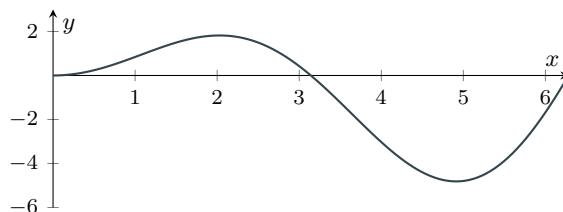


162.  *GDC* Let $f(x) = \ln x - 0.3x$ on $0.5 \leq x \leq 8$.

- (a) Find and classify the stationary point. [4 marks]
- (b) Find the greatest and least values on the closed interval. [3 marks]
- (c) Find all x -intercepts. [3 marks]

163.  *GDC* Let $f(x) = x \sin x$ on $0 \leq x \leq 2\pi$.

- (a) Find f' and both interior stationary points. [4 marks]
- (b) Determine the greatest and least function values. [3 marks]
- (c) Use numerical integration to find $\int_0^{2\pi} f(x) dx$. [2 marks]



164.  *GDC* For $0 \leq x \leq 10$, let $f(x) = (x + 1)e^{-0.4x}$.

- (a) Find f' and locate the maximum point. [4 marks]
- (b) Find f'' and the point of inflexion. [4 marks]
- (c) State the intervals of concavity. [2 marks]

165.  *GDC* Consider $y = e^{x^2}$ at $x = 0.6$.

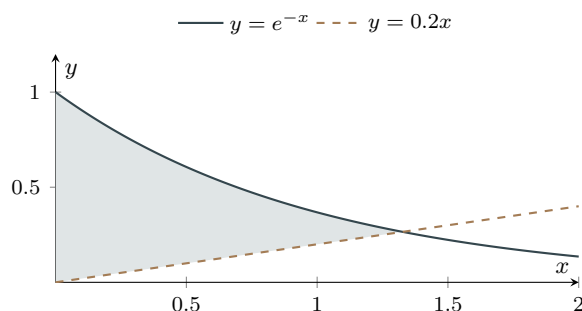
- (a) Find the coordinates of the point and the tangent gradient. [3 marks]
- (b) Find equations of the tangent and normal. [4 marks]
- (c) Find the x -intercept of the normal. [2 marks]

166.  *GDC* The tangent to $y = \ln(1 + x^2)$ at a positive x -coordinate a passes through the origin.

- Show that a satisfies $2a^2/(1 + a^2) = \ln(1 + a^2)$. [3 marks]
- Find a and the tangent equation. [4 marks]
- Find the area under the curve between $x = 0$ and $x = a$. [3 marks]

167.  *GDC* A region is enclosed by $y = e^{-x}$, $y = 0.2x$ and the y -axis.

- Find the intersection of the two graphs. [3 marks]
- Write a definite integral for the region and calculate its area. [4 marks]



168.  *GDC* Find the total area between $y = e^{-x^2}$ and $y = 0.4x$ for $0 \leq x \leq 2$.

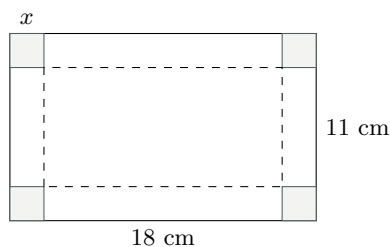
- Find where the upper curve changes. [3 marks]
- Write a sum of two definite integrals. [2 marks]
- Calculate the total area. [2 marks]

169.  *GDC* Let $f(x) = x^3 - 2x - 2$ on $[0, 2]$.

- Find its zero in the interval. [2 marks]
- Find and classify its interior stationary point. [3 marks]
- Find the total area between the graph and the x -axis. [4 marks]

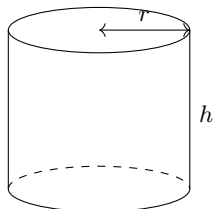
170.  *GDC* An open box is made by cutting squares of side x cm from the corners of an 18 cm by 11 cm card and folding up the sides.

- Find $V(x)$ and its physical domain. [3 marks]
- Find the value of x giving the greatest volume. [3 marks]
- Give the maximum volume and the box dimensions. [3 marks]



171.  *GDC* A closed cylindrical can has volume 330 cm^3 .

- (a) Express its surface area S in terms of its radius r . [3 marks]
- (b) Find the dimensions requiring the least material. [4 marks]
- (c) Find the minimum surface area. [2 marks]



172.  *GDC* A closed container has a square base of side $x \text{ cm}$ and volume 1200 cm^3 . The top and base each cost $0.02 \text{ euros per cm}^2$; the four sides cost $0.01 \text{ euros per cm}^2$.

- (a) Show that its material cost is $C(x) = 0.04x^2 + 48/x$. [4 marks]
- (b) Find the dimensions minimizing the cost. [4 marks]
- (c) Find the minimum cost to the nearest cent. [1 mark]

173.  *GDC* A firm models price per item as $p(q) = 50e^{-0.02q}$ euros and cost as $C(q) = 4q + 100$ euros, treating output q as continuous on $0 \leq q \leq 150$.

- (a) Form the profit function and its derivative. [4 marks]
- (b) Find the output and price maximizing profit. [4 marks]
- (c) Find the two break-even outputs. [3 marks]

174.  *GDC* The profit model $P(n) = -0.3n^3 + 8n^2 + 10n - 50$ euros applies for integer output $0 \leq n \leq 25$.

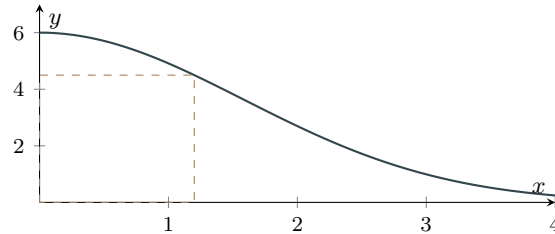
- (a) Find the maximizing output if n is treated as continuous. [3 marks]
- (b) Find the feasible integer output giving the greatest profit. [3 marks]
- (c) State the greatest profit to the nearest cent. [1 mark]

175.  *GDC* A point P lies on $y = \ln x$ with $0.2 \leq x \leq 5$. Let $A = (2, 0)$.

- (a) Write the square of the distance AP as a function of x . [2 marks]
- (b) Find the coordinates of the point closest to A . [4 marks]
- (c) Find the least distance. [2 marks]

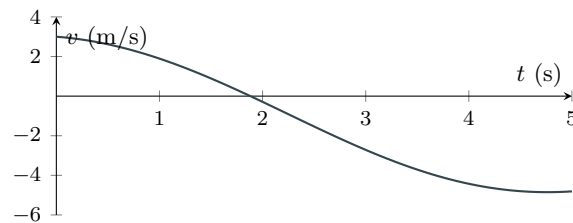
176.  *GDC* A rectangle in the first quadrant has vertices $(0, 0)$, $(x, 0)$, (x, y) , $(0, y)$, with its upper-right vertex on $y = 6e^{-0.2x^2}$.

- (a) Write its area $A(x)$ and find $A'(x)$. [4 marks]
 (b) Find the maximum area and corresponding dimensions. [4 marks]



177.  *GDC* A particle moves with velocity $v(t) = 3\cos(0.7t) - 0.4t$ m/s for $0 \leq t \leq 5$.

- (a) Find the time at which it changes direction. [3 marks]
 (b) Find its displacement and total distance over the five seconds. [4 marks]
 (c) Find its acceleration at the direction change. [2 marks]



178.  *GDC* A particle has $v(t) = 2 + 0.5t - 3e^{-t}$ m/s on $[0, 5]$, with initial position 2 m.

- (a) Find its position function. [4 marks]
 (b) Find its time of rest and least position. [4 marks]
 (c) Find its total distance travelled. [3 marks]

179.  *GDC* A particle starts at the origin with position $s(t) = 4\ln(t+1) - t^2/2$ metres, for $0 \leq t \leq 5$.

- (a) Find the time and position at which it reverses direction. [4 marks]
 (b) Find its positive return time to the origin. [2 marks]
 (c) Find its average speed over the five seconds. [3 marks]

180.  *GDC* Two particles have positions $s_P(t) = t^2$ and $s_Q(t) = 8(1 - e^{-t})$ metres, for $t \geq 0$.

- (a) Find their positive meeting time. [3 marks]
 (b) Find the velocity of each particle at that time. [3 marks]
 (c) State which particle is farther along the positive axis immediately after this meeting. [2 marks]

181.  *GDC* A tank initially contains 20 litres. Its net rate of change is $V'(t) = 12e^{-0.3t} - 1$ litres per minute for $0 \leq t \leq 12$.

(a) Find $V(t)$. [4 marks]

(b) Find when the volume is greatest and state that volume. [4 marks]

(c) Find the volume after 12 minutes. [2 marks]

182.  *GDC* The area under $y = e^{-x^2}$ above the x -axis from $x = 0$ to $x = b$, where $b > 0$, is 0.7.

(a) Write the equation defining b . [2 marks]

(b) Use your GDC to find b . [3 marks]

(c) Explain why there can be at most one such value of b . [2 marks]

183.  *GDC* For $k > 0$, it is given that $\int_0^2 e^{-kx} dx = 0.9$.

(a) Show that $(1 - e^{-2k})/k = 0.9$. [3 marks]

(b) Find k numerically. [3 marks]

(c) Find the height of the curve at $x = 2$. [1 mark]

184.  *GDC* Let $f(x) = x^2 + a \sin x$ on $[0, 6]$. Its tangent at $x = 2$ is horizontal.

(a) Find a . [3 marks]

(b) Find and classify both stationary points. [5 marks]

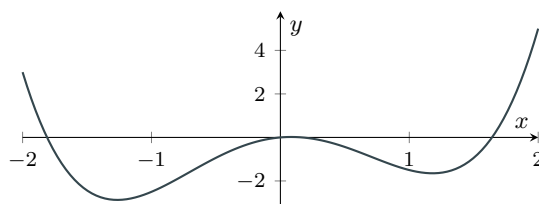
(c) Find the greatest value of f on the interval. [3 marks]

185.  *GDC* Let $f(x) = x^4 - 3x^2 + 0.5x$ on $[-2, 2]$.

(a) Find all stationary points and classify them. [5 marks]

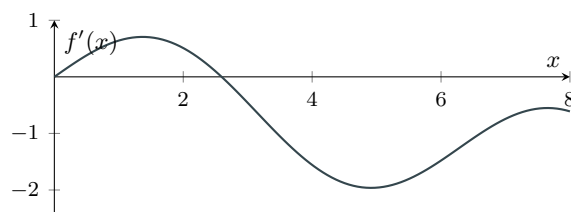
(b) Find both points of inflexion. [3 marks]

(c) Determine the greatest and least function values. [3 marks]



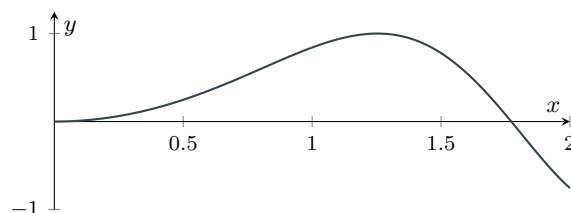
186.  *GDC* A function has $f'(x) = \sin x - 0.2x$ on $[0, 8]$, with $f(0) = 3$.

- Find $f(x)$. [3 marks]
- Find its interior stationary point and classify it. [3 marks]
- Find all intervals on which f is concave up. [4 marks]



187.  *GDC* Let $f(x) = \sin(x^2)$ on $[0, 2]$.

- Find the interior zero of f . [2 marks]
- Calculate its signed integral over the interval. [2 marks]
- Write an expression for the total area with the x -axis and calculate it. [4 marks]



188.  *GDC* The area under $y = 2 + \sin(3x)$ above the x -axis between 0 and $a > 0$ is 5.

- Derive an equation for a by integrating analytically. [3 marks]
- Solve for a numerically. [3 marks]
- Explain why this equation has only one positive solution. [2 marks]

189.  *GDC* Consider $y = \sqrt{x^2 + 4}$ at $x = 1.3$.

- Find the tangent gradient analytically and evaluate it. [3 marks]
- Find the normal equation. [3 marks]
- Find where the normal meets the x -axis. [2 marks]

190.  *GDC* Find the two tangents to $y = x^3 - 3x^2 - 2x + 4$ that are parallel to $y = 1.7x - 2$.

- Write an equation for the possible x -coordinates of contact. [2 marks]
- Find both points of contact. [3 marks]
- Find both tangent equations. [3 marks]

191.  *GDC* For $x > -2/3$, $F'(x) = \sqrt{3x + 2}$ and $F(0) = 1$.

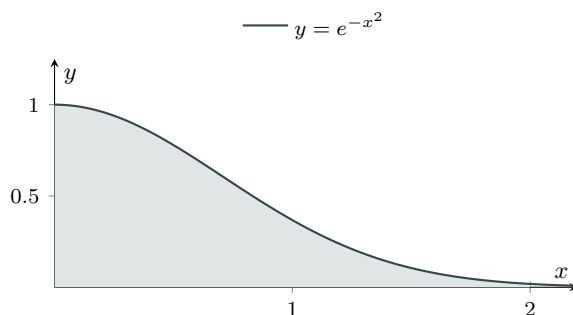
- Find an exact expression for $F(x)$. [4 marks]
- Use your GDC to calculate $F(2)$ to 3 significant figures. [2 marks]
- Check numerically that the derivative of your expression at $x = 2$ equals $\sqrt{8}$. [2 marks]

192.  *GDC* A function satisfies $F'(x) = 2xe^{-x^2}$ and $F(0) = 5$.

- (a) Find $F(x)$ analytically. [4 marks]
- (b) Use your GDC to find the positive solution of $F(x) = 5.8$. [3 marks]
- (c) Explain why there cannot be two positive solutions. [2 marks]

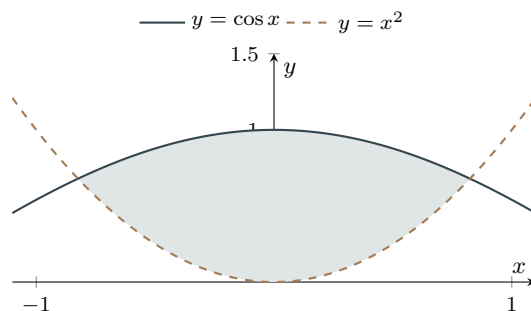
193.  *GDC* Let $f(x) = e^{-x^2}$.

- (a) Write an integral for the area under the curve above the x -axis on $0 \leq x \leq 2$. [2 marks]
- (b) Use your GDC to evaluate the area to 3 significant figures. [2 marks]
- (c) Explain why no splitting at an x -intercept is needed. [1 mark]



194.  *GDC* The curves $y = \cos x$ and $y = x^2$ enclose a region.

- (a) Use your GDC to find the two intersection x -coordinates. [3 marks]
- (b) Write a definite integral for the enclosed area. [2 marks]
- (c) Calculate this area to 3 significant figures. [2 marks]

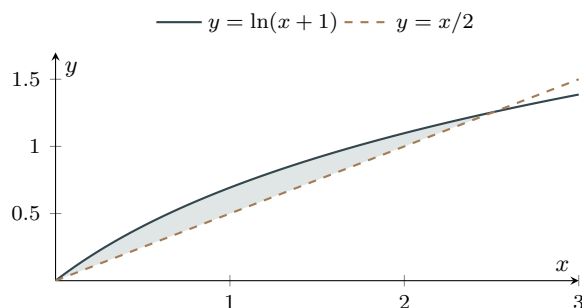


195.  *GDC* A flat panel has lower edge $y = 0$ and upper edge $y = \sqrt{1 + x^3}$ for $0 \leq x \leq 3$. Both coordinates are measured in metres.

- (a) Write a definite integral for the area of the panel. [2 marks]
- (b) Use your GDC to find its area. [2 marks]
- (c) A coating costs 12 euros per square metre. Find the cost to the nearest cent. [2 marks]

196.  *GDC* The graphs $y = \ln(x + 1)$ and $y = x/2$ enclose a region for $x \geq 0$.

- State one intersection and use your GDC to find the other. [3 marks]
- Write a definite integral for the area enclosed by the graphs. [2 marks]
- Use your GDC to calculate the area. [2 marks]



197.  *GDC* A particle has position $s(t) = t^3 - 4t^2 + 3t$ metres for $0 \leq t \leq 4$. Use technology.

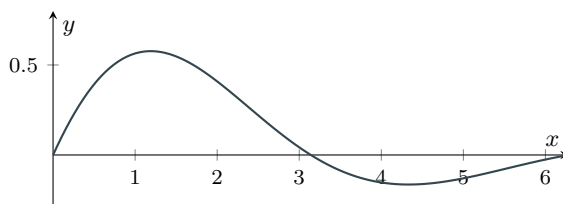
- Find all times it passes the origin. [2 marks]
- Find its interior times of rest. [3 marks]
- Find its total distance travelled. [4 marks]

198.  *GDC* A particle has velocity $v(t) = 2 \sin t - 0.4t$ m/s for $0 \leq t \leq 4$. Use radians and technology.

- Find the interior time at which it changes direction. [3 marks]
- Find its displacement and total distance. [4 marks]
- Find its acceleration at $t = 2$. [2 marks]

199.  *GDC* Let $f(x) = e^{-0.4x} \sin x$ on $[0, 2\pi]$.

- Find the greatest and least values of f . [5 marks]
- Find all points of inflexion. [4 marks]
- Find the total area between f and the x -axis. [3 marks]



200.  *GDC* Let $f(x) = x \ln x - x$ and $g(x) = 0.4x$, for $x > 0$.

- Find f' and f'' . [3 marks]
- Find and classify the stationary point of f , and give its normal equation. [3 marks]
- Find where the graphs of f and g intersect. [3 marks]
- Write an integral for the area between the two graphs from $x = 1$ to their intersection, and calculate it. [4 marks]

Answer key and solution outlines

Equivalent forms and correct alternative methods are accepted. The examination-style questions are original practice items, not official IB questions.

1. Short response (Q1–60)

1. The values approach 2. The table does not give the value at $x = 1$; a limit and the function value need not coincide.
2. $f'(2) \approx [7.0401 - 6.9601]/(2.01 - 1.99) = 4$.
3. At an output of 50 items, cost is increasing at approximately €3.20 per additional item.
4. At 8 minutes, the volume is decreasing at 4.5 litres per minute.
5. Gradient $= (13 - 7)/(5 - 2) = 2$.
6. No: the left-hand and right-hand limiting values differ.
7. f decreases for $x < -2$ and increases for $x > -2$.
8. $f' > 0$ between its roots, so f increases for $1 < x < 5$.
9. No. The function increases on both sides, so there is no local extremum.
10. Yes, for example $f(x) = x - 10$ at $x = 0$: $f(0) = -10$, but $f' = 1 > 0$.
11. The derivative is below the axis for $x < 2$, so f decreases there.
12. $f(3) > f(5)$ because f decreases as x increases.
13. Using $\frac{d}{dx}(ax^n) = anx^{n-1}$, $f'(x) = 10x^4$.
14. $f'(x) = -12x^3 + 5$, hence $f'(1) = -7$.
15. Using $\frac{d}{dx}(ax^n) = anx^{n-1}$, $f'(x) = -6x^{-2}$.
16. $f'(x) = 3 - 8x^{-3}$, hence $f'(1) = -5$.
17. Using $\frac{d}{dx}(ax^n) = anx^{n-1}$, $f'(x) = 10x + 6x^{-4}$.
18. $f'(x) = 0$, hence $f'(1) = 0$.
19. $y_0 = 5$ and $m = f'(2) = 4$. Thus $y - 5 = 4(x - 2)$, or $y = 4x - 3$.
20. $y_0 = -1$ and $m = f'(1) = 1$. Thus $y - -1 = 1(x - 1)$, or $y = x - 2$.
21. $m_n = -1/m_t = -\frac{1}{2}$, so $y = -\frac{1}{2}x + \frac{9}{2}$.
22. Tangent $y = 8$; normal $x = 3$.
23. $2x^3 + C$, using $\int ax^n dx = \frac{a}{n+1}x^{n+1} + C$.
24. $-6x^{-1} + C$, using $\int ax^n dx = \frac{a}{n+1}x^{n+1} + C$.
25. $f'(x) = \frac{3}{2}x^{-1/2} + x^{-3/2}$.
26. $dy/dx = 4 \cos x + 3 \sin x$.
27. $f' = 2e^x + 5/x$.
28. The chain rule gives $dy/dx = 6(2x - 1)^5 \cdot 2 = 12(2x - 1)^5$.
29. The product rule gives $f' = 2xe^x + x^2e^x = e^x(x^2 + 2x)$.
30. The quotient rule gives $f' = [(x - 1) - (x + 1)]/(x - 1)^2 = -2/(x - 1)^2$.
31. $y' = 4x^3 - 12x^2$, so $y'' = 12x^2 - 24x$.
32. $f' = -2e^{-2x}$ and $f'' = 4e^{-2x}$.
33. $f' = 2/(2x + 1)$, so $f'' = -4/(2x + 1)^2 < 0$. The graph is concave down throughout its domain.
34. $f'' = 2x$. The gradient increases for $x > 0$; f is concave down for $x < 0$.
35. $f'(x) = 2x - 6 = 0 \Rightarrow x = 3$. Since $f'' = 2 > 0$, $(3, -4)$ is a local minimum.

36. $f'' = 6x$ changes from negative to positive at 0. The point is (0,0) and $f'(0) = 0$, so the tangent is horizontal.
37. $f'' = 12x^2 > 0$ on both sides of 0: concavity does not change. It is a minimum, not an inflexion point.
38. Concave down for $x < 2$; concave up for $x > 2$. The concavity changes at $x = 2$.
39. $f'' = 6x$ changes sign at 0; inflexion (0,2) with gradient $f'(0) = -3$.
40. No. For x^4 it is a minimum; for x^3 it is a horizontal inflexion. Both have first and second derivatives zero at 0.
41. $v = s' = 3t^2 - 8t + 5$ m/s; $a = v' = 6t - 8$ m/s².
42. Speed 7 m/s. It is slowing down because velocity and acceleration have opposite signs.
43. $v = 3t^2 + C$; $C = 4$, so $v = 3t^2 + 4$ m/s.
44. No. It is instantaneously at rest at 2, but $v > 0$ on both sides, so it continues in the positive direction.
45. Displacement $= \int_a^b v(t) dt$; total distance $= \int_a^b |v(t)| dt$.
46. Displacement $12 - 5 + 3 = 10$ m. Total distance $12 + 5 + 3 = 20$ m.
47. Use $\int x^n dx = x^{n+1}/(n+1) + C$, $n \neq -1$. The result is $\frac{9}{5}x^{5/3} + 8x^{-1/2} + 2x + C$, for $x > 0$.
48. $8 \ln |x| + C$, on an interval excluding 0. The power rule with $n \neq -1$ cannot be used when $n = -1$.
49. With $u = 2x - 1$, $du = 2 dx$: $\frac{1}{2} \int u^4 du = \frac{1}{10}(2x - 1)^5 + C$.
50. $-\frac{1}{2}e^{4-2x} + C$, because the inner derivative is -2 .
51. $u = x^2 + 1$, $du = 2x dx$, so $3 \int u^2 du = u^3 + C = (x^2 + 1)^3 + C$.
52. $u = \cos x$, $du = -\sin x dx$. Hence $-\int du/u = -\ln |\cos x| + C$, where $\cos x \neq 0$.
53. $[x^3 - x^2 + x]_{-1}^2 = 6 - (-3) = 9$.
54. $[\frac{1}{2}e^{2x}]_0^{\ln 3} = \frac{1}{2}(9 - 1) = 4$.
55. $[\frac{1}{2} \sin(2x)]_0^{\pi/6} = \frac{\sqrt{3}}{4}$.
56. $[x^2 + x]_1^0 = -2$. Reversing the limits changes the sign of a definite integral.
57. (a) $g(3) - g(-1) = 7 - 4 = 3$. (b) $g(1) - g(3) = -2 - 7 = -9$.
58. $\int_0^5 f = 5 - 3 = 2$, so the required value is $2(2) + (5 - 0) = 9$.
59. (a) $[x^2/2 - x]_0^3 = \frac{3}{2}$. (b) $A = \int_0^1 (1 - x) dx + \int_1^3 (x - 1) dx = \frac{1}{2} + 2 = \frac{5}{2}$.
60. $A = \int_0^2 (2x - x^2) dx = [x^2 - x^3/3]_0^2 = \frac{4}{3}$.

2. Extended response (Q61–120)

61. The values approach 12. Direct substitution gives 0/0, which is undefined; the table describes values close to, but not at, $x = 2$. No formal limit manipulation is needed.
62. Average rate $= [F(4) - F(3)]/1 = 5$ L/min. $F'(t) = -2t + 12$; $F'(3) = 6$ L/min, so the volume is increasing.
63. $f(4 + h) \approx f(4) + hf'(4)$, so $f(4.08) \approx 12 + 0.08(2.5) = 12.2$. The tangent only closely follows a smooth curve near the point of contact.
64. $f'(3) \approx (6.0401 - 5.9601)/0.02 = 4$. The positive gradient, and increasing table values, indicate that the function is increasing locally.
65. $f' = 2x \ln x + x$; $f'' = 2 \ln x + 3$. At $x = 1$, $f = 0$ and $f' = 1$, so the tangent is $y = x - 1$.
66. $f' = e^{2x}(2x - 1)$ and $f'' = 4xe^{2x}$. The stationary value is $x = 1/2$, with $f = -e/2$. Since $f''(1/2) = 2e > 0$, it is a minimum.
67. $f' = [(x + 2) - x]/(x + 2)^2 = 2/(x + 2)^2$. At (0,0), the tangent gradient is $1/2$. Tangent $y = x/2$; normal $y = -2x$.
68. $f' = 3 \cos 3x - 2 \sin 2x$, $f'' = -9 \sin 3x - 4 \cos 2x$. At 0, $f = 1$ and $f' = 3$, so $y = 3x + 1$.

- 69.** $f' = (2x + 3)^{-1/2} > 0$, and $f'' = -(2x + 3)^{-3/2} < 0$. It is increasing and concave down throughout its domain.
- 70.** $y' = 3/(3x + 1)$; at the origin the tangent is $y = 3x$. For $3x + 1 = 1.06$, $x = .02$, giving 0.06. Since $y'' = -9/(3x + 1)^2 < 0$, the concave-down curve lies below its tangent, so this is an overestimate.
- 71.** $f' = x(x + 2)/(x + 1)^2$. Its zeros give $(-2, -4)$ and $(0, 0)$. From $f = x - 1 + (x + 1)^{-1}$, $f' = 1 - (x + 1)^{-2}$, so $f'' = 2/(x + 1)^3$.
- 72.** $f' = 2x/(x^2 + 1)$ and $f'' = 2(1 - x^2)/(x^2 + 1)^2$. The stationary point $(0, 0)$ is a minimum. Concave up for $-1 < x < 1$ and concave down for $x < -1$ or $x > 1$.
- 73.** $f' = 6x(x^2 + 1)^2$. By the product and chain rules, $f'' = 6(x^2 + 1)^2 + 24x^2(x^2 + 1) = 6(x^2 + 1)(5x^2 + 1) > 0$, so the graph is concave up everywhere.
- 74.** The point gives $a = 2$. Since $y' = ae^x + b$, at $x = \ln 2$ we have $4 + b = 0$, so $b = -4$. The horizontal tangent is $y = 4 - 4 \ln 2$.
- 75.** The product rule gives $f' = \cos^2 x - \sin^2 x = \cos 2x$. Alternatively $f = \frac{1}{2} \sin 2x$, giving the same derivative. Thus $f'' = -2 \sin 2x$.
- 76.** $f' = -2xe^{-x^2}$, $f'' = (4x^2 - 2)e^{-x^2}$. It changes sign at $x = \pm 1/\sqrt{2}$, so the points of inflexion are $(\pm 1/\sqrt{2}, e^{-1/2})$.
- 77.** $f'(x) = 3x^2 - 4x$, so $f(2) = 4$, $m_t = 4$. Tangent: $y = 4x - 4$. Normal gradient $m_n = -1/m_t = -\frac{1}{4}$; normal: $y = -\frac{1}{4}x + \frac{9}{2}$.
- 78.** $f'(x) = 6x - 2x^{-2}$, so $f(1) = 5$, $m_t = 4$. Tangent: $y = 4x + 1$. Normal gradient $m_n = -1/m_t = -\frac{1}{4}$; normal: $y = -\frac{1}{4}x + \frac{21}{4}$.
- 79.** Point condition: $1a + 1b = 5$. Gradient condition from $y' = 2ax + b$: $2a + b = 7$. Solving gives $a = 2$, $b = 3$.
- 80.** $m = 2x = 4$ and the point is $(2, 10)$. Tangent $y = 4x + 2$, so intercepts are $(-\frac{1}{2}, 0)$ and $(0, 2)$. Triangle area $= \frac{1}{2} |-\frac{1}{2}| (2) = \frac{1}{2}$ square units.
- 81.** $f(x) = 2x^3 + 2x + C$. Substituting $x = 0$ gives $C = 3$, hence $f(x) = 2x^3 + 2x + 3$ and $f(2) = 23$.
- 82.** $f(x) = 3x - 8x^{-1} + C$. Substituting $x = 1$ gives $C = 9$, hence $f(x) = 3x + 9 - 8x^{-1}$ and $f(2) = 11$.
- 83.** $f' = 4x(x^2 - 4)$ and $f'' = 12x^2 - 16$. Minima $(\pm 2, -16)$; maximum $(0, 0)$. Inflexions $(\pm 2/\sqrt{3}, -80/9)$, since f'' changes sign. Concave up for $|x| > 2/\sqrt{3}$; down for $|x| < 2/\sqrt{3}$.
- 84.** Signs of f' are $-, +, +$ across $-2, 1$: minimum at -2 ; neither extremum at 1 . Expanding gives $f' = x^3 - 3x + 2$, so $f'' = 3(x - 1)(x + 1)$. It changes sign at both -1 and 1 ; $f'(1) = 0$, so 1 is a horizontal inflexion. The other concavity change is at $x = -1$.
- 85.** $f' = 1 - 4/x^2 = 0 \Rightarrow x = 2$. It is negative for $0 < x < 2$, positive for $x > 2$, so $(2, 4)$ is the absolute minimum. $f'' = 8/x^3 > 0$: concave up throughout.
- 86.** $f' = e^{-x}(1 - x)$ changes $+$ $\rightarrow$ $-$ at 1 , giving maximum $(1, e^{-1})$. $f'' = e^{-x}(x - 2)$ changes sign at 2 , giving inflexion $(2, 2e^{-2})$. Concave down for $0 < x < 2$, up for $x > 2$.
- 87.** $f' = 1/x - 1/2 = 0$ at $x = 2$. $f'' = -1/x^2 < 0$ for all $x > 0$, so the maximum is $(2, \ln 2 - 1)$ and concavity never changes.
- 88.** $f' = 3(x + 1)(x - 2) = 3x^2 - 3x - 6$. Comparing with $3x^2 + 2ax + b$ gives $a = -3/2$, $b = -6$. Since $f'' = 6x - 3$, $x = -1$ gives a maximum and $x = 2$ a minimum.
- 89.** $h = 64/x^2$, so $S = 2x^2 + 4xh = 2x^2 + 256/x$. $S' = 4x - 256/x^2 = 0 \Rightarrow x^3 = 64 \Rightarrow x = 4$, $h = 4$. $S'' = 4 + 512/x^3 > 0$; minimum area 96 cm^2 .
- 90.** Let x be each perpendicular side; the parallel side is $40 - 2x$. $A = 40x - 2x^2$, $0 < x < 20$. $A' = 40 - 4x = 0$ at 10 ; $A'' = -4 < 0$. Dimensions 10 m by 20 m ; area 200 m^2 .
- 91.** $V = x(12 - 2x)^2$, $0 < x < 6$. $V' = 12(x - 2)(x - 6)$; it changes from $+$ to $-$ at $x = 2$. Dimensions $8 \times 8 \times 2 \text{ cm}$; maximum volume 128 cm^3 .
- 92.** $A = 2x(12 - x^2) = 24x - 2x^3$, $0 < x < \sqrt{12}$. $A' = 24 - 6x^2 = 0 \Rightarrow x = 2$; $A'' = -12x < 0$. Height 8 , width 4 ; maximum area 32 square units.
- 93.** $D^2 = x^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$. Its derivative is $2x(2x^2 - 5)$; minima at $x = \pm\sqrt{5/2}$. Points $(\pm\sqrt{5/2}, 5/2)$; distance $\sqrt{11}/2$. The square root is increasing, so minimizing D^2 minimizes D .

94. f' changes $+$ $\rightarrow$ $-$ at -1 (maximum), then $- \rightarrow +$ at 2 (minimum). f increases for $x < -1$ and $x > 2$. The derivative graph decreases for $x < 1/2$ and increases for $x > 1/2$; hence f is concave down then up, with inflexion at $x = 1/2$.

95. Rest times in the interior: 2, 4 s. The velocity signs on successive intervals are positive, negative, positive. $s(t) = \frac{1}{3}t^3 - 3t^2 + 8t + 3$; $s(5) = \frac{29}{3}$ m, displacement $\frac{20}{3}$ m. Splitting $\int_0^5 |v(t)| dt$ at each rest time gives distance $\frac{28}{3}$ m.

t	0	2	4	5		
$v(t)$		+	0	-	0	+

96. $v = 2t^2 - 6t + 4 = 2(t-1)(t-2)$; $s = \frac{2}{3}t^3 - 3t^2 + 4t + 2$. Rest at 1, 2 s; signs $+$, $-$, $+$. Displacement $s(3) - s(0) = 3$ m. Positions $s(0) = 2$, $s(1) = 11/3$, $s(2) = 10/3$, $s(3) = 5$. Distance $5/3 + 1/3 + 5/3 = 11/3$ m.

97. $s = 3\sin t + 2$. The velocity changes $+$ $\rightarrow$ $-$ at $t = \pi/2$. Displacement 0 m; distance $3 + 3 = 6$ m; average speed $6/\pi$ m/s.

98. $s = e^t - 2t - 1$. Rest at $t = \ln 2$, with signs $- \rightarrow +$. Displacement $3 - 4\ln 2$ m. At rest $s = 1 - 2\ln 2$; distance $-(1 - 2\ln 2) + (2 - 2\ln 2) = 1$ m.

99. $v = (t-1)(t-3)$ and $a = 2t - 4$. Speed increases when $va > 0$: $1 < t < 2$ and $3 < t < 4$. Candidate speeds at $t = 0, 1, 2, 3, 4$ are 3, 0, 1, 0, 3 m/s; greatest 3 m/s. At $t = 2$, acceleration is zero but velocity is -1 m/s.

100. At the origin: $t = 0, 3$. $v = 3(t-1)(t-3)$, so rest at 1, 3. At 3 it reaches the origin and reverses from negative to positive motion. Positions at 0, 1, 3, 4 are 0, 4, 0, 4; distance $4 + 4 + 4 = 12$ m.

101. $t^2 = 6t - 5$ gives $t = 1, 5$ s. $v_P = 2t$, $v_Q = 6$: velocities (2, 6) m/s at 1 and (10, 6) m/s at 5. Separation at 3: $|9 - 13| = 4$ m.

102. $v = u + at$ and $s = ut + \frac{1}{2}at^2$. Thus $2u + 2a = 10$ and $4u + 8a = 28$. Solving: $a = 2$ m/s², $u = 3$ m/s; $v(3) = 9$ m/s.

103. Direction changes at $t = 3$. Areas: $+4, +2, -2, -4$ m. Displacement 0 m; total distance 12 m. Acceleration $(-4 - 4)/(4 - 2) = -4$ m/s².

104. $v = 3(t-2)^2 \geq 0$, positive on both sides of 2, so there is no direction change. $a = 6t - 12$ gives $a(2) = 0$. Since motion is nonnegative, distance $s(4) - s(0) = 15 - (-1) = 16$ m.

105. Integrate powers: $f = 2x^{3/2} + 2x^{-1} + C$. Since $5 = 2 + 2 + C$, $C = 1$. Thus $f(x) = 2x^{3/2} + 2/x + 1$; substitution gives $f(1) = 5$.

106. $f = 2\ln|2x-3| + C$. Since $2x-3 > 0$, this is $2\ln(2x-3) + C$. At $x = 2$, $7 = 2\ln 1 + C$, so $f = 2\ln(2x-3) + 7$.

107. $f = \frac{5}{2}\sin(2x) + 3\cos x + C$. The condition gives $1 = 0 + 3 + C$, so $C = -2$. Thus $f(\pi/2) = -2$.

108. The proposed derivative is $8(2x+5)^3$, twice the required integrand. The chain-rule factor 2 was not compensated for. The correct answer is $\frac{1}{2}(2x+5)^4 + C$.

109. $du = 2x dx$, so the integral becomes $2 \int u^{-2} du = -2u^{-1} + C = -2/(x^2+3) + C$.

110. $u = x^3 + 2$ and $du = 3x^2 dx$. Thus $\int u^{1/2} du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(x^3+2)^{3/2} + C$, where $x^3+2 > 0$.

111. The numerator is the derivative of the denominator, so the integral is $\ln|x^2-x+5| + C$. Since $x^2-x+5 = (x-\frac{1}{2})^2 + \frac{19}{4} > 0$, it is $\ln(x^2-x+5) + C$.

112. Let $u = 2 + \cos x$, so $du = -\sin x dx$. The result is $-u^5/5 + C = -\frac{1}{5}(2 + \cos x)^5 + C$. Differentiating multiplies by $5(2 + \cos x)^4(-\sin x)$, giving the required positive integrand.

113. $du = 3dx$; $x = 0 \Rightarrow u = -1$, $x = 2 \Rightarrow u = 5$. Hence $\frac{1}{3} \int_{-1}^5 u^2 du = [u^3/9]_{-1}^5 = 14$.

114. $u = 2x^2 + 1$, $du = 4x dx$; limits 1, 3. The integral is $\int_1^3 du/u = \ln 3$. Since $2x^2 + 1 \geq 1$, the argument is positive.

115. Zeros: $-1, 0, 1$. With $F = x^4/4 - x^2/2$, the integral is $F(2) - F(-1) = \frac{9}{4}$. The signs are $+, -, +$, so $A = \int_{-1}^0 f dx - \int_0^1 f dx + \int_1^2 f dx = \frac{1}{4} + \frac{1}{4} + \frac{9}{4} = \frac{11}{4}$.

- 116.** $x^2 = x + 2 \Rightarrow (x - 2)(x + 1) = 0$, giving $(-1, 1)$ and $(2, 4)$. The line is upper, so $A = \int_{-1}^2 (x + 2 - x^2) dx = [x^2/2 + 2x - x^3/3]_{-1}^2 = \frac{9}{2}$.
- 117.** Intersections satisfy $x^2 = 4$, so $x = \pm 2$. Thus $A = \int_{-2}^2 [(-x^2 - 1) - (-5)] dx = \int_{-2}^2 (4 - x^2) dx = \frac{32}{3}$. The difference measures a positive vertical separation.
- 118.** $\sin x = \cos x$ at $x = \pi/4$. Thus $A = \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx = 2(\sqrt{2} - 1) = 2\sqrt{2} - 2$.
- 119.** The total area is $\int_0^4 x dx = 8$. Hence $\int_0^c x dx = 4$, so $c^2/2 = 4$ and $c = 2\sqrt{2}$.
- 120.** The integrals are $6 - 2 = 4$ and $6 - 5 = 1$. The table determines net changes, but not any additional sign changes or cancellation within the subintervals. Therefore it does not determine $\int_{-1}^4 |F'(x)| dx$.

3. Paper 1-style examination practice (Q121–160)

- 121.** $f' = x(2 \ln x + 1)$ and $f'' = 2 \ln x + 3$. The stationary point is $(e^{-1/2}, -1/(2e))$, a minimum because $f'' = 2$ there. Concavity changes at $x = e^{-3/2}$, giving inflexion $(e^{-3/2}, -3/(2e^3))$.
- 122.** $f' = [e^x(x + 1) - e^x]/(x + 1)^2 = xe^x/(x + 1)^2$. Its sign changes $- \rightarrow +$ at 0, giving minimum $(0, 1)$. At 1, $f = e/2$, $f' = e/4$, so $y - e/2 = (e/4)(x - 1)$.
- 123.** $f' = \cos x + \cos 2x = 2 \cos^2 x + \cos x - 1 = (2 \cos x - 1)(\cos x + 1)$. The interior solution is $x = \pi/3$. At the endpoints $f = 0$; at $\pi/3$, $f = 3\sqrt{3}/4$, the maximum.
- 124.** From the value and derivatives: $a + b + c = -2$, $3a + 2b + c = 0$, $6a + 2b = 6$. Solving gives $a = 1, b = 0, c = -3$. Since $f''(1) > 0$, $(1, 0)$ is a local minimum.
- 125.** $f(1) = 2$, $f' = 1/(2\sqrt{x + 3})$, so the gradient is $1/4$. Tangent $y - 2 = (x - 1)/4$; normal $y - 2 = -4(x - 1)$. The normal meets the axis at $(3/2, 0)$. Since $f'' = -1/[4(x + 3)^{3/2}] < 0$, the graph is concave down.
- 126.** $f' > 0$ on $(-1, 3)$ and is negative outside. Integrating $-x^2 + 2x + 3$ gives $f = -x^3/3 + x^2 + 3x + 2$. Since $f'' = -2x + 2$ changes sign at 1, the point of inflexion is $(1, 17/3)$.
- 127.** $v = 2/(t + 1) - 1$ and $a = -2/(t + 1)^2$. Direction changes from positive to negative at $t = 1$. Since $s(0) = 0$, $s(1) = 2 \ln 2 - 1$, $s(3) = 4 \ln 2 - 3$, total distance is $[s(1) - s(0)] + [s(1) - s(3)] = 1$ m.
- 128.** $f = x^4 + 2x^2 + 1 = (x^2 + 1)^2$. At $(1, 4)$ the gradient is 8, giving tangent $y = 8x - 4$. The curve is upper because $f'' = 12x^2 + 4 > 0$. Hence $A = \int_0^1 (x^4 + 2x^2 - 8x + 5) dx = 1/5 + 2/3 - 4 + 5 = 28/15$.
- 129.** (a) $f' = 3(x + 3)(x - 1)$; points $(-3, 31)$ and $(1, -1)$. (b) $f'' = 6x + 6$ is negative at -3 and positive at 1: maximum then minimum. (c) f'' changes sign at -1 , giving $(-1, 15)$; $f'(-1) = -12 \neq 0$. (d) $y - 15 = -12(x + 1)$, so $y = -12x + 3$.
- 130.** (a) $f' = 4x^2(x - 3)$: $(0, 0)$, $(3, -27)$. (b) f' has signs $-, -, +$, so 0 is neither extremum; 3 is a minimum. (c) $f'' = 12x(x - 2)$ changes sign at 0, 2: inflexions $(0, 0)$ and $(2, -16)$. Concave up for $x < 0$ and $x > 2$.
- 131.** (a) $-2 < x < 3$. (b) Maximum at -2 ($+$ $\rightarrow$ $-$); minimum at 3 ($-$ $\rightarrow$ $+$). (c) $f'' = 2x - 1$ changes sign at $x = 1/2$, so this is the inflexion abscissa.

x	$-\infty$	-2	3	$+\infty$		
$f'(x)$		$+$	0	$-$	0	$+$
$f(x)$			Max	Min		

- 132.** (a) $x^2 h = 108 \Rightarrow h = 108/x^2$. Without a lid, $S = x^2 + 4xh = x^2 + 432/x$. (b) $S' = 2x - 432/x^2 = 0 \Rightarrow x^3 = 216$, so $x = 6$, $h = 3$. (c) $S'' = 2 + 864/x^3 > 0$; minimum $S = 108 \text{ cm}^2$.
- 133.** (a) $\pi r^2 h = 54\pi \Rightarrow h = 54/r^2$; $S = 2\pi r^2 + 2\pi r h$. (b) $S' = 4\pi r - 108\pi/r^2 = 0 \Rightarrow r^3 = 27$. Thus $r = 3$, $h = 6$ cm. (c) $S'' = 4\pi + 216\pi/r^3 > 0$ for $r > 0$; minimum $54\pi \text{ cm}^2$.
- 134.** (a) Printed height $100/x$; $A = (x + 2)(100/x + 2) = 104 + 2x + 200/x$. (b) $A' = 2 - 200/x^2 = 0 \Rightarrow x = 10$. Poster dimensions 12 cm by 12 cm. (c) $A'' = 400/x^3 > 0$.

- 135.** (a) Product rule: $f' = e^{-2x} - 2xe^{-2x}$. (b) The derivative changes $+$ $\rightarrow$ $-$ at $x = 1/2$; maximum $(1/2, 1/(2e))$. (c) $f'' = 4e^{-2x}(x - 1)$ changes sign at 1; inflexion $(1, e^{-2})$. Concave down for $0 < x < 1$, up for $x > 1$.
- 136.** (a) Quotient rule gives $f' = (1 - \ln x)/x^2$. (b) $x = e$; sign changes $+$ $\rightarrow$ $-$, so maximum $(e, 1/e)$. (c) $2 \ln x - 3 = 0 \Rightarrow x = e^{3/2}$. The numerator increases through zero and $x^3 > 0$, so concavity changes; point $(e^{3/2}, 3/(2e^{3/2}))$.
- 137.** (a) $f''(1) = 6 + 2a = 0 \Rightarrow a = -3$. $f'(0) = b = 0$. $f(1) = 1 - 3 + c = 4 \Rightarrow c = 6$. (b) $f' = 3x(x - 2)$, $f'' = 6x - 6$; maximum $(0, 6)$, minimum $(2, 2)$.
- 138.** (a) Denominator positive; signs $-$, $+$, $-$ across $-1, 1$. Minimum $(-1, -1/2)$; maximum $(1, 1/2)$. (b) y'' has signs $-$, $+$, $-$, $+$ across $-\sqrt{3}, 0, \sqrt{3}$. Inflexions $(-\sqrt{3}, -\sqrt{3}/4), (0, 0), (\sqrt{3}, \sqrt{3}/4)$. (c) None: the gradients are $-1/8, 1, -1/8$.
- 139.** $a(t) = 6t - 18$ m/s². Rest at 2, 4 s. $s(t) = t^3 - 9t^2 + 24t + 2$ m. Displacement $\int_0^5 v(t) dt = 20$ m. Split at the rest times and add absolute changes in position: total distance 28 m.
- 140.** $v = 6t^2 - 12t + 6 = 6(t - 1)^2$; $s = 2t^3 - 6t^2 + 6t + 1$. Rest only at 1 s; velocity stays positive on either side, so no change of direction. Distance $s(3) - s(0) = 19 - 1 = 18$ m.
- 141.** $v = 3(t - 1)(t - 3)$ and $a = 6(t - 2)$. Rest at 1, 3 s; positions 8, 4 m. Speed increases when $va > 0$: $1 < t < 2$ and $3 < t < 4$. Candidate speeds at 0, 1, 2, 3, 4 are 9, 0, 3, 0, 9 m/s; maximum speed 9 m/s at 0, 4.
- 142.** $a = 4 \cos t$; $s = 7 - 4 \cos t$. Direction changes at $t = \pi$ (positive to negative). Displacement is 0; distance $8 + 8 = 16$ m; average speed $16/(2\pi) = 8/\pi$ m/s.
- 143.** $s = 3 - 3e^{-t} - t$. Direction changes at $\ln 3$. Displacement at $\ln 9$ is $8/3 - 2 \ln 3$ m. The turning position is $2 - \ln 3$; distance $2(2 - \ln 3) - (8/3 - 2 \ln 3) = 4/3$ m.
- 144.** Meet when $t^2 - 6t + 5 = 0$, so $t = 1, 5$ s. $v_P = 2t - 2$, $v_Q = 4$; velocities $(0, 4)$ and $(8, 4)$ m/s. On $[1, 5]$, separation $s_Q - s_P = -t^2 + 6t - 5$ is greatest at $t = 3$: 4 m.
- 145.** Reversal at $t = 3$. Signed areas 4, +1, -1, -4 m: displacement 0, distance 10 m. Accelerations 0, -2, 0 m/s² on $(0, 2), (2, 4), (4, 6)$. At $t = 2, 4$ the one-sided slopes differ, so the graph has no derivative.
- 146.** $s = 6 - 6/(t + 1) - t$. The positive turning time is $r = \sqrt{6} - 1$; $s(r) = 7 - 2\sqrt{6}$. Final position $s(3) = 3/2$. Distance $2(7 - 2\sqrt{6}) - 3/2 = 25/2 - 4\sqrt{6}$ m.
- 147.** Integrating term by term gives $f = x^4 - 6\sqrt{x} + 2 \ln x + C$. From $f(1) = 2$, $2 = 1 - 6 + C$, so $C = 7$. The tangent gradient is $f'(1) = 4 - 3 + 2 = 3$; tangent $y - 2 = 3(x - 1)$, or $y = 3x - 1$.
- 148.** $f = 2 \ln(3x - 1) + C$. Since $4 = 2 \ln 2 + C$, $f = 2 \ln[(3x - 1)/2] + 4$. Setting $f = 6$ gives $\ln[(3x - 1)/2] = 1$, hence $x = (1 + 2e)/3$.
- 149.** $y = -4e^{-2x} + \sin(3x) + C$. The point gives $1 = -4 + C$, so $C = 5$. At 0, the gradient is $8 + 3 = 11$; tangent $y = 11x + 1$.
- 150.** Let $u = x^3 - 2x$, so $du = (3x^2 - 2)dx$. Then $F = e^{x^3 - 2x} + C$; $3 = 1 + C$ gives $C = 2$. Hence $F(2) = e^4 + 2$. The chain rule gives $F' = e^{x^3 - 2x}(3x^2 - 2)$.
- 151.** $u = 1 + e^{2x}$, $du = 2e^{2x}dx$, so $F = \frac{1}{2} \ln(1 + e^{2x}) + C$. The condition gives $C = 2 - \frac{1}{2} \ln 2$. At $x = \ln 2$, $e^{2x} = 4$, hence $F(\ln 2) = 2 + \frac{1}{2} \ln(5/2)$.
- 152.** $F = a \ln(x + 1) + x^2 + C$. From $F(0) = 1$, $C = 1$. From $F(1) = 4$, $a \ln 2 + 1 + 1 = 4$, giving $a = 2/\ln 2$. Hence $F = \frac{2}{\ln 2} \ln(x + 1) + x^2 + 1$.
- 153.** $f = 2(x - 1)^2 + 1$, with minimum 1. The integral is $[2x^3/3 - 2x^2 + 3x]_{-1}^2 = 10/3 - (-17/3) = 9$. Since $f > 0$ throughout, signed and total area coincide.
- 154.** The intersections are $(-3, 0), (3, 0)$. Since the curve is below the axis, $A = \int_{-3}^3 (9 - x^2) dx = [9x - x^3/3]_{-3}^3 = 36$.
- 155.** $8x - 2x^2 = 2x \Rightarrow 2x(3 - x) = 0$, giving $(0, 0), (3, 6)$. The area is $A = \int_0^3 (6x - 2x^2) dx = [3x^2 - 2x^3/3]_0^3 = 9$. For the tangent, $f' = 8 - 4x = 2$ gives $x = 3/2$, $y = 15/2$. Thus $y = 2x + 9/2$.
- 156.** $\sqrt{x} = x^2$ gives $x = 0$ or $x = 1$, so $(0, 0), (1, 1)$. On $0 < x < 1$, $\sqrt{x} > x^2$. Thus $A = \int_0^1 (x^{1/2} - x^2) dx = [\frac{2}{3}x^{3/2} - \frac{1}{3}x^3]_0^1 = \frac{1}{3}$.

- 157.** $8/3 - 2k = 0$, so $k = 4/3$. The zero is $a = 2/\sqrt{3}$. The area is $A = \int_0^a (4/3 - x^2) dx + \int_a^2 (x^2 - 4/3) dx$. The signed integral is zero, so $A = 2 \int_0^a (4/3 - x^2) dx = \frac{32\sqrt{3}}{27}$.
- 158.** $\int_0^{3\pi/2} \cos x dx = [\sin x]_0^{3\pi/2} = -1$. The area is $A = \int_0^{\pi/2} \cos x dx - \int_{\pi/2}^{3\pi/2} \cos x dx = 1 - (-2) = 3$.
- 159.** $[x^3/3 - x^2]_0^3 = 0$. The integrand changes sign at $x = 2$, so positive and negative contributions cancel. The area is $A = \int_0^2 (2x - x^2) dx + \int_2^3 (x^2 - 2x) dx = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}$.
- 160.** Let $a = \sqrt{k/2}$; intersections occur at $x = \pm a$. Then $A = \int_{-a}^a (k - 2x^2) dx = 2[kx - 2x^3/3]_0^a = \frac{4}{3}ka$. Setting this equal to $64/3$ gives $k\sqrt{k/2} = 16$, hence $k^3 = 512$ and $k = 8$.

4. Paper 2-style examination practice (Q161–200)

- 161.** $f' = e^x - 3 = 0$ at $x = \ln 3 \approx 1.10$; the minimum value is $3 - 3 \ln 3 \approx -0.296$. Zeros $r_1 \approx 0.619$, $r_2 \approx 1.51$. Then $A = \int_0^{r_1} f dx - \int_{r_1}^{r_2} f dx + \int_{r_2}^2 f dx \approx 0.741$.
- 162.** $f' = 1/x - 0.3 = 0$ at $x = 10/3 \approx 3.33$; $f'' < 0$ gives a maximum, with value 0.204. Endpoint values are $f(0.5) \approx -0.843$ and $f(8) \approx -0.321$; the minimum is -0.843 at 0.5. Zeros $x \approx 1.63, 5.94$.
- 163.** $f' = \sin x + x \cos x$. Interior stationary points: (2.03, 1.82) and (4.91, -4.81). Endpoint values are zero, so the greatest value is 1.82 and the least is -4.81. The integral is ≈ -6.28 .
- 164.** $f' = e^{-0.4x}(0.6 - 0.4x)$ gives maximum (1.50, 1.37) after checking endpoints. Also $f'' = e^{-0.4x}(0.16x - 0.64)$, so the inflexion is (4.00, 1.01). Concave down on (0, 4) and up on (4, 10).
- 165.** The point is (0.600, 1.43) and $m = 2xe^{x^2} \approx 1.72$. Tangent $y - 1.43 = 1.72(x - 0.6)$; normal $y - 1.43 = -0.581(x - 0.6)$ (rounded coefficients). Its x -intercept is $x = 0.6 + my_0 \approx 3.07$, using the unrounded point height y_0 and gradient.
- 166.** The tangent equation at a passes through (0, 0) when $f(a) = af'(a)$, giving the stated equation. Its positive solution is $a \approx 1.98$; the tangent is $y \approx 0.805x$. Area $A = \int_0^a \ln(1 + x^2) dx \approx 1.40$.
- 167.** The intersection is (1.33, 0.265). If its x -coordinate is r , then $A = \int_0^r (e^{-x} - 0.2x) dx \approx 0.559$.
- 168.** The curves cross at $r \approx 0.972$. Thus $A = \int_0^r (e^{-x^2} - 0.4x) dx + \int_r^2 (0.4x - e^{-x^2}) dx \approx 1.01$.
- 169.** The zero is $r \approx 1.77$. Since $f' = 3x^2 - 2$, the stationary point is (0.816, -3.09), a minimum. Area $A = -\int_0^r f dx + \int_r^2 f dx \approx 4.44$.
- 170.** $V = x(18 - 2x)(11 - 2x)$, $0 < x < 5.5$. Solve $V' = 198 - 116x + 12x^2 = 0$ in the domain: $x \approx 2.21$ cm. The derivative changes $+$ $\rightarrow$ $-$ and endpoint volumes are zero. Maximum $V \approx 197$ cm³; dimensions $13.6 \times 6.57 \times 2.21$ cm.
- 171.** From $h = 330/(\pi r^2)$, $S = 2\pi r^2 + 660/r$. Solve $S' = 4\pi r - 660/r^2 = 0$: $r \approx 3.74$ cm and $h \approx 7.49$ cm. Since $S'' = 4\pi + 1320/r^3 > 0$, this is a minimum; $S \approx 264$ cm².
- 172.** The height is $h = 1200/x^2$. Top and base cost $0.04x^2$; sides cost $0.01(4xh) = 48/x$. Then $C' = 0.08x - 48/x^2 = 0$ gives $x^3 = 600$. Thus $x \approx 8.43$ cm, $h \approx 16.9$ cm. Since $C'' = 0.08 + 96/x^3 > 0$, the minimum cost is 8.54 euros.
- 173.** $P = 50qe^{-0.02q} - 4q - 100$; $P' = 50e^{-0.02q}(1 - 0.02q) - 4$. The maximum occurs at $q \approx 40.9$, with price $p \approx 22.1$ euros. Check endpoints as well. Break-even outputs are $q \approx 2.28, 117$.
- 174.** The interior maximum solves $P' = -0.9n^2 + 16n + 10 = 0$, giving $n \approx 18.4$. Compare neighbours: $P(18) = 972.40$, $P(19) = 970.30$, and the endpoints. The best integer output is 18 with profit 972.40 euros.
- 175.** $D^2 = (x - 2)^2 + (\ln x)^2$. Solve $d(D^2)/dx = 2(x - 2) + 2 \ln x/x = 0$ and compare endpoints. The minimum is at $P \approx (1.69, 0.524)$, with $D \approx 0.609$. Minimizing D^2 also minimizes non-negative D .
- 176.** $A = 6xe^{-0.2x^2}$, so $A' = 6e^{-0.2x^2}(1 - 0.4x^2)$. Its sign changes $+$ $\rightarrow$ $-$ at $x = \sqrt{2.5} \approx 1.58$. Height $y \approx 3.64$; maximum area $A \approx 5.75$.
- 177.** The direction changes at $r \approx 1.88$ s, from positive to negative velocity. Displacement $\int_0^5 v dt \approx -6.50$ m. Distance $\int_0^r v dt - \int_r^5 v dt \approx 13.4$ m. Acceleration $a = -2.1 \sin(0.7t) - 0.4$ gives $a(r) \approx -2.43$ m/s².

- 178.** Integrating and using $s(0) = 2$ gives $s = 2 + 2t + 0.25t^2 + 3(e^{-t} - 1)$. Rest occurs at $r \approx 0.327$ s. Since $v' = 0.5 + 3e^{-t} > 0$, velocity changes $- \rightarrow +$ there, giving least position 1.84 m. Distance $-\int_0^r v dt + \int_r^5 v dt \approx 13.6$ m.
- 179.** $v = 4/(t+1) - t = 0$ gives $r = (\sqrt{17} - 1)/2 \approx 1.56$ s and $s(r) \approx 2.54$ m. The positive root of $s = 0$ is $t \approx 3.46$ s. Average speed is $[\int_0^r v dt - \int_r^5 v dt]/5 \approx 2.08$ m/s.
- 180.** Solve $t^2 = 8(1 - e^{-t})$: the positive solution is $t \approx 2.74$ s. Velocities are $v_P = 2t \approx 5.47$ m/s and $v_Q = 8e^{-t} \approx 0.519$ m/s. Particle P has the greater velocity at the crossing, so it is farther along immediately afterwards.
- 181.** $V = 20 + 40(1 - e^{-0.3t}) - t$. The rate changes $+ \rightarrow -$ at $t = \ln 12/0.3 \approx 8.28$ min, giving maximum volume 48.4 L. Finally $V(12) \approx 46.9$ L.
- 182.** $\int_0^b e^{-x^2} dx = 0.7$ gives $b \approx 0.886$. Since the integrand is positive, extending the right-hand boundary strictly increases the area, so the solution is unique.
- 183.** Integrating gives $[-e^{-kx}/k]_0^2 = (1 - e^{-2k})/k$. Solving for $k > 0$ gives $k \approx 0.942$. The height at 2 is $e^{-2k} \approx 0.152$.
- 184.** $f' = 2x + a \cos x$; $4 + a \cos 2 = 0$, so $a \approx 9.61$. Stationary points are $(2.00, 12.7)$, a maximum, and $(3.80, 8.56)$, a minimum, as verified by $f'' = 2 - a \sin x$. Comparing endpoints gives greatest value $f(6) \approx 33.3$.
- 185.** $f' = 4x^3 - 6x + 0.5$. Stationary points, in order, are $(-1.26, -2.87)$, $(0.0837, 0.0209)$, $(1.18, -1.65)$, respectively minimum, maximum, minimum. Since $f'' = 12x^2 - 6$, the inflexion points are $(-0.707, -1.60)$ and $(0.707, -0.896)$. Checking $f(-2) = 3$, $f(2) = 5$ and all stationary values gives maximum 5 and minimum -2.87 .
- 186.** $f = 4 - \cos x - 0.1x^2$. The interior stationary point is $(2.60, 4.18)$, a maximum because f' changes $+ \rightarrow -$. Since $f'' = \cos x - 0.2$, the graph is concave up on $(0, 1.37)$ and $(4.91, 7.65)$.
- 187.** The interior zero is $r = \sqrt{\pi} \approx 1.77$. The signed integral is $\int_0^2 f dx \approx 0.805$. The area is $A = \int_0^r f dx - \int_r^2 f dx \approx 0.985$.
- 188.** $\int_0^a (2 + \sin 3x) dx = 2a + (1 - \cos 3a)/3 = 5$. The solution is $a \approx 2.42$. The integrand is at least 1, so accumulated area strictly increases; it also grows without bound.
- 189.** $y' = x/\sqrt{x^2 + 4}$, giving $m \approx 0.545$ and point $(1.30, 2.39)$. The normal is $y - 2.39 = -1.83(x - 1.3)$, with rounded coefficients. Using unrounded values, its x -intercept is $x = 1.3 + my_0 = 2.60$.
- 190.** The equation is $3x^2 - 6x - 2 = 1.7$. Contact points are $(-0.494, 4.13)$ and $(2.49, -4.13)$. Tangents: $y \approx 1.7x + (4.98)$ and $y \approx 1.7x + (-8.38)$.
- 191.** $F = \frac{2}{9}(3x + 2)^{3/2} + C$. The condition gives $C = 1 - 4\sqrt{2}/9$. Thus $F(2) = 1 + 28\sqrt{2}/9 \approx 5.40$. The numerical derivative at 2 is ≈ 2.83 , agreeing with $\sqrt{8}$.
- 192.** With $u = -x^2$, $du = -2x dx$, integration gives $F = -e^{-x^2} + C$. Thus $C = 6$ and $F = 6 - e^{-x^2}$. Solve $e^{-x^2} = 0.2$: $x = \sqrt{\ln 5} \approx 1.27$. For $x > 0$, $F' > 0$, so F is strictly increasing and the positive solution is unique.
- 193.** $A = \int_0^2 e^{-x^2} dx \approx 0.882$. Since $e^{-x^2} > 0$, the curve never crosses the axis.
- 194.** The intersections have $x \approx \pm 0.824$. If r is the positive root, then $A = \int_{-r}^r (\cos x - x^2) dx = 2 \int_0^r (\cos x - x^2) dx \approx 1.09$. Retain the unrounded root in the calculation.
- 195.** $A = \int_0^3 \sqrt{1 + x^3} dx \approx 7.34$ m². Using the unrounded area, the cost is $12A \approx 88.10$ euros.
- 196.** One intersection is $(0, 0)$; the other has $x = b \approx 2.51$, $y \approx 1.26$. The logarithmic curve is upper, so $A = \int_0^b [\ln(x+1) - x/2] dx \approx 0.322$. Use the unrounded value of b .
- 197.** Origin: $t(t-1)(t-3) = 0$, so $t = 0, 1, 3$ s. $v = 3t^2 - 8t + 3 = 0$ gives $t = (4 \pm \sqrt{7})/3 \approx 0.451, 2.22$ s. Split $\int_0^4 |v| dt$ at both times: distance ≈ 17.5 m.
- 198.** Besides $t = 0$, the velocity root is $r \approx 2.60$ s, with signs $+ \rightarrow -$. Displacement $\int_0^4 v dt = 2 - 2 \cos 4 - 3.2 \approx 0.107$ m. Splitting at r gives distance ≈ 4.62 m. $a = 2 \cos t - 0.4$; $a(2) \approx -1.23$ m/s².
- 199.** $f' = e^{-0.4x}(\cos x - 0.4 \sin x)$. Interior stationary values occur at $x \approx 1.19, 4.33$. After comparing endpoints, greatest value 0.577; least -0.164 . Now $f'' = e^{-0.4x}(-0.84 \sin x - 0.8 \cos x)$, giving inflexion points $(2.38, 0.266)$, $(5.52, -0.0757)$. Area $A = \int_0^\pi f dx - \int_\pi^{2\pi} f dx \approx 1.42$.
- 200.** $f' = \ln x$, $f'' = 1/x > 0$. Thus $(1, -1)$ is a minimum and its normal is $x = 1$. For an intersection, $x \ln x - x = 0.4x$ and $x > 0$ give $\ln x = 1.4$, so $b = e^{1.4} \approx 4.06$ and $y \approx 1.62$. On $[1, b]$, g is upper, so $A = \int_1^b [0.4x - (x \ln x - x)] dx \approx 3.16$.