Discrete and continuous random variables

IB Mathematics AA HL ▪ Section 4.14

50 original questions: mean, spread, densities, transformations and fair games

Learning goal	Calculate and interpret mean and spread; use density curves, medians, modes, linear transformations and fair-game models.
Syllabus link	AHL 4.14: both discrete and continuous random variables. This is **HL-only content**; it is not part of the AA SL syllabus.
Structure	Q1–15 short response; Q16–30 extended response; Q31–50 AA HL Paper 1-style. Worked solutions begin on page 16.
Technology	Non-calculator throughout. The IB AA formula booklet may be used. Give exact answers unless stated otherwise.

Instructions. Work on separate paper and show your reasoning. For $\mathrm{E}(X^2)$, square each discrete value before weighting, or show x^2 multiplying the density $f(x)$ inside the integral. Use $\mathrm{Var}(X) = \mathrm{E}(X^2) - [\mathrm{E}(X)]^2$ and take its square root for standard deviation. Check probability and density parameters against their allowed ranges. Include units, and distinguish payouts from net gains.

1 Short response

Build fluency with means, squared values, variance, standard deviation and density areas.

1. From mean to standard deviation

A score X has the distribution

x	-1	1	3
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(a) Find $\mathrm{E}(X)$ and $\mathrm{E}(X^2)$.

(b) Hence find $\mathrm{Var}(X)$ and the standard deviation of X.

2. Recover the expected square

A random variable X has $\mathrm{E}(X) = -2$ and $\mathrm{Var}(X) = 5$.

(a) Find $\mathrm{E}(X^2)$.

(b) Explain why $\mathrm{E}(X^2)$ is not equal to $[\mathrm{E}(X)]^2$ here.

3. Square a negative value correctly

A score Y has the distribution

y	-2	0	4
$\mathrm{P}(Y = y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

(a) Calculate $\mathrm{E}(Y)$ and $\mathrm{E}(Y^2)$.

(b) Find the standard deviation.

4. Same mean, different spread

A score A always equals 2. A second score B equals 0 or 4, each with probability $\frac{1}{2}$.

(a) Find the mean of each score.

(b) Find both variances and interpret the difference.

5. A negative multiplier

Given $\mathrm{E}(X) = 3$ and $\mathrm{Var}(X) = 4$, define $Y = 5 - 2X$. Find the mean, variance and standard deviation of Y.

6. A shift and a rescaling

A random variable X has variance 9. Define $U = X + 7$ and $V = \frac{1}{3}X$.

(a) Find the variance and standard deviation of U.

(b) Find the variance and standard deviation of V.

7. Keep track of units

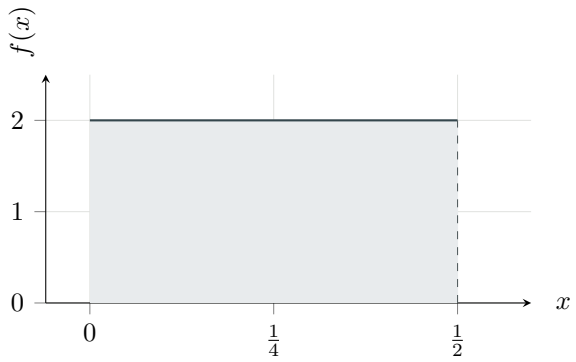
The length X of a manufactured part, in centimetres, has mean 12 and standard deviation $\frac{3}{10}$. Let Y be its length in millimetres.

(a) Write Y in terms of X.

(b) Find the mean, variance and standard deviation of Y, including units.

8. Density height is not probability

The graph shows $y = f(x)$ for a continuous random variable X. The density is 2 on $[0, \frac{1}{2}]$ and zero elsewhere.



(a) Explain why this is a valid probability density function even though its height exceeds 1.

(b) Find $\mathrm{P}(X = \frac{1}{4})$.

9. Normalize a density

A continuous random variable X has density

$$f(x) = \begin{cases} k(1+x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find k, and check non-negativity.

10. Find an area

A continuous random variable X has density

$$f(x) = \begin{cases} \frac{x}{2} & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{P}(\frac{1}{2} < X \le \frac{3}{2})$.

(b) Would including both endpoints change this probability? Explain.

11. Multiply the density by the correct power

A continuous random variable X has density

$$f(x) = \begin{cases} 3x^2 & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{E}(X)$, showing the product $xf(x)$.

(b) Find $\mathrm{E}(X^2)$, showing the product $x^2 f(x)$.

12. A median divides the area equally

A continuous random variable X has density

$$f(x) = \begin{cases} 2(1-x) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find its median m, giving an exact answer.

13. A mode is a location

A continuous random variable X has density

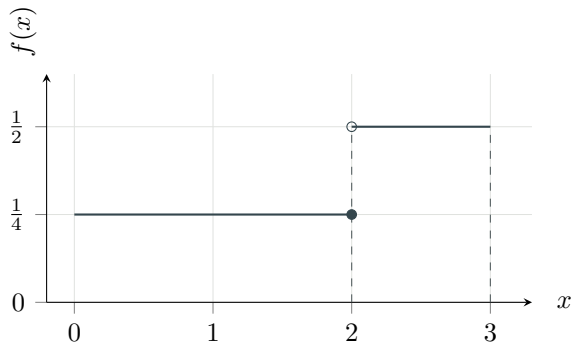
$$f(x) = \begin{cases} \frac{3}{4}(1-x^2) & -1 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

Find its mode. Distinguish the mode from the maximum density height.

14. Split an interval at a formula change

The density of X is

$$f(x) = \begin{cases} \frac{1}{4} & 0 \le x \le 2, \\ \frac{1}{2} & 2 < x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$



Find $\mathrm{P}(1 < X < \frac{5}{2})$.

15. A fee changes the mean, not the spread

On one play, a computer chooses a real number X with mean $\frac{1}{2}$ and variance $\frac{1}{12}$. The payout is $R = 6X$ euros. The entry fee is c euros, so the player's net gain is $G = 6X - c$ euros.

(a) Find the fair entry fee.

(b) Find the standard deviation of G.

2 Extended response

Combine methods, find unknown parameters and explain your results in context.

16. Use a mean before finding variance

The number X of repairs completed in one interval has distribution

x	0	1	3
$\mathrm{P}(X = x)$	p	$2p$	$1 - 3p$

It is given that $\mathrm{E}(X) = 1$.

(a) Find p and check that the resulting distribution is valid.

(b) Calculate $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$.

(c) Find the standard deviation.

17. Reject an impossible parameter

A discrete random variable X has distribution

x	-1	0	2
$\mathrm{P}(X = x)$	p	$1 - 2p$	p

(a) Find the allowed range of p.

(b) Express $\mathrm{E}(X)$, $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$ in terms of p.

(c) Given $\mathrm{Var}(X) = \frac{24}{25}$, find p.

18. Recover a distribution from mean and variance

A random variable X takes values 0, 2 and 5 with probabilities a, b, c, respectively. Its mean is 2 and its variance is 3.

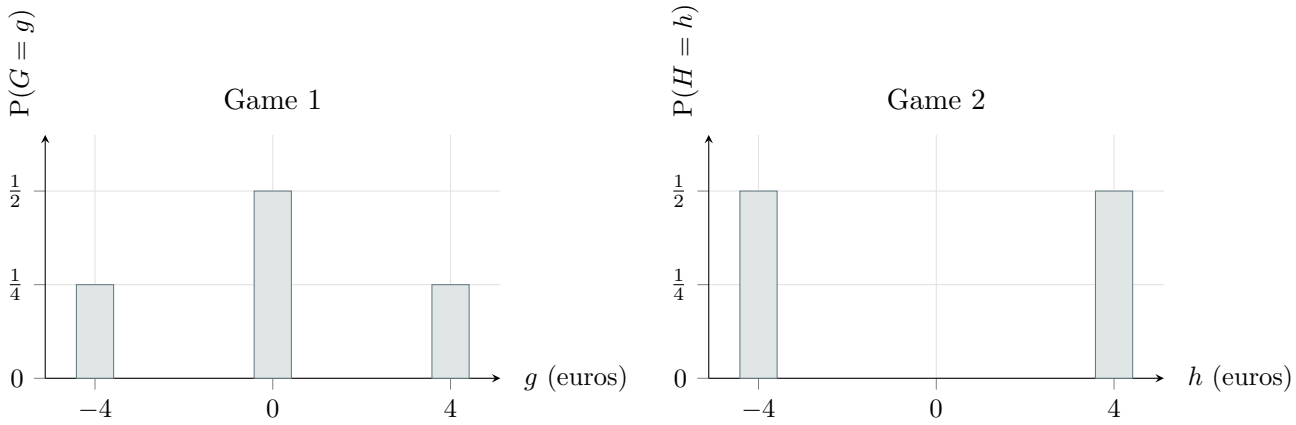
(a) Find $\mathrm{E}(X^2)$.

(b) Form three equations in a, b, c and solve them.

(c) Find $\mathrm{P}(X > \mathrm{E}(X))$.

19. Compare two net-gain distributions

The charts show net gains G and H, in euros per play, for two games. Each chart lists all possible gains.



(a) Use the charts to show that both games are fair.

(b) Find and compare their variances.

(c) Find the probability of losing money in each game.

20. Reorder values and probabilities together

A score X has distribution

x	0	2	5
$\mathrm{P}(X = x)$	$\frac{3}{10}$	$\frac{1}{2}$	$\frac{1}{5}$

An adjusted score is $Y = 10 - 2X$.

(a) Construct the distribution of Y in increasing order of its values.

(b) Find the mean and variance of Y.

(c) Find $\mathrm{P}(Y < 6)$.

21. Find both possible transformations

A random variable X has mean -1 and variance 9. The variable $Y = aX + b$ has mean 5 and standard deviation 6.

(a) Find all possible pairs (a, b).

(b) Which pair makes Y decrease as X increases?

22. Fair entry fee and organiser profit

The payout R, in euros on one play, has distribution

r	0	6	12
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

The player pays c euros. Define the player's net gain as $G = R - c$ and the organiser's profit as $H = c - R$. Ignore other costs.

(a) Find $\mathrm{E}(R)$ and $\mathrm{Var}(R)$.

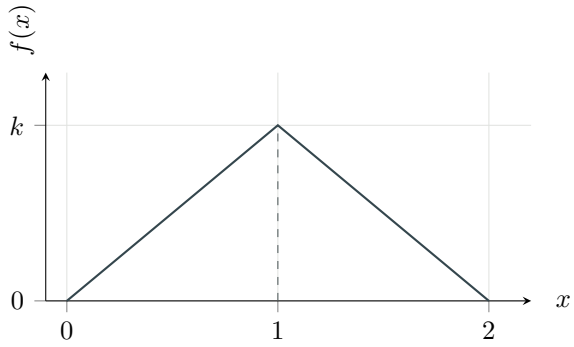
(b) Find the fair fee and the standard deviation of the player's net gain.

(c) Find the fee giving the organiser an expected profit of €1 per play. Does this alter the variance of G?

23. A symmetric triangular density

A continuous random variable X has density

$$f(x) = \begin{cases} kx & 0 \le x \le 1, \\ k(2-x) & 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Find k.

(b) Find $\mathrm{P}(X > \frac{1}{2})$.

(c) Use symmetry to state the mean and median, and state the mode.

(d) Calculate $\mathrm{E}(X^2)$ and hence the variance.

24. Locate the median in the correct interval

A continuous random variable X has density

$$f(x) = \begin{cases} kx & 0 \le x \le 1, \\ k & 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find k and $\mathrm{P}(X \le 1)$.

(b) Find the median, showing why it lies above 1.

(c) Find the mean.

25. Conditional probability and area cutoffs

A waiting time X, in minutes, has density

$$f(x) = \begin{cases} \frac{x}{8} & 0 \le x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $\mathrm{P}(X > 3 \mid X > 2)$.

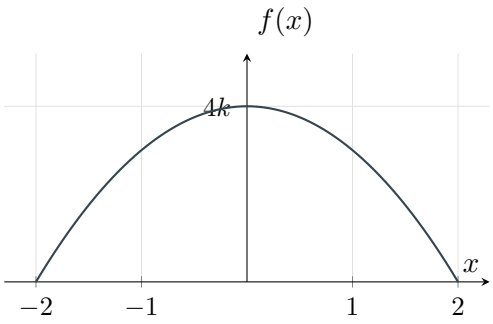
(b) Find the median waiting time.

(c) Find q such that $\mathrm{P}(X \le q) = \frac{3}{4}$.

26. Symmetry and squared values

The signed error X of a sensor, in millimetres, has density

$$f(x) = \begin{cases} k(4 - x^2) & -2 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Find k.

(b) State the mean and mode.

(c) Find the variance and standard deviation.

27. Two unknown density heights

A continuous random variable X has density

$$f(x) = \begin{cases} a & 0 \le x \le 1, \\ b & 1 < x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

Its mean is $\frac{4}{3}$.

(a) Find a and b.

(b) Find the median.

(c) Find the variance.

28. Convert a continuous time measurement

A journey time X, in minutes, has density $f(x) = \frac{1}{4}$ for $2 \le x \le 6$, and zero otherwise. A fixed 30-second check is added. Let T be the total time, in seconds.

(a) Write T in terms of X.

(b) Find the mean and variance of X by integration.

(c) Find the mean and standard deviation of T.

(d) Find $\mathrm{P}(T < 210)$.

29. A continuous game with a fair fee

On one play, a marker lands on a 1-metre strip. Let X be its distance in metres from the left end. Its density is $f(x) = 2x$ on $[0, 1]$, and zero elsewhere. The payout is $R = 9X + 1$ euros. With entry fee c euros, the player's net gain is $G = R - c$.

(a) Find the mean and variance of X.

(b) Find the fair fee and the standard deviation of G.

(c) At that fee, find the probability of making a profit.

30. A valid family of densities

A proposed density is

$$f(x) = \begin{cases} 1 + a(2x - 1) & 0 \le x \le 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that its total integral is 1 for every a.

(b) Find the range of a for which it is non-negative throughout $[0, 1]$.

(c) Given $\mathrm{E}(X) = \frac{7}{12}$, find a and then $\mathrm{P}(X > \frac{1}{2})$.

3 AA HL exam-style / Paper 1

Original non-calculator questions. Use the mark allocations to guide the depth of your working.

31. Delivery time

[Maximum mark: 9]

The number X of deliveries on one route has distribution

x	0	1	4
$\mathrm{P}(X = x)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

A simplified model gives the total route time as $T = 8 + 4X$ minutes, including preparation.

(a) Find $\mathrm{E}(X)$. [2 marks]

(b) Find $\mathrm{Var}(X)$. [3 marks]

(c) Find the mean and standard deviation of T. [3 marks]

(d) Find $\mathrm{P}(T > 12)$. [1 mark]

32. A variance determines probabilities

[Maximum mark: 9]

A discrete random variable X has distribution

x	0	1	2
$\mathrm{P}(X = x)$	p	$1 - 3p$	$2p$

It is given that $\mathrm{Var}(X) = \frac{11}{16}$.

(a) Find the allowed range of p. [2 marks]

(b) Show that $\mathrm{Var}(X) = 3p - p^2$. [3 marks]

(c) Find p, explaining any rejected solution. [3 marks]

(d) Find the standard deviation of X. [1 mark]

33. Squaring is not a linear transformation [Maximum mark: 9]

A score X has distribution

x	0	2	4
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

A new score is defined by $Y = X^2$.

(a) Find $\mathrm{E}(X)$, $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$. [3 marks]

(b) Construct the probability distribution of Y. [2 marks]

(c) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [3 marks]

(d) Explain why $\mathrm{Var}(Y)$ cannot be found by simply squaring $\mathrm{Var}(X)$. [1 mark]

34. Prize money and net gain [Maximum mark: 8]

The payout R, in euros for one play, has distribution

r	0	3	15
$\mathrm{P}(R = r)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

The entry fee is €4. Let G be the player's net gain, in euros, after paying the fee.

(a) Write G in terms of R and calculate $\mathrm{E}(G)$. [3 marks]

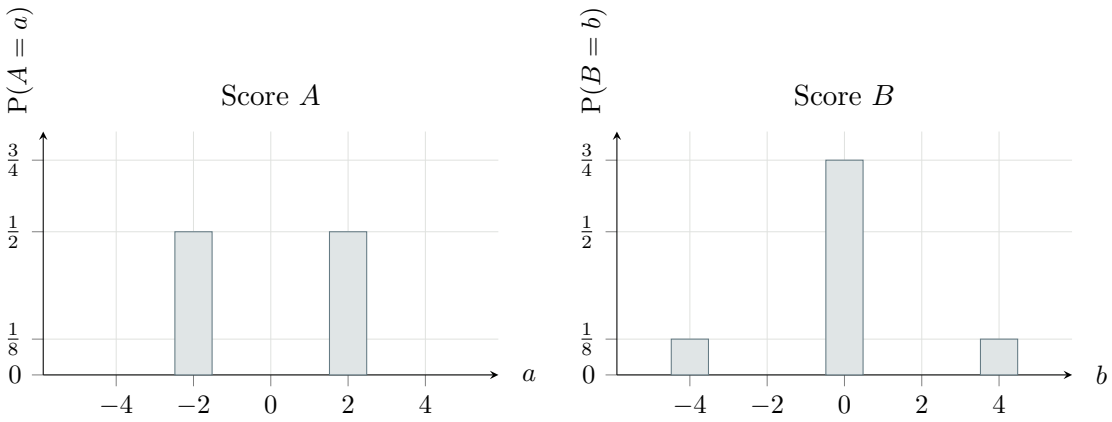
(b) Find $\mathrm{Var}(G)$ and the standard deviation of G. [3 marks]

(c) Find the probability of making a profit. [1 mark]

(d) Find the fee that would make the game fair. [1 mark]

35. Equal mean and variance do not determine a distribution [Maximum mark: 7]

The charts give two score distributions, A and B.



(a) Show that both means are zero. [2 marks]

(b) Show that both variances equal 4. [3 marks]

(c) Find $\mathrm{P}(A = 0)$ and $\mathrm{P}(B = 0)$. [1 mark]

(d) A student says that equal means and variances make the two distributions identical. Comment on this statement. [1 mark]

36. Three unknown probabilities [Maximum mark: 8]

A random variable X takes values 1, 3 and 6 with probabilities a, b, c. It has mean 3 and variance 3.

(a) Find $\mathrm{E}(X^2)$. [1 mark]

(b) Form and solve three equations to find a, b, c. [4 marks]

(c) Find $\mathrm{P}(X \geq 3)$. [1 mark]

(d) Find the mean and standard deviation of $Y = 2X - 5$. [2 marks]

37. An unknown negative multiplier [Maximum mark: 7]

A continuous random variable X has mean 4 and standard deviation 3. Define $Y = aX + b$, where $a < 0$. The mean of Y is 7 and its variance is 36. It is also known that $\mathrm{P}(X < 4) = \frac{3}{5}$.

(a) Find a and b. [3 marks]

(b) Find $\mathrm{P}(Y > 7)$. [2 marks]

(c) Explain why adding b does not affect the variance. [2 marks]

38. Expected profit and variability [Maximum mark: 8]

For one insured delivery, the compensation payment X, in euros, is 0, 20 or 80 with probabilities $\frac{3}{4}, \frac{1}{5}, \frac{1}{20}$, respectively. The company charges €10 per delivery and has no other costs. Let $H = 10 - X$ be its profit in euros on one delivery.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(b) Find the mean and standard deviation of H. [2 marks]

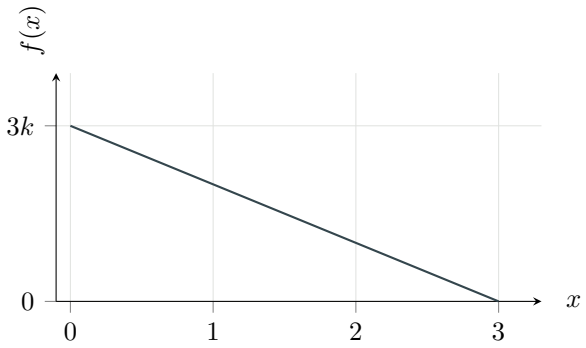
(c) Find the probability that the company loses money on a delivery. [1 mark]

(d) Using the same model for 600 deliveries, find the expected total profit. [1 mark]

39. A decreasing triangular density [Maximum mark: 12]

A continuous random variable X has density

$$f(x) = \begin{cases} k(3 - x) & 0 \leq x \leq 3, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Find k. [2 marks]

(b) Find $\mathrm{P}(1 < X < 2)$. [2 marks]

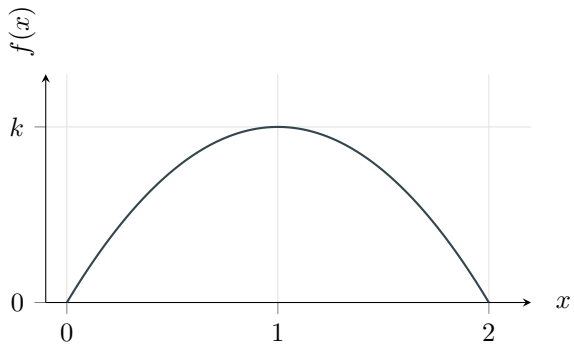
(c) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(d) Find the median and state the mode. [4 marks]

40. An interior mode [Maximum mark: 9]

A continuous random variable X has density

$$f(x) = \begin{cases} kx(2-x) & 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$



(a) Show that $k = \frac{3}{4}$. [2 marks]

(b) Find the mode using differentiation. [2 marks]

(c) Find the mean and variance. [3 marks]

(d) Find $\mathrm{P}(X > \frac{3}{2})$. [2 marks]

41. A piecewise density with a jump [Maximum mark: 9]

A continuous random variable X has density

$$f(x) = \begin{cases} kx & 0 \le x \le 2, \\ k(3-x) & 2 < x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find k. [2 marks]

(b) Find $\mathrm{P}(X > 2)$. [1 mark]

(c) Determine which interval contains the median, then find it. [3 marks]

(d) Find the mean. [3 marks]

42. Recover a density from an area [Maximum mark: 10]

The signed error X of a measuring device, in millimetres, has density

$$f(x) = \begin{cases} a & -1 \le x \le 1, \\ b & 1 < x \le 3, \\ 0 & \text{otherwise.} \end{cases}$$

It is given that $\mathrm{P}(X \le 1) = \frac{1}{3}$.

(a) Find a and b. [2 marks]

(b) Find the mean and variance of X. [4 marks]

(c) A reported score is $Y = 3X - 2$ points. Find its mean and standard deviation. [2 marks]

(d) Find the median error. [2 marks]

43. A shifted density and a reversed event [Maximum mark: 10]

A continuous random variable X has density

$$f(x) = \begin{cases} 2(x-1) & 1 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Define $T = 12 - 6X$.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [4 marks]

(b) Find the mean and variance of T. [2 marks]

(c) Find $\mathrm{P}(T > 3)$. [2 marks]

(d) Find the median of X. [2 marks]

44. A trigonometric density [Maximum mark: 7]

A continuous random variable X has density

$$f(x) = \begin{cases} k\cos x & 0 \le x \le \frac{\pi}{2}, \\ 0 & \text{otherwise.} \end{cases}$$

Angles are measured in radians.

(a) Find k. [2 marks]

(b) Find $\mathrm{P}(\frac{\pi}{6} < X < \frac{\pi}{3})$. [2 marks]

(c) Find the median. [2 marks]

(d) State the mode and justify your answer. [1 mark]

45. A fair game with a continuous payout [Maximum mark: 9]

On one play, a computer chooses a real number X between 0 and 1 with density $f(x) = 3x^2$, and zero density elsewhere. The player receives $R = 8X + 2$ euros and pays a fixed fee of c euros. Let $G = R - c$ be the player's net gain.

(a) Find $\mathrm{E}(X)$ and $\mathrm{Var}(X)$. [3 marks]

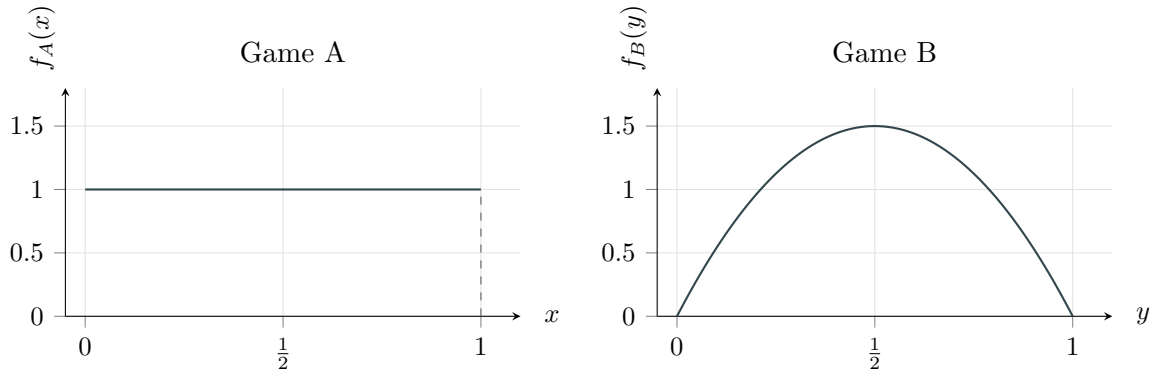
(b) Find the fair entry fee. [2 marks]

(c) Find the standard deviation of G. [2 marks]

(d) At the fair fee, find the probability of making a profit. [2 marks]

46. Compare two continuous fair games [Maximum mark: 10]

In game A, a computer selects X with density $f_A(x) = 1$ on $[0, 1]$. In game B, it selects Y with density $f_B(y) = 6y(1 - y)$ on $[0, 1]$. Both densities are zero elsewhere. The respective payouts are $R_A = 12X$ and $R_B = 12Y$ euros. Each game costs €6, so the net gains are $G_A = 12X - 6$ and $G_B = 12Y - 6$ euros.



(a) Use symmetry to show that both games are fair. [2 marks]

(b) Find $\text{Var}(X)$ and $\text{Var}(Y)$. [4 marks]

(c) Compare the standard deviations of the net gains. [3 marks]

(d) Find the probability of a profit in each game. [1 mark]

47. Reflect a triangular model [Maximum mark: 9]

A continuous random variable X has density

$$f(x) = \begin{cases} \frac{x}{4} & 0 \le x \le 2, \\ \frac{4-x}{4} & 2 < x \le 4, \\ 0 & \text{otherwise.} \end{cases}$$

Define $W = 10 - 2X$.

(a) Use symmetry to state the mean and median of X. [1 mark]

(b) Calculate $\text{E}(X^2)$ and hence $\text{Var}(X)$. [4 marks]

(c) Find the mean and standard deviation of W. [2 marks]

(d) Find $\text{P}(W > 8)$. [2 marks]

48. An unknown endpoint [Maximum mark: 10]

A continuous random variable X has density

$$f(x) = \begin{cases} kx^3 & 0 \le x \le a, \\ 0 & \text{otherwise.} \end{cases}$$

where $a > 0$ and $k > 0$.

(a) Express k in terms of a. [2 marks]

(b) Show that $\text{E}(X) = \frac{4a}{5}$ and $\text{E}(X^2) = \frac{2a^2}{3}$. [3 marks]

(c) Given $\text{Var}(X) = \frac{1}{6}$, find a and k. [3 marks]

(d) Find the median for these values of a and k. [2 marks]

49. Two parameters and a transformed score [Maximum mark: 11]

A continuous random variable X has density

$$f(x) = \begin{cases} ax & 0 \le x \le 1, \\ b & 1 < x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

Its mean is $\frac{7}{6}$.

(a) Find a and b. [3 marks]

(b) Find the median. [3 marks]

(c) Find $\mathrm{E}(X^2)$ and $\mathrm{Var}(X)$. [3 marks]

(d) A score is defined by $Y = 30 - 6X$. Find its mean and variance. [2 marks]

50. Discrete and continuous models with the same summaries [Maximum mark: 9]

A discrete random variable X has distribution

x	-1	1	3
$\mathrm{P}(X = x)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

A continuous random variable Y has density

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{6}} & 1 - \sqrt{6} \le y \le 1 + \sqrt{6}, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Show that $\mathrm{E}(X) = 1$ and $\mathrm{Var}(X) = 2$. [2 marks]

(b) Use symmetry to state $\mathrm{E}(Y)$. By integrating $(y-1)^2 f_Y(y)$, show that $\mathrm{Var}(Y) = 2$. [3 marks]

(c) Find $\mathrm{P}(X = 1)$ and $\mathrm{P}(Y = 1)$. [1 mark]

(d) Find $\mathrm{P}(X \ge 2)$ and $\mathrm{P}(Y \ge 2)$. Explain what this comparison shows. [3 marks]

Related tutorials

Variance and standard deviation
youtu.be/mQrY17mmFi0

Continuous variables and probability
youtu.be/uWwHelQHRHk

Scope and provenance. IB *Mathematics: analysis and approaches guide*, AHL 4.14, p. 60. All questions and solutions are original. Project AA HL Paper 1 references informed the command terms, subpart structure and mark allocations. The worksheet assumes SL 4.7 probability foundations and the calculus needed to integrate the stated densities; no cumulative distribution function methods are required.

Answer key and worked solutions

Try the questions before reading the solutions. The sections and question numbers match the worksheet. $\mathrm{E}(X^2)$ is the **second moment about zero**: the expected value of the squared outcomes. Variance is the mean squared deviation from the mean; standard deviation is the square root of variance.

Section 1: Short response — Questions 1–15

1. Solution

(a) $\mathrm{E}(X) = (-1)(\frac{1}{4}) + 1(\frac{1}{2}) + 3(\frac{1}{4}) = 1$. Square each value before weighting: $\mathrm{E}(X^2) = (-1)^2(\frac{1}{4}) + 1^2(\frac{1}{2}) + 3^2(\frac{1}{4}) = 3$.

(b) $\mathrm{Var}(X) = 3 - 1^2 = 2$, so $\sigma_X = \sqrt{2}$ points.

2. Solution

(a) $\mathrm{E}(X^2) = \mathrm{Var}(X) + [\mathrm{E}(X)]^2 = 5 + (-2)^2 = 9$.

(b) $[\mathrm{E}(X)]^2 = 4$. Their difference is the non-zero variance, $9 - 4 = 5$. The expected square and the square of the expected value are different quantities.

3. Solution

(a) $\mathrm{E}(Y) = (-2)(\frac{1}{2}) + 0(\frac{1}{4}) + 4(\frac{1}{4}) = 0$. Also $\mathrm{E}(Y^2) = (-2)^2(\frac{1}{2}) + 0^2(\frac{1}{4}) + 4^2(\frac{1}{4}) = 6$.

(b) $\mathrm{Var}(Y) = 6 - 0^2 = 6$, so $\sigma_Y = \sqrt{6}$ points.

4. Solution

(a) $\mathrm{E}(A) = 2$ and $\mathrm{E}(B) = 0(\frac{1}{2}) + 4(\frac{1}{2}) = 2$.

(b) $\mathrm{Var}(A) = 0$. For B, $\mathrm{E}(B^2) = 8$, so $\mathrm{Var}(B) = 8 - 2^2 = 4$. The first score is constant; the second varies despite having the same mean.

5. Solution

$\mathrm{E}(Y) = 5 - 2(3) = -1$. $\mathrm{Var}(Y) = (-2)^2(4) = 16$. Thus $\sigma_Y = 4$; a negative multiplier does not make a standard deviation negative.

6. Solution

(a) $\mathrm{Var}(U) = 9$ and $\sigma_U = 3$. Adding a constant shifts all values equally and leaves the spread unchanged.

(b) $\mathrm{Var}(V) = (\frac{1}{3})^2(9) = 1$ and $\sigma_V = \frac{1}{3}(3) = 1$.

7. Solution

(a) $Y = 10X$.

(b) $\mathrm{E}(Y) = 120$ mm; $\sigma_Y = 10(\frac{3}{10}) = 3$ mm; $\mathrm{Var}(Y) = 3^2 = 9$ mm^2.

8. Solution

(a) $f(x) \geq 0$ and the total area is $\frac{1}{2} \times 2 = 1$. It is probability, not density height, that must lie between 0 and 1.

(b) $\mathrm{P}(X = \frac{1}{4}) = 0$: a single point has zero area under a continuous density.

9. Solution

$1 = \int_0^1 k(1 + x)\,\mathrm{d}x = k[x + \frac{x^2}{2}]_0^1 = \frac{3}{2}k$, so $k = \frac{2}{3}$. Then $\frac{2}{3}(1 + x) \geq 0$ on $[0, 1]$.

Section 1: Short response — Questions 1–15 (continued)

10. Solution

(a) $\mathrm{P}(\frac{1}{2} < X \leq \frac{3}{2}) = \int_{1/2}^{3/2} \frac{x}{2}\,\mathrm{d}x = [\frac{x^2}{4}]_{1/2}^{3/2} = \frac{9}{16} - \frac{1}{16} = \frac{1}{2}$.

(b) No. Each endpoint has probability zero, so strict and non-strict inequalities give the same area.

11. Solution

(a) $\mathrm{E}(X) = \int_0^1 x(3x^2)\,\mathrm{d}x = 3\int_0^1 x^3\,\mathrm{d}x = \frac{3}{4}$.

(b) $\mathrm{E}(X^2) = \int_0^1 x^2(3x^2)\,\mathrm{d}x = 3\int_0^1 x^4\,\mathrm{d}x = \frac{3}{5}$. The probability density is still $3x^2$.

12. Solution

$\int_0^m 2(1-x)\,\mathrm{d}x = \frac{1}{2}$, so $2m - m^2 = \frac{1}{2}$. Thus $m = 1 \pm \frac{1}{\sqrt{2}}$. Only $\boxed{m = 1 - \dfrac{1}{\sqrt{2}}}$ lies in $[0, 1]$.

13. Solution

$f'(x) = -\frac{3}{2}x = 0$ at $x = 0$. The parabola opens downwards and $f(-1) = f(1) = 0$. The mode is the location $\boxed{0}$; the maximum height is $f(0) = \frac{3}{4}$.

14. Solution

Split at $x = 2$: $\mathrm{P}(1 < X < \frac{5}{2}) = \int_1^2 \frac{1}{4}\,\mathrm{d}x + \int_2^{5/2} \frac{1}{2}\,\mathrm{d}x = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

15. Solution

(a) Fairness means $\mathrm{E}(G) = 0$. Thus $6(\frac{1}{2}) - c = 0$, giving $c = 3$ euros.

(b) $\mathrm{Var}(G) = 6^2(\frac{1}{12}) = 3$, so $\sigma_G = \sqrt{3}$ euros. The fixed fee has no effect on this spread.

Section 2: Extended response — Questions 16–30

16. Solution

(a) $0p + 1(2p) + 3(1-3p) = 1$ gives $3 - 7p = 1$, so $p = \frac{2}{7}$. The probabilities $\frac{2}{7}, \frac{4}{7}, \frac{1}{7}$ are non-negative and sum to 1.

(b) $\mathrm{E}(X^2) = 0^2(\frac{2}{7}) + 1^2(\frac{4}{7}) + 3^2(\frac{1}{7}) = \frac{13}{7}$. Hence $\mathrm{Var}(X) = \frac{13}{7} - 1^2 = \frac{6}{7}$.

(c) $\sigma_X = \sqrt{\frac{6}{7}} = \frac{\sqrt{42}}{7}$ repairs.

17. Solution

(a) $p \geq 0$ and $1 - 2p \geq 0$, so $0 \leq p \leq \frac{1}{2}$. The total probability is already 1.

(b) $\mathrm{E}(X) = -p + 2p = p$; $\mathrm{E}(X^2) = p + 4p = 5p$; $\mathrm{Var}(X) = 5p - p^2$.

(c) $5p - p^2 = \frac{24}{25}$ gives $(5p-1)(5p-24) = 0$. The roots are $\frac{1}{5}$ and $\frac{24}{5}$. Only $\boxed{p = \dfrac{1}{5}}$ is allowed.

18. Solution

(a) $\mathrm{E}(X^2) = 3 + 2^2 = 7$.

(b) $a + b + c = 1$, $2b + 5c = 2$, $4b + 25c = 7$. Subtract twice the mean equation from the last: $15c = 3$. Thus $c = \frac{1}{5}$, $b = \frac{1}{2}$, $a = \frac{3}{10}$. All probabilities are valid.

(c) Only $X = 5$ is greater than 2, so $\mathrm{P}(X > \mathrm{E}(X)) = \frac{1}{5}$.

Section 2: Extended response — Questions 16–30 (continued)

19. Solution

(a) $\mathrm{E}(G) = (-4)(\frac{1}{4}) + 0(\frac{1}{2}) + 4(\frac{1}{4}) = 0$. $\mathrm{E}(H) = (-4)(\frac{1}{2}) + 4(\frac{1}{2}) = 0$. Both expected net gains are zero.

(b) $\mathrm{Var}(G) = 16(\frac{1}{4}) + 16(\frac{1}{4}) = 8$ euros^2; $\mathrm{Var}(H) = 16(\frac{1}{2}) + 16(\frac{1}{2}) = 16$ euros^2. H has twice the variance and $\sqrt{2}$ times the standard deviation.

(c) $\mathrm{P}(G < 0) = \frac{1}{4}$ and $\mathrm{P}(H < 0) = \frac{1}{2}$. Equal expected gains do not require equal chances of a loss.

20. Solution

(a)

y	0	6	10
$\mathrm{P}(Y = y)$	$\frac{1}{5}$	$\frac{1}{2}$	$\frac{3}{10}$

(b) $\mathrm{E}(X) = 2$, $\mathrm{E}(X^2) = 7$, so $\mathrm{Var}(X) = 3$. Hence $\mathrm{E}(Y) = 10 - 2(2) = 6$ and $\mathrm{Var}(Y) = (-2)^2(3) = 12$.

(c) Only $Y = 0$ is below 6, so the probability is $\frac{1}{5}$. Equivalently, $10 - 2X < 6$ means $X > 2$.

21. Solution

(a) $36 = 9a^2$, so $a = \pm 2$. The mean equation is $5 = -a + b$, so $b = 5 + a$. Thus $(a, b) = (2, 7)$ or $(-2, 3)$.

(b) A negative coefficient reverses the order, so $(a, b) = (-2, 3)$.

22. Solution

(a) $\mathrm{E}(R) = 6(\frac{1}{3}) + 12(\frac{1}{6}) = 4$. $\mathrm{E}(R^2) = 36(\frac{1}{3}) + 144(\frac{1}{6}) = 36$, so $\mathrm{Var}(R) = 36 - 4^2 = 20$.

(b) $c = \mathrm{E}(R) = 4$ euros. $\sigma_G = \sqrt{20} = 2\sqrt{5}$ euros.

(c) $\mathrm{E}(H) = c - 4 = 1$, so $c = 5$ euros. The variance of G remains 20 euros^2 because subtracting a fixed fee does not change variance.

23. Solution

(a) The triangle has base 2 and height k. Its area is k, so $k = 1$.

(b) $\mathrm{P}(X > \frac{1}{2}) = 1 - \int_0^{1/2} x \, \mathrm{d}x = 1 - \frac{1}{8} = \frac{7}{8}$.

(c) The density is symmetric about $x = 1$, so the mean and median are 1. Its unique maximum is also at 1, so the mode is 1.

(d) $\mathrm{E}(X^2) = \int_0^1 x^2(x) \, \mathrm{d}x + \int_1^2 x^2(2 - x) \, \mathrm{d}x = \frac{1}{4} + \frac{11}{12} = \frac{7}{6}$. Thus $\mathrm{Var}(X) = \frac{7}{6} - 1^2 = \frac{1}{6}$.

24. Solution

(a) $k \int_0^1 x \, \mathrm{d}x + k \int_1^2 1 \, \mathrm{d}x = \frac{3}{2}k = 1$, so $k = \frac{2}{3}$. Hence $\mathrm{P}(X \leq 1) = \frac{1}{3}$.

(b) The area up to 1 is only $\frac{1}{3} < \frac{1}{2}$. Thus $\frac{1}{3} + \int_1^m \frac{2}{3} \, \mathrm{d}x = \frac{1}{2}$, giving $m = \frac{5}{4}$.

(c) $\mathrm{E}(X) = \int_0^1 x(\frac{2}{3}x) \, \mathrm{d}x + \int_1^2 x(\frac{2}{3}) \, \mathrm{d}x = \frac{2}{9} + 1 = \frac{11}{9}$.

25. Solution

(a) $\mathrm{P}(X > 3 \mid X > 2) = \dfrac{\int_3^4 x/8 \, \mathrm{d}x}{\int_2^4 x/8 \, \mathrm{d}x} = \dfrac{7/16}{12/16} = \frac{7}{12}$.

(b) $\int_0^m \frac{x}{8} \, \mathrm{d}x = \frac{m^2}{16} = \frac{1}{2}$, so $m = 2\sqrt{2}$ minutes.

(c) $\frac{q^2}{16} = \frac{3}{4}$, so $q = 2\sqrt{3}$ minutes. Use the positive roots because waiting times are non-negative.

Section 2: Extended response — Questions 16–30 (continued)

26. Solution

(a) $1 = k\int_{-2}^{2}(4-x^2)\,\mathrm{d}x = k\frac{32}{3}$, so $k = \frac{3}{32}$.

(b) Symmetry gives $\mathrm{E}(X) = 0$ mm. The unique maximum is at $x = 0$, so the mode is 0 mm.

(c) $\mathrm{E}(X^2) = \int_{-2}^{2} x^2\left(\frac{3}{32}(4-x^2)\right)\mathrm{d}x = \frac{3}{32}\left[\frac{4x^3}{3} - \frac{x^5}{5}\right]_{-2}^{2} = \frac{4}{5}$. Thus $\mathrm{Var}(X) = \frac{4}{5}$ mm^2 and $\sigma_X = \frac{2\sqrt{5}}{5}$ mm.

27. Solution

(a) Total area: $a + 2b = 1$. Mean: $\frac{a}{2} + 4b = \frac{4}{3}$. Solving gives $a = \frac{4}{9}$, $b = \frac{5}{18}$, both non-negative.

(b) Area up to 1 is $\frac{4}{9} < \frac{1}{2}$. Hence $\frac{4}{9} + \frac{5}{18}(m-1) = \frac{1}{2}$, so $m = \frac{6}{5}$.

(c) $\mathrm{E}(X^2) = \int_0^1 x^2\left(\frac{4}{9}\right)\mathrm{d}x + \int_1^3 x^2\left(\frac{5}{18}\right)\mathrm{d}x = \frac{4}{27} + \frac{65}{27} = \frac{23}{9}$. Thus $\mathrm{Var}(X) = \frac{23}{9} - \left(\frac{4}{3}\right)^2 = \frac{7}{9}$.

28. Solution

(a) $T = 60X + 30$.

(b) $\mathrm{E}(X) = \int_2^6 x\left(\frac{1}{4}\right)\mathrm{d}x = 4$ minutes. $\mathrm{E}(X^2) = \int_2^6 x^2\left(\frac{1}{4}\right)\mathrm{d}x = \frac{52}{3}$, so $\mathrm{Var}(X) = \frac{52}{3} - 16 = \frac{4}{3}$ minutes^2.

(c) $\mathrm{E}(T) = 60(4) + 30 = 270$ seconds. $\sigma_T = 60\sqrt{\frac{4}{3}} = 40\sqrt{3}$ seconds.

(d) $T < 210 \iff X < 3$. Hence $\mathrm{P}(T < 210) = \int_2^3 \frac{1}{4}\,\mathrm{d}x = \frac{1}{4}$.

29. Solution

(a) $\mathrm{E}(X) = \int_0^1 x(2x)\,\mathrm{d}x = \frac{2}{3}$. $\mathrm{E}(X^2) = \int_0^1 x^2(2x)\,\mathrm{d}x = \frac{1}{2}$. Thus $\mathrm{Var}(X) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.

(b) $\mathrm{E}(R) = 9\left(\frac{2}{3}\right) + 1 = 7$, so $c = 7$ euros. $\mathrm{Var}(G) = 9^2\left(\frac{1}{18}\right) = \frac{9}{2}$, so $\sigma_G = \frac{3\sqrt{2}}{2}$ euros.

(c) $G = 9X - 6 > 0 \iff X > \frac{2}{3}$. Thus $\mathrm{P}(G > 0) = \int_{2/3}^1 2x\,\mathrm{d}x = 1 - \frac{4}{9} = \frac{5}{9}$. Fairness does not require a profit probability of $\frac{1}{2}$.

30. Solution

(a) $\int_0^1 [1 + a(2x-1)]\,\mathrm{d}x = [x + a(x^2 - x)]_0^1 = 1$.

(b) The graph is a straight line, so check the endpoint heights: $1 - a \geq 0$ and $1 + a \geq 0$. Thus $-1 \leq a \leq 1$.

(c) $\mathrm{E}(X) = \int_0^1 x[1 + a(2x-1)]\,\mathrm{d}x = \frac{1}{2} + \frac{a}{6} = \frac{7}{12}$ gives $a = \frac{1}{2}$. Then $f(x) = x + \frac{1}{2}$ and $\mathrm{P}(X > \frac{1}{2}) = \int_{1/2}^1 (x + \frac{1}{2})\,\mathrm{d}x = \frac{5}{8}$.

Section 3: AA HL exam-style / Paper 1 — Questions 31–50

31. Solution

(a) $\mathrm{E}(X) = 0\left(\frac{1}{2}\right) + 1\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) = \frac{5}{4}$.

(b) $\mathrm{E}(X^2) = 0^2\left(\frac{1}{2}\right) + 1^2\left(\frac{1}{4}\right) + 4^2\left(\frac{1}{4}\right) = \frac{17}{4}$. Thus $\mathrm{Var}(X) = \frac{17}{4} - \left(\frac{5}{4}\right)^2 = \frac{43}{16}$.

(c) $\mathrm{E}(T) = 8 + 4\left(\frac{5}{4}\right) = 13$ minutes. $\mathrm{Var}(T) = 4^2\left(\frac{43}{16}\right) = 43$, so $\sigma_T = \sqrt{43}$ minutes.

(d) $T > 12 \iff X > 1$. Only $X = 4$ qualifies, so the probability is $\frac{1}{4}$.

Section 3: AA HL exam-style / Paper 1 — Questions 31–50 (continued)

32. Solution

(a) Non-negativity gives $p \geq 0$ and $1-3p \geq 0$, so $0 \leq p \leq \frac{1}{3}$.

(b) $\mathrm{E}(X) = 1-3p+4p = 1+p$ and $\mathrm{E}(X^2) = 1-3p+8p = 1+5p$. Hence $\mathrm{Var}(X) = 1+5p-(1+p)^2 = 3p-p^2$.

(c) $3p-p^2 = \frac{11}{16}$ gives $(4p-1)(4p-11) = 0$. Thus $p = \frac{1}{4}$ or $\frac{11}{4}$; the latter is outside $[0, \frac{1}{3}]$. Therefore $p = \frac{1}{4}$.

(d) $\sigma_X = \sqrt{\frac{11}{16}} = \frac{\sqrt{11}}{4}$.

33. Solution

(a) $\mathrm{E}(X) = 2$; $\mathrm{E}(X^2) = 0^2(\frac{1}{4}) + 2^2(\frac{1}{2}) + 4^2(\frac{1}{4}) = 6$; $\mathrm{Var}(X) = 6 - 2^2 = 2$.

(b)

y	0	4	16
$\mathrm{P}(Y=y)$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{1}{4}$

(c) $\mathrm{E}(Y) = 6$. $\mathrm{E}(Y^2) = 0^2(\frac{1}{4}) + 4^2(\frac{1}{2}) + 16^2(\frac{1}{4}) = 72$. Thus $\mathrm{Var}(Y) = 72 - 6^2 = 36$.

(d) The rule $\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)$ requires a linear expression. Here $Y = X^2$ is not of that form; indeed $[\mathrm{Var}(X)]^2 = 4 \neq 36$.

34. Solution

(a) $G = R - 4$. $\mathrm{E}(R) = 3(\frac{1}{3}) + 15(\frac{1}{6}) = \frac{7}{2}$, so $\mathrm{E}(G) = \frac{7}{2} - 4 = -\frac{1}{2}$ euro.

(b) $\mathrm{E}(R^2) = 9(\frac{1}{3}) + 225(\frac{1}{6}) = \frac{81}{2}$. Thus $\mathrm{Var}(G) = \mathrm{Var}(R) = \frac{81}{2} - (\frac{7}{2})^2 = \frac{113}{4}$ euros^2, and $\sigma_G = \frac{\sqrt{113}}{2}$ euros.

(c) Only the €15 payout exceeds the fee, so $\mathrm{P}(G > 0) = \frac{1}{6}$.

(d) The fair fee is $\mathrm{E}(R) = \frac{7}{2}$ euros, or €3.50.

35. Solution

(a) $\mathrm{E}(A) = (-2)(\frac{1}{2}) + 2(\frac{1}{2}) = 0$. $\mathrm{E}(B) = (-4)(\frac{1}{8}) + 0(\frac{3}{4}) + 4(\frac{1}{8}) = 0$.

(b) $\mathrm{Var}(A) = 4(\frac{1}{2}) + 4(\frac{1}{2}) = 4$. $\mathrm{Var}(B) = 16(\frac{1}{8}) + 16(\frac{1}{8}) = 4$. Both means are zero, so these second moments equal the variances.

(c) $\mathrm{P}(A = 0) = 0$ and $\mathrm{P}(B = 0) = \frac{3}{4}$.

(d) The statement is false. Mean and variance summarize a distribution but do not specify every value and probability. The probabilities in part (c) provide a counterexample.

36. Solution

(a) $\mathrm{E}(X^2) = 3 + 3^2 = 12$.

(b) $a+b+c = 1$, $a+3b+6c = 3$, $a+9b+36c = 12$. Subtracting the first equation from the next two gives $2b+5c = 2$ and $8b+35c = 11$. Hence $15c = 3$, so $c = \frac{1}{5}$, $b = \frac{1}{2}$, $a = \frac{3}{10}$.

(c) $\mathrm{P}(X \geq 3) = \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$.

(d) $\mathrm{E}(Y) = 2(3) - 5 = 1$, and $\sigma_Y = 2\sqrt{3}$.

Section 3: AA HL exam-style / Paper 1 — Questions 31–50 (continued)

37. Solution

(a) $36 = a^2(3^2)$, so $a^2 = 4$. Since $a < 0$, $a = -2$. Then $7 = -2(4) + b$ gives $b = 15$.

(b) $Y > 7 \iff -2X + 15 > 7 \iff X < 4$, reversing the inequality when dividing by -2. Thus the probability is $\frac{3}{5}$.

(c) $Y - \mathrm{E}(Y) = aX + b - (a\,\mathrm{E}(X) + b) = a(X - \mathrm{E}(X))$. The shift cancels from each deviation from the mean.

38. Solution

(a) $\mathrm{E}(X) = 20(\frac{1}{5}) + 80(\frac{1}{20}) = 8$. $\mathrm{E}(X^2) = 400(\frac{1}{5}) + 6400(\frac{1}{20}) = 400$, so $\mathrm{Var}(X) = 400 - 8^2 = 336$.

(b) $\mathrm{E}(H) = 10 - 8 = 2$ euros. $\mathrm{Var}(H) = (-1)^2(336) = 336$, so $\sigma_H = \sqrt{336} = 4\sqrt{21}$ euros.

(c) $H < 0 \iff X > 10$, so the probability is $\frac{1}{5} + \frac{1}{20} = \frac{1}{4}$.

(d) $600\,\mathrm{E}(H) = 600(2) = €1200$. This is an expected total, not a guaranteed profit.

39. Solution

(a) $1 = k\int_0^3 (3 - x)\,\mathrm{d}x = \frac{9}{2}k$, so $k = \frac{2}{9}$.

(b) $\mathrm{P}(1 < X < 2) = \frac{2}{9}[3x - \frac{x^2}{2}]_1^2 = \frac{1}{3}$.

(c) $\mathrm{E}(X) = \int_0^3 x(\frac{2}{9}(3 - x))\,\mathrm{d}x = 1$. $\mathrm{E}(X^2) = \int_0^3 x^2(\frac{2}{9}(3 - x))\,\mathrm{d}x = \frac{3}{2}$. Thus $\mathrm{Var}(X) = \frac{3}{2} - 1 = \frac{1}{2}$.

(d) $\frac{2}{9}(3m - \frac{m^2}{2}) = \frac{1}{2}$ gives $m = 3 \pm \frac{3}{\sqrt{2}}$. The root in $[0, 3]$ is $m = 3 - \frac{3}{\sqrt{2}}$. The density is greatest at the left endpoint, so the mode is 0.

40. Solution

(a) $1 = k\int_0^2 (2x - x^2)\,\mathrm{d}x = \frac{4}{3}k$, so $k = \frac{3}{4}$.

(b) $f'(x) = \frac{3}{4}(2 - 2x)$ vanishes at $x = 1$. Also $f''(x) = -\frac{3}{2} < 0$ and both endpoint heights are zero. Thus the mode is 1.

(c) Symmetry about 1 gives $\mathrm{E}(X) = 1$. Also $\mathrm{E}(X^2) = \int_0^2 x^2(\frac{3}{4}x(2 - x))\,\mathrm{d}x = \frac{6}{5}$. Thus $\mathrm{Var}(X) = \frac{6}{5} - 1 = \frac{1}{5}$.

(d) $\mathrm{P}(X > \frac{3}{2}) = \frac{3}{4}[x^2 - \frac{x^3}{3}]_{3/2}^2 = \frac{5}{32}$.

41. Solution

(a) $1 = k\int_0^2 x\,\mathrm{d}x + k\int_2^3 (3 - x)\,\mathrm{d}x = 2k + \frac{k}{2}$, so $k = \frac{2}{5}$.

(b) $\mathrm{P}(X > 2) = \frac{2}{5}\int_2^3 (3 - x)\,\mathrm{d}x = \frac{1}{5}$.

(c) $\mathrm{P}(X \le 2) = \frac{4}{5} > \frac{1}{2}$, so $0 < m < 2$. Thus $\int_0^m \frac{2}{5}x\,\mathrm{d}x = \frac{m^2}{5} = \frac{1}{2}$, giving $m = \sqrt{\frac{5}{2}} = \frac{\sqrt{10}}{2}$.

(d) $\mathrm{E}(X) = \int_0^2 x(\frac{2}{5}x)\,\mathrm{d}x + \int_2^3 x(\frac{2}{5}(3 - x))\,\mathrm{d}x = \frac{2}{5}(\frac{8}{3} + \frac{7}{6}) = \frac{23}{15}$.

42. Solution

(a) $2a = \frac{1}{3}$, so $a = \frac{1}{6}$. Total area gives $2a + 2b = 1$, so $b = \frac{1}{3}$.

(b) $\mathrm{E}(X) = \int_{-1}^1 x(\frac{1}{6})\,\mathrm{d}x + \int_1^3 x(\frac{1}{3})\,\mathrm{d}x = \frac{4}{3}$. Also $\mathrm{E}(X^2) = \frac{1}{9} + \frac{26}{9} = 3$. Thus $\mathrm{Var}(X) = 3 - (\frac{4}{3})^2 = \frac{11}{9}$ mm^2.

(c) $\mathrm{E}(Y) = 3(\frac{4}{3}) - 2 = 2$ points. $\sigma_Y = 3\sqrt{\frac{11}{9}} = \sqrt{11}$ points.

(d) The area to 1 is $\frac{1}{3}$, so the median is above 1. Solve $\frac{1}{3} + \frac{1}{3}(m - 1) = \frac{1}{2}$: $m = \frac{3}{2}$ mm.

Section 3: AA HL exam-style / Paper 1 — Questions 31–50 (continued)

43. Solution

(a) $\mathrm{E}(X) = \int_1^2 x(2(x-1))\,\mathrm{d}x = \frac{5}{3}$. $\mathrm{E}(X^2) = \int_1^2 x^2(2(x-1))\,\mathrm{d}x = \frac{17}{6}$. Thus $\mathrm{Var}(X) = \frac{17}{6} - \frac{25}{9} = \frac{1}{18}$.

(b) $\mathrm{E}(T) = 12 - 6(\frac{5}{3}) = 2$; $\mathrm{Var}(T) = (-6)^2(\frac{1}{18}) = 2$.

(c) $T > 3 \iff X < \frac{3}{2}$. Hence $\mathrm{P}(T > 3) = \int_1^{3/2} 2(x-1)\,\mathrm{d}x = \frac{1}{4}$.

(d) $\int_1^m 2(x-1)\,\mathrm{d}x = (m-1)^2 = \frac{1}{2}$. The root in $[1,2]$ is $m = 1 + \frac{1}{\sqrt{2}}$.

44. Solution

(a) $1 = k\int_0^{\pi/2} \cos x\,\mathrm{d}x = k[\sin x]_0^{\pi/2} = k$, so $k = 1$. The cosine is non-negative on the given interval.

(b) $[\sin x]_{\pi/6}^{\pi/3} = \frac{\sqrt{3}-1}{2}$.

(c) $\int_0^m \cos x\,\mathrm{d}x = \sin m = \frac{1}{2}$. In $[0, \frac{\pi}{2}]$, this gives $m = \frac{\pi}{6}$.

(d) The cosine decreases from 1 to 0 on this interval, so the density has its maximum at $x = 0$. The mode is 0.

45. Solution

(a) $\mathrm{E}(X) = \int_0^1 x(3x^2)\,\mathrm{d}x = \frac{3}{4}$. $\mathrm{E}(X^2) = \int_0^1 x^2(3x^2)\,\mathrm{d}x = \frac{3}{5}$. Thus $\mathrm{Var}(X) = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{80}$.

(b) $c = \mathrm{E}(R) = 8(\frac{3}{4}) + 2 = 8$ euros.

(c) $\mathrm{Var}(G) = 8^2(\frac{3}{80}) = \frac{12}{5}$, so $\sigma_G = \sqrt{\frac{12}{5}} = \frac{2\sqrt{15}}{5}$ euros.

(d) At $c = 8$, $G = 8X - 6 > 0 \iff X > \frac{3}{4}$. Thus $\mathrm{P}(G > 0) = \int_{3/4}^1 3x^2\,\mathrm{d}x = 1 - \frac{27}{64} = \frac{37}{64}$.

46. Solution

(a) Both densities are symmetric about $\frac{1}{2}$. Thus $\mathrm{E}(X) = \mathrm{E}(Y) = \frac{1}{2}$, and both expected net gains are $12(\frac{1}{2}) - 6 = 0$.

(b) $\mathrm{E}(X^2) = \int_0^1 x^2(1)\,\mathrm{d}x = \frac{1}{3}$, so $\mathrm{Var}(X) = \frac{1}{12}$. Also $\mathrm{E}(Y^2) = \int_0^1 y^2(6y(1-y))\,\mathrm{d}y = 6(\frac{1}{4} - \frac{1}{5}) = \frac{3}{10}$, so $\mathrm{Var}(Y) = \frac{1}{20}$.

(c) $\sigma_{G_A} = 12\sqrt{\frac{1}{12}} = 2\sqrt{3}$ euros; $\sigma_{G_B} = 12\sqrt{\frac{1}{20}} = \frac{6\sqrt{5}}{5}$ euros. Their variances are 12 and $\frac{36}{5}$, respectively, so game B has less spread.

(d) A profit requires $X > \frac{1}{2}$ or $Y > \frac{1}{2}$, respectively. By symmetry each probability is $\frac{1}{2}$. Equal fairness and profit probabilities do not imply equal variability.

47. Solution

(a) The density is symmetric about 2, so both the mean and median are 2.

(b) $\mathrm{E}(X^2) = \int_0^2 x^2(\frac{x}{4})\,\mathrm{d}x + \int_2^4 x^2(\frac{4-x}{4})\,\mathrm{d}x = 1 + \frac{11}{3} = \frac{14}{3}$. Thus $\mathrm{Var}(X) = \frac{14}{3} - 4 = \frac{2}{3}$.

(c) $\mathrm{E}(W) = 10 - 2(2) = 6$. $\mathrm{Var}(W) = 4(\frac{2}{3}) = \frac{8}{3}$, so $\sigma_W = \sqrt{\frac{8}{3}} = \frac{2\sqrt{6}}{3}$.

(d) $W > 8 \iff X < 1$, so $\mathrm{P}(W > 8) = \int_0^1 \frac{x}{4}\,\mathrm{d}x = \frac{1}{8}$.

48. Solution

(a) $1 = \int_0^a kx^3\,\mathrm{d}x = \frac{ka^4}{4}$, so $k = \frac{4}{a^4}$.

(b) $\mathrm{E}(X) = \int_0^a x(\frac{4}{a^4}x^3)\,\mathrm{d}x = \frac{4}{a^4}\frac{a^5}{5} = \frac{4a}{5}$. Similarly, $\mathrm{E}(X^2) = \int_0^a x^2(\frac{4}{a^4}x^3)\,\mathrm{d}x = \frac{4}{a^4}\frac{a^6}{6} = \frac{2a^2}{3}$.

(c) $\mathrm{Var}(X) = \frac{2a^2}{3} - (\frac{4a}{5})^2 = \frac{2a^2}{75} = \frac{1}{6}$, so $a^2 = \frac{25}{4}$. Since $a > 0$, $a = \frac{5}{2}$ and $k = \frac{64}{625}$.

(d) $\int_0^m \frac{4}{a^4}x^3\,\mathrm{d}x = \frac{m^4}{a^4} = \frac{1}{2}$. Hence $m = \frac{a}{\sqrt[4]{2}} = \frac{5}{2\sqrt[4]{2}}$.

Section 3: AA HL exam-style / Paper 1 — Questions 31–50 (continued)

49. Solution

(a) Area gives $\frac{a}{2}+b=1$. The mean gives $\frac{a}{3}+\frac{3b}{2}=\frac{7}{6}$. Substituting $b=1-\frac{a}{2}$ yields $\frac{3}{2}-\frac{5a}{12}=\frac{7}{6}$. Hence $a=\frac{4}{5}$ and $b=\frac{3}{5}$, both non-negative.

(b) The area to 1 is $\frac{a}{2}=\frac{2}{5}<\frac{1}{2}$. Thus $\frac{2}{5}+\frac{3}{5}(m-1)=\frac{1}{2}$, giving $m=\frac{7}{6}$.

(c) $\mathrm{E}(X^2)=\int_0^1 x^2(\frac{4}{5}x)\,\mathrm{d}x+\int_1^2 x^2(\frac{3}{5})\,\mathrm{d}x=\frac{1}{5}+\frac{7}{5}=\frac{8}{5}$. Thus $\mathrm{Var}(X)=\frac{8}{5}-(\frac{7}{6})^2=\frac{43}{180}$.

(d) $\mathrm{E}(Y)=30-6(\frac{7}{6})=23$ points. $\mathrm{Var}(Y)=(-6)^2(\frac{43}{180})=\frac{43}{5}$ points^2.

50. Solution

(a) $\mathrm{E}(X)=(-1)(\frac{1}{4})+1(\frac{1}{2})+3(\frac{1}{4})=1$. $\mathrm{E}(X^2)=3$, so $\mathrm{Var}(X)=3-1^2=2$.

(b) The density is symmetric about 1, so $\mathrm{E}(Y)=1$. Put $u=y-1$: $\mathrm{Var}(Y)=\frac{1}{2\sqrt{6}}\int_{-\sqrt{6}}^{\sqrt{6}}u^2\,\mathrm{d}u=\frac{1}{2\sqrt{6}}\left[\frac{u^3}{3}\right]_{-\sqrt{6}}^{\sqrt{6}}=2$.

(c) $\mathrm{P}(X=1)=\frac{1}{2}$, whereas $\mathrm{P}(Y=1)=0$ because Y is continuous.

(d) $\mathrm{P}(X\geq 2)=\frac{1}{4}$. Also $\mathrm{P}(Y\geq 2)=\int_2^{1+\sqrt{6}}\frac{1}{2\sqrt{6}}\,\mathrm{d}y=\frac{\sqrt{6}-1}{2\sqrt{6}}$. These probabilities differ: identical mean and variance do not imply identical distributions or event probabilities.