



Optimisation

IB Mathematics: Applications and Interpretation SL | SL 5.7

50 questions: fluency, interpretation and IB-style practice

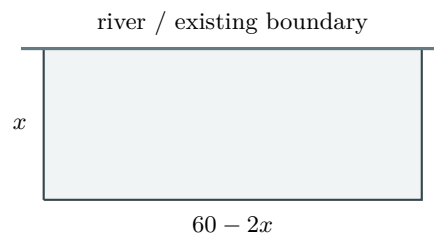
Learning goal	Build and optimize contextual models; check constraints, endpoints and integer decisions.
Syllabus link	AI SL 5.7: optimisation in context. Uses first-derivative and graphical methods from SL 5.6; no kinematics.
Structure	1–15 short response; 16–30 extended response; 31–50 IB exam-style.
Technology	A GDC is permitted throughout, as in AI SL examinations.  marks tasks requiring numerical or graphical technology.

Instructions. Work on separate paper and show your method. Give exact answers unless a decimal is requested or a numerical method is used; give numerical results to 3 significant figures. State your model and feasible domain. Justify each optimum using derivative signs, a graph or endpoint comparisons. Include units and check any integer constraints. Geometric diagrams are not drawn to scale.

1. Short response

15 questions

1. A rectangular garden has perimeter 40 m. If its width is x m, express its length in terms of x .
2. For the garden in Question 1, write its area $A(x)$ and the feasible domain for a non-degenerate garden.
3. Find the value of x that maximizes $A(x) = 20x - x^2$, for $0 \leq x \leq 20$.
4. A farmer fences three sides of a rectangular field beside a straight river. The two sides perpendicular to the river each have length x m; 60 m of fence is used. Express the third side and the area in terms of x .



5. A box is made by cutting squares of side x cm from the corners of a 30 cm by 20 cm sheet and folding up the sides. State the three box dimensions and the feasible domain.
6. For the box in Question 5, write the volume $V(x)$ as a product.
7. A company sells q items for € p each. Its total cost is $C(q)$ euros. Write expressions for revenue and profit.
8. The selling price is $p = 50 - 0.5q$ euros per item. Write the revenue $R(q)$ and state the range of q for which price is nonnegative.
9. A function modelling cost satisfies $C'(x) < 0$ for $x < 6$ and $C'(x) > 0$ for $x > 6$, with $C'(6) = 0$. Explain why $x = 6$ minimizes the cost.

10. A profit model on $[0, 10]$ has only one interior stationary point. The candidate values are shown. Find the greatest profit.

q	0	6	10
$P(q)$ (euros)	-20	160	120

11. A continuous revenue model has its maximum at $q = 12.4$, but only whole items can be sold. Which two integer values should be compared if the model increases up to 12.4 and decreases after it?

12. A closed cylinder has radius r cm, height h cm and volume 500 cm^3 . Express h in terms of r .

13. Using the cylinder in Question 12, show that its total surface area is $S(r) = 2\pi r^2 + 1000/r$.

14. A model gives a stationary point at $x = -4$ for the width of a box. Explain why this cannot be accepted as the optimum.

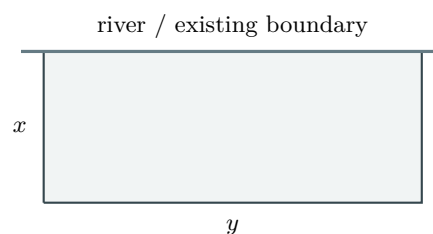
15. A rectangular design has area $A(x) = 30x - 2x^2$, where x is in metres. State the units of $A(x)$ and $A'(x)$.

2. Extended response

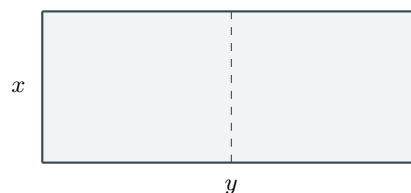
15 questions

16. A rectangle has perimeter 64 cm. Find its dimensions for maximum area, and justify the maximum.

17. A farmer uses 80 m of fencing for three sides of a rectangular enclosure beside a wall. Find the dimensions that maximize the area.



18. Two identical rectangular pens share a dividing fence perpendicular to the total length y . A total of 120 m of fencing is used. Find the overall dimensions that maximize the total enclosed area.



19. Two positive numbers have sum 30. Find the numbers that maximize their product, showing a suitable model and domain.

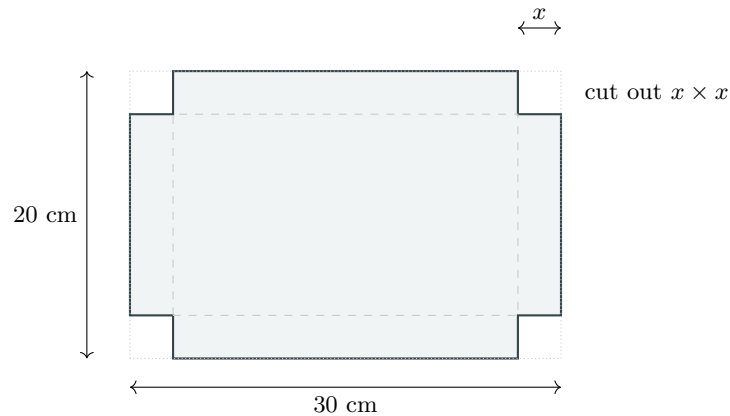
20.  A rectangular poster has fixed area 600 cm^2 . Find the dimensions that minimize its perimeter.

21. A company models price by $p = 80 - 2q$ euros and total cost by $C(q) = 200 + 10q$ euros, for $0 \leq q \leq 40$. Find the production level maximizing continuous-model profit and the corresponding price and profit.

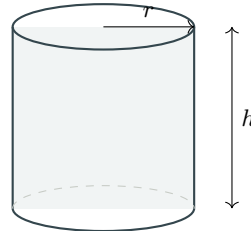
22. Use the model in Question 21 when q must be a whole number. Find every profit-maximizing production level and the profit.

23. A square sheet of side 24 cm is made into an open box by cutting equal corner squares of side x cm. Find the cut size and dimensions giving the maximum volume.

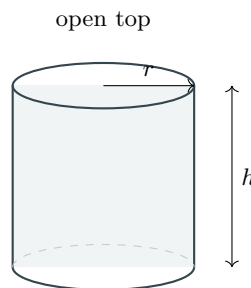
24.  A box without a lid is made from a 30 cm by 20 cm sheet by cutting out corner squares and folding. Use your GDC to find the cut size and maximum volume.



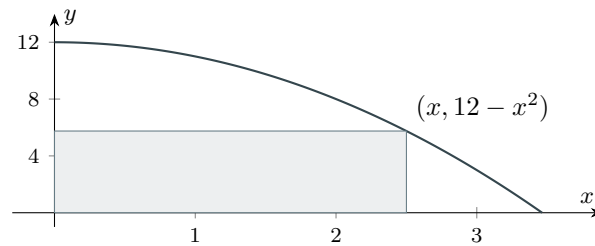
25.  A closed cylindrical can holds 750 cm^3 . Find the radius and height that minimize its total surface area. State the function minimized.



26.  An open cylindrical container holds 400 cm^3 . Find the radius and height that minimize the material area, ignoring thickness.



27. A rectangle lies under $y = 12 - x^2$ in the first quadrant, with one vertex at the origin and opposite vertex $(x, 12 - x^2)$. Find its maximum area.



28. A firm models average cost per item by $A(q) = 0.02q + 200/q + 8$ euros, for $10 \leq q \leq 200$. Find the production level giving the least average cost.

29. A ticket price p euros gives predicted attendance $N(p) = 300 - 5p$, for $0 \leq p \leq 60$. The hall holds at most 180 people. All predicted demand must be accommodated. Find the feasible price range and the price maximizing revenue.

30. A profit model is $P(q) = -q^2 + 50q - 100$, but capacity restricts production to $0 \leq q \leq 20$. Find the greatest possible profit and explain why the stationary point cannot be used.

3. IB exam-style practice

20 questions

31.

[Maximum mark: 8]

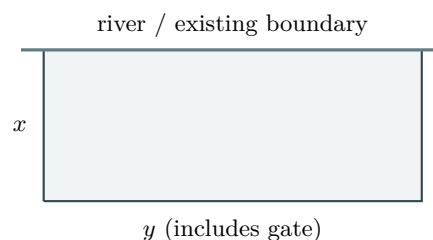
A rectangular garden is enclosed by four fences. The two sides of length x metres cost €3 per metre; the two sides of length y metres cost €6 per metre. The budget is €360.

- Show that $y = 30 - 0.5x$ when the full budget is used. [2]
- Write an area function and its feasible domain. [2]
- Find the dimensions giving maximum area and justify your answer. [4]

32.

[Maximum mark: 8]

A rectangular field borders a straight wall. No fence is needed along the wall. A 4 m gate is installed on the opposite side, and 100 m of fencing encloses the remaining boundary. Each side perpendicular to the wall has length x m.

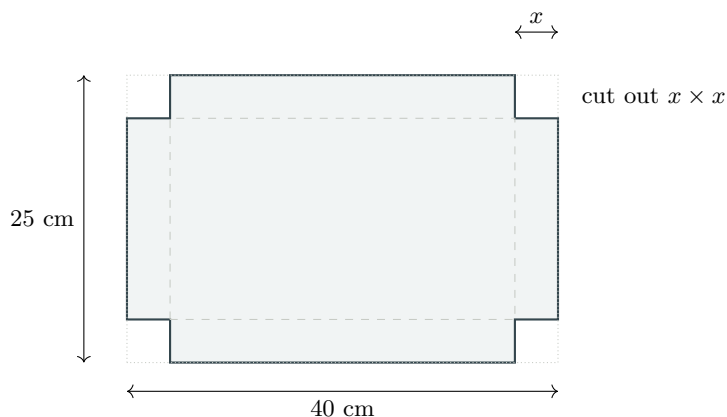


- Show that the side parallel to the wall has length $104 - 2x$ metres. [2]
- Find the feasible domain, taking account of the gate width. [2]
- Find the dimensions and greatest area of the field. [4]

33.

[Maximum mark: 9]

An open box is made from a 40 cm by 25 cm sheet by removing four corner squares of side x cm.

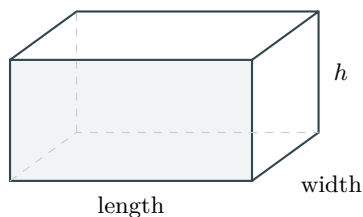


- Write the volume $V(x)$ and its feasible domain. [3]
- Find the cut size giving the greatest volume. [3]
- State the resulting box dimensions and volume, and justify that it is a maximum. [3]

34. 

[Maximum mark: 9]

A closed box with a square base has volume 800 cm^3 . Its base side is x cm and height is h cm.



- Express h in terms of x , and show that surface area is $S(x) = 2x^2 + 3200/x$. [3]
- Find the dimensions that minimize S . [3]
- Find the least surface area and justify the minimum. [3]

35.

[Maximum mark: 9]

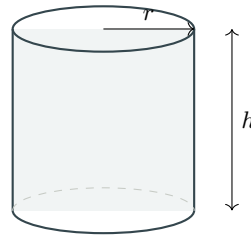
An open box with a square base uses 108 cm^2 of material. Its base side is x cm and height is h cm.

- Show that $h = (108 - x^2)/(4x)$ and state the feasible domain for x . [3]
- Show that its volume is $V(x) = 27x - x^3/4$. [2]
- Find the greatest volume and the corresponding dimensions. [4]

36. 

[Maximum mark: 9]

A closed cylindrical container holds 500 cm^3 . Material for the two circular ends costs €0.04 per cm^2 , and material for the curved side costs €0.02 per cm^2 .



- Show that the material cost is $C(r) = 0.08\pi r^2 + 20/r$ euros. [3]
- Find the radius and height minimizing the cost. [4]
- Find the least cost, to the nearest cent. [2]

37.

[Maximum mark: 9]

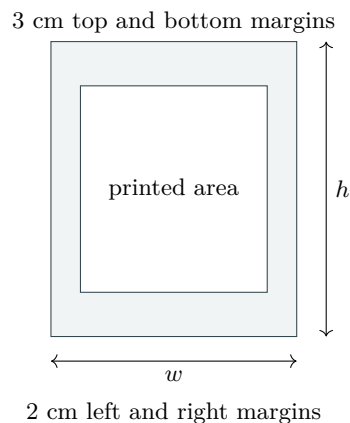
An open cylinder is made using $300\pi \text{ cm}^2$ of material. Its radius is $r \text{ cm}$ and height is $h \text{ cm}$.

- Show that $h = (300 - r^2)/(2r)$ and state the feasible domain. [3]
- Show that $V(r) = \frac{\pi}{2}(300r - r^3)$. [2]
- Find the radius, height and greatest volume. [4]

38. 

[Maximum mark: 9]

A poster must contain 600 cm^2 of printed material. The left and right margins are each 2 cm; the top and bottom margins are each 3 cm. The overall width is $w \text{ cm}$ and height is $h \text{ cm}$.

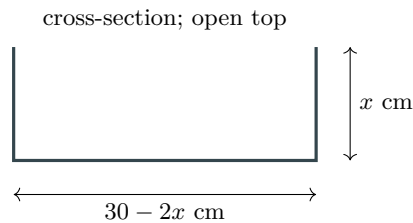


- Express h in terms of w and state the feasible domain. [3]
- Show that total paper area is $A(w) = 6w + 600w/(w - 4)$. [2]
- Find the dimensions that use the least paper and the least area. [4]

39.

[Maximum mark: 8]

A 30 cm wide metal strip is folded along its length to form a rectangular gutter. Each vertical side has height x cm. The gutter is 2 m long.



- Write the cross-sectional area $A(x)$ and the feasible domain. [2]
- Find x that maximizes the cross-sectional area. [3]
- Find the greatest volume of water the gutter could contain, in litres. [3]

40.

[Maximum mark: 8]

A parcel has a square cross-section of side x cm and length L cm. The delivery service requires $L + 4x = 120$.

- Write its volume as a function of x and state the feasible domain. [3]
- Find the dimensions giving the greatest volume. [4]
- Find that volume. [1]

41.

[Maximum mark: 7]

At a ticket price of €20, an event expects 500 visitors. Each €1 increase is predicted to reduce attendance by 15. Let n be the number of €1 increases, with $n \in \{0, 1, \dots, 30\}$.

- Show that revenue is $R(n) = 10000 + 200n - 15n^2$. [2]
- Find the maximizing value when n is temporarily treated as continuous. [2]
- Find the best permitted ticket price and its predicted revenue, showing an integer comparison. [3]

42. 

[Maximum mark: 9]

A business models profit by $P(q) = -0.02q^3 + 1.2q^2 + 12q - 80$ euros, where q is kilograms produced and $0 \leq q \leq 50$.

- Find $P'(q)$. [2]
- Find the production level giving the greatest profit and the profit achieved. [4]
- A new restriction limits production to 40 kg. Find the new optimum and explain the change. [3]

43. 

[Maximum mark: 8]

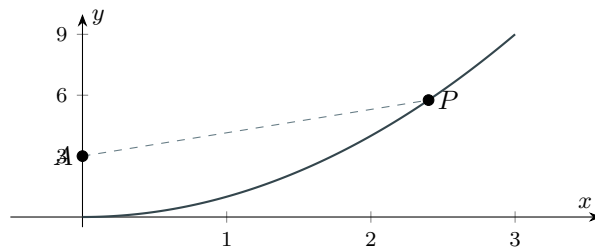
A manufacturer sells each item for €15. Total cost is $C(q) = 0.001q^3 - 0.15q^2 + 12q + 200$ euros, for integer q with $0 \leq q \leq 130$.

- Write the profit function. [2]
- Find the continuous-model production level maximizing profit. [3]
- Find the best whole-number production level and maximum profit, to the nearest cent. [3]

44. 

[Maximum mark: 9]

A connection point $P = (x, x^2)$, with $0 \leq x \leq 3$, lies on a curved rail. A straight brace joins P to $A = (0, 3)$. Coordinates are in metres.

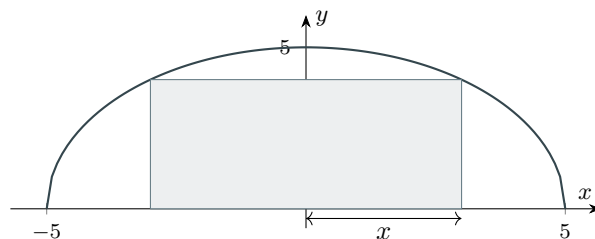


- Show that the squared brace length is $D^2 = x^4 - 5x^2 + 9$. [3]
- Find the position P giving the shortest brace. [3]
- Find the shortest length and explain why minimizing D^2 also minimizes D . [3]

45. 

[Maximum mark: 8]

A rectangle is inscribed in a semicircle of radius 5 cm. Its base lies on the diameter and its upper vertices are on the semicircle. Let x cm be half the rectangle width.



- Express the rectangle height and area in terms of x . [3]
- Find the dimensions giving greatest area. [3]
- Find the greatest area and justify that it is the maximum. [2]

46. 

[Maximum mark: 8]

A closed cylindrical can must hold 500 cm^3 , and its radius is restricted to $2 \leq r \leq 3$ cm. Its surface area is $S(r) = 2\pi r^2 + 1000/r$.

- Find the stationary radius of this model without applying the restriction. [2]
- Find the permitted radius and height minimizing surface area. [3]
- Find the least permitted surface area and explain why the stationary radius is rejected. [3]

47.

[Maximum mark: 6]

A production model $P(q)$ is continuous on $[0, 100]$. The only interior solutions of $P'(q) = 0$ are $q = 20$ and $q = 70$. Its values and derivative signs are shown.

q	0	20	70	100
$P(q)$ (euros)	-100	320	150	450

q	0	20	70	100	
$P'(q)$	+	0	-	0	+

- Identify and classify the local extrema. [2]
- Find the production level giving greatest profit on the domain. [2]
- Explain why solving $P'(q) = 0$ alone does not finish this optimisation problem. [2]

48. 

[Maximum mark: 8]

The percentage response to an advertising campaign is modelled by $E(x) = 40xe^{-0.2x}$, where x is spending in hundreds of euros and $0 \leq x \leq 12$.

- Use your GDC to find the spending that gives the greatest response percentage. [3]
- Find that percentage and justify your optimum. [3]
- Find the difference between this response and the response at the maximum permitted spending. [2]

49. 

[Maximum mark: 8]

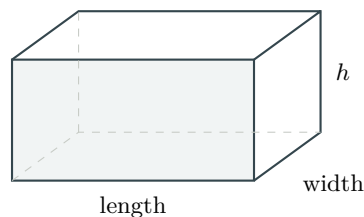
A firm models profit by $P(q) = \frac{120q}{q+10} - 2q - 20$ euros, where q is kilograms sold and $0 \leq q \leq 60$.

- Use technology to find the interior stationary point. [3]
- Find the quantity maximizing profit, and the maximum profit. [3]
- Compare with the profit at $q = 0$ and $q = 60$ to justify the result. [2]

50. 

[Maximum mark: 10]

A closed rectangular tank holds 32 m^3 . Its base width is $x \text{ m}$, its base length is $2x \text{ m}$, and its height is $h \text{ m}$. Material thickness is ignored.



- Show that $h = 16/x^2$ and the surface area is $S(x) = 4x^2 + 96/x$. [3]
- Find the dimensions giving the least surface area when $x > 0$. [3]
- The tank must now be no more than 2 m high. Find the new dimensions giving the least area and justify the boundary choice. [4]

Answer key and solution outlines

AI SL 5.7

Equivalent forms are accepted. Numerical coordinates are rounded to 3 significant figures; use unrounded values for comparisons.

Section 1: Short response

- $2x + 2\ell = 40$, so $\ell = 20 - x$ m.
- $A(x) = x(20 - x) = 20x - x^2$; $0 < x < 20$.
- $A'(x) = 20 - 2x = 0$ at $x = 10$. It changes from positive to negative, so the maximum occurs at $x = 10$.
- Third side $= 60 - 2x$; area $A(x) = x(60 - 2x)$ m².
- Dimensions x , $30 - 2x$, $20 - 2x$ cm; $0 < x < 10$.
- $V(x) = x(30 - 2x)(20 - 2x)$ cm³.
- Revenue $R(q) = pq$; profit $P(q) = pq - C(q)$.
- $R(q) = q(50 - 0.5q) = 50q - 0.5q^2$; $0 \leq q \leq 100$.
- The cost decreases up to 6 and increases after 6, so $x = 6$ is a minimum.
- The greatest profit is €160, at $q = 6$.
- Compare the revenues at $q = 12$ and $q = 13$; do not assume that rounding the continuous optimum gives the correct decision.
- $\pi r^2 h = 500 \Rightarrow h = 500/(\pi r^2)$, with $r > 0$.
- $S = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r [500/(\pi r^2)] = 2\pi r^2 + 1000/r$.
- A width must be positive. The stationary point lies outside the feasible domain and must be rejected.
- $A(x)$ is in m²; $A'(x)$ is in m² per metre (equivalently metres).

Section 2: Extended response

- Let width x , length $32 - x$. $A = 32x - x^2$, $0 < x < 32$; $A' = 32 - 2x = 0$ gives $x = 16$. Signs $+$ $\rightarrow$ $-$ show a maximum: 16 cm by 16 cm, area 256 cm².
- $2x + y = 80 \Rightarrow y = 80 - 2x$. $A = 80x - 2x^2$, $0 < x < 40$. $A' = 80 - 4x = 0$ gives $x = 20$, $y = 40$ m; maximum area 800 m² ($+$ $\rightarrow$ $-$).
- $3x + 2y = 120 \Rightarrow y = 60 - 1.5x$. $A = 60x - 1.5x^2$, $0 < x < 40$. $A' = 60 - 3x = 0$ gives $x = 20$, $y = 30$ m; maximum total area 600 m². The derivative changes $+$ $\rightarrow$ $-$ at $x = 20$.
- Numbers x , $30 - x$, with $0 < x < 30$. $P = x(30 - x)$; $P' = 30 - 2x = 0$ at $x = 15$, with $+$ $\rightarrow$ $-$. The numbers are 15 and 15; maximum product 225.
- Let width $x > 0$; height $600/x$. $P = 2x + 1200/x$; $P' = 2 - 1200/x^2 = 0$ gives $x = \sqrt{600} = 10\sqrt{6}$. Signs $-$ $\rightarrow$ $+$: square of side $10\sqrt{6} \approx 24.5$ cm; minimum perimeter $40\sqrt{6} \approx 98.0$ cm.
- $P = q(80 - 2q) - (200 + 10q) = -2q^2 + 70q - 200$. $P' = -4q + 70 = 0$ at $q = 17.5$, with $+$ $\rightarrow$ $-$. Price €45; maximum profit €412.50.
- Compare $P(17) = 412$ and $P(18) = 412$. Both 17 and 18 items maximize profit, giving €412. The model increases before 17.5 and decreases after it.
- $V = x(24 - 2x)^2$, $0 < x < 12$. $V' = 12(x - 4)(x - 12)$. The interior root $x = 4$ gives $+$ $\rightarrow$ $-$; volume tends to 0 at the two endpoints. Dimensions $16 \times 16 \times 4$ cm; maximum 1024 cm³.
- $V = x(30 - 2x)(20 - 2x)$, $0 < x < 10$. GDC maximum at $x = 3.9237478 \dots \approx 3.92$ cm; greatest volume 1056.3059 $\dots \approx 1.06 \times 10^3$ cm³. The graph increases then decreases, and endpoint limits are zero.
- $h = 750/(\pi r^2)$, so $S = 2\pi r^2 + 1500/r$, $r > 0$. GDC minimum: $r = 4.92$ cm, $h = 9.85$ cm; $S_{\min} = 457$ cm². S decreases then increases at $r = (375/\pi)^{1/3}$.
- $h = 400/(\pi r^2)$; $S = \pi r^2 + 800/r$, $r > 0$. Minimum at $r = (400/\pi)^{1/3} \approx 5.03$ cm, with $h = r \approx 5.03$ cm. S decreases then increases; minimum area 239 cm².
- $A = x(12 - x^2) = 12x - x^3$, $0 \leq x \leq \sqrt{12}$. $A' = 12 - 3x^2 = 0$ gives $x = 2$, $y = 8$; derivative $+$ $\rightarrow$ $-$ and endpoint areas 0. Maximum area 16 square units.
- $A' = 0.02 - 200/q^2 = 0$ gives $q = 100$. The derivative changes $-$ $\rightarrow$ $+$; minimum average cost $A(100) = €12$ per item.
- Capacity: $300 - 5p \leq 180 \Rightarrow p \geq 24$, so $24 \leq p \leq 60$. $R = p(300 - 5p)$; $R' = 300 - 10p = 0$ at $p = 30$, with $+$ $\rightarrow$ $-$. Attendance 150; maximum revenue €4500.
- $P' = 50 - 2q = 0$ at $q = 25$, outside the domain. On $[0, 20]$, $P' > 0$, so the maximum is at $q = 20$: $P(20) = €500$.

Section 3: IB exam-style practice

31. (a) $2(3)x + 2(6)y = 360 \Rightarrow y = 30 - x/2$. (b) $A = 30x - x^2/2$, $0 < x < 60$. (c) $A' = 30 - x = 0$ gives $x = 30$, $y = 15$ m. Signs $+$ $\rightarrow$ $-$, so maximum area 450 m^2 .
32. (a) $2x + (y - 4) = 100$, so $y = 104 - 2x$. (b) $x > 0$, $y \geq 4 \Rightarrow 0 < x \leq 50$. (c) $A = 104x - 2x^2$; $A' = 104 - 4x = 0$ at $x = 26$. Derivative signs $+$ $\rightarrow$ $-$ give $x = 26$, $y = 52$ m; maximum area 1352 m^2 .
33. (a) $V = x(40 - 2x)(25 - 2x)$, $0 < x < 12.5$. (b) $V' = 1000 - 260x + 12x^2 = 4(3x - 50)(x - 5)$. Only $x = 5$ is feasible. (c) $30 \times 15 \times 5 \text{ cm}$; volume 2250 cm^3 . The derivative is positive before 5 and negative after; volume tends to zero at the endpoints.
34. (a) $h = 800/x^2$; $S = 2x^2 + 4xh = 2x^2 + 3200/x$. (b) $S' = 4x - 3200/x^2 = 0$ gives $x^3 = 800$. Thus $x = h = \sqrt[3]{800} \approx 9.28$ cm. (c) Minimum area $6(800)^{2/3} \approx 517 \text{ cm}^2$; derivative changes $- \rightarrow +$ for $x > 0$.
35. (a) $x^2 + 4xh = 108$, so $h = (108 - x^2)/(4x)$; $0 < x < \sqrt{108}$. (b) $V = x^2h = 27x - x^3/4$. (c) $V' = 27 - 3x^2/4 = 0$ at $x = 6$. Signs $+$ $\rightarrow$ $-$; endpoint limits are zero. $h = 3$ cm; greatest volume 108 cm^3 .
36. (a) $h = 500/(\pi r^2)$; cost $0.04(2\pi r^2) + 0.02(2\pi rh) = 0.08\pi r^2 + 20/r$. (b) $C' = 0.16\pi r - 20/r^2 = 0$ gives $r = (125/\pi)^{1/3} \approx 3.41$ cm and $h = 4r \approx 13.7$ cm; signs $- \rightarrow +$. (c) Minimum cost €8.79.
37. (a) $\pi r^2 + 2\pi rh = 300\pi \Rightarrow h = (300 - r^2)/(2r)$; $0 < r < \sqrt{300}$. (b) $V = \pi r^2h = \frac{\pi}{2}(300r - r^3)$. (c) $V' = \frac{\pi}{2}(300 - 3r^2) = 0$ at $r = 10$, with $+$ $\rightarrow$ $-$. $h = 10$ cm; greatest volume $1000\pi \text{ cm}^3$.
38. (a) $(w - 4)(h - 6) = 600 \Rightarrow h = 6 + 600/(w - 4)$, $w > 4$. (b) $A = wh = 6w + 600w/(w - 4)$. (c) GDC minimum at $w = 24$, $h = 36$ cm, area 864 cm^2 . The area decreases before 24 and increases after it.
39. (a) $A = x(30 - 2x)$, $0 < x < 15$. (b) $A' = 30 - 4x = 0$ gives $x = 7.5$ cm, with $+$ $\rightarrow$ $-$; base 15 cm. (c) $A = 112.5 \text{ cm}^2$; length 200 cm, so volume $22500 \text{ cm}^3 = 22.5$ litres.
40. (a) $L = 120 - 4x$; $V = x^2(120 - 4x)$, $0 < x < 30$. (b) $V' = 12x(20 - x)$: $+$ $\rightarrow$ $-$ at $x = 20$. Length 40 cm; cross-section 20×20 cm. (c) Maximum volume 16000 cm^3 .
41. (a) $R = (20 + n)(500 - 15n) = 10000 + 200n - 15n^2$. (b) $R' = 200 - 30n = 0 \Rightarrow n = 20/3$. (c) $R(6) = 10660$, $R(7) = 10665$; the quadratic increases then decreases. Choose $n = 7$: price €27, attendance 395, revenue €10665.
42. (a) $P' = -0.06q^2 + 2.4q + 12$. (b) Feasible root $q = 20 + 10\sqrt{6} \approx 44.5$ kg, with $+$ $\rightarrow$ $-$. Profit $1067.8775 \dots \approx €1.07 \times 10^3$; endpoints give -80 and 1020 . (c) On $[0, 40]$, P is increasing, so choose 40 kg, giving €1040; the unrestricted maximum is infeasible.
43. (a) $P = -0.001q^3 + 0.15q^2 + 3q - 200$. (b) $P' = -0.003q^2 + 0.3q + 3 = 0$ gives feasible $q = 50 + 10\sqrt{35} \approx 109.161$, with $+$ $\rightarrow$ $-$. (c) Compare $P(109) = 614.121$ and $P(110) = 614.000$: choose 109 items; maximum €614.12. Endpoints give -200 and 528 .
44. (a) $D^2 = (x - 0)^2 + (x^2 - 3)^2 = x^4 - 5x^2 + 9$. (b) GDC minimum on $[0, 3]$ at $x = \sqrt{5/2} \approx 1.58$, with $y = 2.5$: $P = (\sqrt{5/2}, 2.5)$. (c) $D^2 = 11/4$, so $D = \sqrt{11}/2 \approx 1.66$ m. Square root is increasing on nonnegative values. Endpoints give $D^2 = 9, 45$, both larger.
45. (a) $h = \sqrt{25 - x^2}$; $A = 2x\sqrt{25 - x^2}$, $0 \leq x \leq 5$. (b) GDC gives $x = 5/\sqrt{2} \approx 3.54$, so width $5\sqrt{2} \approx 7.07$ cm and height $5/\sqrt{2} \approx 3.54$ cm. (c) Maximum area 25 cm^2 ; graph increases then decreases and endpoint areas are zero.
46. (a) $S' = 4\pi r - 1000/r^2 = 0$ gives $r = (250/\pi)^{1/3} \approx 4.30$ cm. (b) $S' < 0$ throughout $[2, 3]$, so choose $r = 3$ cm; $h = 500/(9\pi) \approx 17.7$ cm. (c) Least permitted area $18\pi + 1000/3 \approx 390 \text{ cm}^2$. The stationary radius exceeds the upper limit.
47. (a) Local maximum at $(20, 320)$; local minimum at $(70, 150)$. (b) Greatest profit €450 at $q = 100$. (c) Interior stationary points are candidates, but endpoints must also be compared on a closed feasible interval.
48. (a) GDC gives $x = 5$, so spend €500. (b) $E(5) = 200/e \approx 73.6\%$; the graph increases up to 5 then decreases. Endpoint responses $E(0) = 0$, $E(12) \approx 43.5\%$ are smaller. (c) $E(5) - E(12) \approx 30.0$ percentage points.
49. (a) Stationary point $(q, P) \approx (14.5, 22.0)$. (b) Choose $q = \sqrt{600} - 10 \approx 14.5$ kg; greatest profit €22.0 (3 s.f.). The derivative changes $+$ $\rightarrow$ $-$. (c) $P(0) = -20$, $P(60) = -260/7 \approx -37.1$ euros, both lower.
50. (a) $2x^2h = 32 \Rightarrow h = 16/x^2$; $S = 2(2x^2 + 2xh + xh) = 4x^2 + 96/x$. (b) $S' = 8x - 96/x^2 = 0 \Rightarrow x = \sqrt[3]{12} \approx 2.29$ m, length 4.58 m, height 3.05 m; signs $- \rightarrow +$. (c) $h \leq 2 \Rightarrow x \geq \sqrt{8}$. Since $\sqrt{8} > \sqrt[3]{12}$, S increases throughout this feasible range. Width $\sqrt{8} \approx 2.83$ m, length $2\sqrt{8} \approx 5.66$ m, height 2 m; area $32 + 24\sqrt{2} \approx 65.9 \text{ m}^2$.